Stat-491-Fall2014-Assignment-IV

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1. (a) Suppose one of the states of an irreducible Markov chain has a self loop. Show that the chain has period 1.

   (b) Let N be a collection of integers closed under addition. Let 100 be the first integer in N for which the immediate successor is also in N. Find an integer after which every integer is present in N.

   (c) Using Euclid's Algorithm, find a, b such that $100a + 81b = 1$.

   Solution

   (a) Let x, y be two states with the selfloop at x. There is a loop of length 1 at y. So the period at y is 1. Since both are recurrent and the chain is irreducible there exists a path of length say l from x to y and a path of length say m from y to x. By going from x to y through the path of length l staying at y for time k or for $k+1$ and returning through the path of length m, we have two loops of length $l+m+k$ and $l+m+k+1$. The gcd of these two numbers is 1. Therefore the period of the chain at x is 1. This holds for all states and therefore the chain has period 1.

   (b) If for some $k \geq 1$, the numbers $k(100), ..., k(100)+k$ are present in N, then because $100, 101$ are present in N, so are $(k+1)100, \ldots, (k+1)(100)+(k+1)$. By induction, since $(1)100, (1)100+1$ are present, therefore, for all $k \geq 1$, $k(100), ..., k(100)+k$ are present in N. Choosing $k = 99$, we see that the length of the blocks becomes at least as long as the spacing between the blocks, and so, 9900 is an integer of the kind specified in the problem.

   (c) By taking repeated division and finding reminder we get

   $19 = 100 - 81; 5 = 81 - 4(100-81); -1 = 19 - 4[81 - 4(100-81)] = (100-81) - 4[81 - 4(100-81)];$

   So $1 = 21 \times 81 - 17 \times 100$.

2. (7 points). At a local 2 year college, 2/3 of freshmen become sophomores, 1/4 remain freshmen, and 1/12 drop out. 2/3 of sophomores graduate, 1/4 remain as sophomores and 1/12 dropout. Take the states to be 'F' for freshmen,'S' for sophomore, 'D' for dropout and 'G' for graduate. Let $q(i, R)$ denote the probability that from state i, we will eventually reach the absorbing state R. Let $l(i, R)$ denote the expected time to reach absorbing state R from state i.

(a) Write the transition matrix for the Markov chain with rows and columns in the order F, S, G, D.

(b) Write the set of equations for $q(i, G)$, $q(i, D)$.

(c) Write the set of equations for $l(i, G \cup D)$.

(d) What fraction of new students eventually graduate?

(e) What is the expected time of graduation for a sophomore student?

Solution

(a) (1 point)

$$\begin{pmatrix} & F & S & G & D \\ F & 1/4 & 2/3 & 0 & 1/12 \\ S & 0 & 1/4 & 2/3 & 1/12 \\ G & 0 & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

(b) (2 points) The equations for $q(i, G)$ are

$$q(F, G) = p(F, G) + p(F, F)q(F, G) + p(F, S)q(S, G).$$

$$q(S, G) = p(S, G) + p(S, S)q(S, G) + p(S, F)q(F, G).$$

Substituting the values we get

$$q(F, G) = 1/4q(F, G) + 2/3q(S, G).$$

$$q(S, G) = 2/3 + 1/4q(S, G).$$

Solving we get

$$q(F, G) = 64/81; q(s, G) = 8/9.$$

The equations for $q(i, D)$ are

$$q(F, D) = p(F, D) + p(F, F)q(F, D) + p(F, S)q(S, D).$$

$$q(S, D) = p(S, D) + p(S, S)q(S, D) + p(S, D)q(F, D).$$

(c) (2 points) The equations for $l(i, GD)$ are

$$l(F, GD) = 1 + p(F, F)l(F, GD) + p(F, S)l(S, GD).$$

$$l(S, GD) = 1 + p(S, S)l(S, GD) + p(S, F)l(F, GD).$$

Substituting the values of $p(i, j)$, we get the equations,

$$l(F, GD) = 1 + 1/4l(F, GD) + 2/3l(S, GD).$$

$$l(S, GD) = 1 + 1/4l(S, GD).$$

So we get $l(s, GD) = 4/3$.

(d) (1 point) As calculated above $q(F, G) = 64/81$.

(e) (1 point) As calculated above $l(F, GD) = 4/3$.

3. (5 points). A certain Markov chain has transition matrix

$$\begin{pmatrix} & A & B & C & D & E \\ A & 0 & 0 & 1/3 & 1/3 & 1/3 \\ B & 1/3 & 1/3 & 1/3 & 0 & 0 \\ C & 0 & 1/3 & 1/3 & 1/3 & 0 \\ D & 0 & 0 & 0 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

(a) Which are the transient and which, the recurrent states? Are there any absorbing states?

(b) Write linear equations for $q(i, k), l(i, k)$, where k is an absorbing state.

Solution

(a) E is an absorbing state and we can reach E from all other states through directed paths but not return. So A, B, C, D are transient.

(b) The equations for $q(i, E)$ are

$$q(A, E) = 1/3 + 1/3q(C, E) + 1/3q(D, E);$$

$$q(B, E) = 1/3q(A, E) + 1/3q(B, E) + 1/3q(C, E);$$

$$q(C, E) = 1/3q(B, E) + 1/3q(C, E) + 1/3q(D, E);$$

$$q(D, E) = 1;$$

The equations for $l(i, E)$ are

$$l(A, E) = 1 + 1/3l(C, E) + 1/3l(D, E);$$

$$l(B, E) = 1 + 1/3l(A, E) + 1/3l(B, E) + 1/3l(C, E);$$

$$l(C, E) = 1 + 1/3l(B, E) + 1/3l(C, E) + 1/3l(D, E);$$

$$l(D, E) = 1;$$

4. (4 points) A certain Markov chain has its states partitioned into T, R_1, R_2, R_3. Let $x \in T$.

(a) How will you compute the probability of starting from x and reaching either R_1 or R_2?

(b) Let $r \in R_1$. How will you compute the probability of starting from x and reaching r *before* any other recurrent state?

Solution

(a) The probability of reaching from $x \in T$ to either R_1 or R_2 is simply the sum of the probabilities associated with all directed paths starting at x and ending at some state y of R_1 or R_2. This latter state y whould also be the node at which the path enters R_1, R_2 for the first time. (Note that the probability associated with $x, x_1, \cdots, x_k = y$, is $p(x, x_1) \times \cdots, \times p(x_{k-1}, x_k)$.) Since these paths and their probabilities would not change even if we merge R_1, R_2 into a single absorbing node leaving all probabilities of edges entering them unchanged, and adding a self loop of probability 1 at the merged node, we do this and write $q(i, k)$ equations for the merged Markov chain.

(b) Arguing as above as long as the paths reaching r from x remain the same any modification we make to the Markov chain would not affect the answer. So we break up R_1 into component single nodes by deleting all edges between nodes in R_1, and add self loops at all these nodes of probability 1 so that they become absorbing states. We then write equations for $q(i, r)$ in the resulting Markov chain.

5. (4 points) Let P be a square matrix with non-negative entries. Suppose all rows of P have row sum strictly less than 1. Show that $I - P$ is invertible.

Solution

(a) We will take the matrix P to be nonegative. The matrix $I - P$ is invertible if the only solution to $(I - P)x = 0$ is $x = 0$. Equivalently, $I - P$ is invertible if $x = Px$ has only the zero solution. We prove this by contradiction. Suppose $x \neq 0$. Let x_i have the maximum magnitude. Without loss of generality we assume x_i to be positive. So $x_i = \Sigma_j p_{i,j} x_j \leq \Sigma_j p_{i,j} |x_j|$. Now $\Sigma_j [p_{i,j}/q_i] |x_j|$, where $q_i = \Sigma_j p_{i,j}$, is an average of the values $|x_j|$ and cannot exceed their maximum magnitude. So $\Sigma_j [p_{i,j}/q_i] |x_j| \leq x_i$. But q_i has been given to be less than 1. So $\Sigma_j p_{i,j} |x_j| < x_i$, a contradiction.

6. A certain branching process has the following one step probability for number of progeny: $p_0 = a, p_1 = b, p_2 = c$. For the following cases

- $a = 1/2, b = 1/4, c = 1/4$;
- $a = 1/4, b = 1/4, c = 1/2$;

(a) Write down the equation the probability of extinction satisfies;

(b) Examine the probability of extinction.

Solution

(a) In this case

$$\rho = 1/2 + 1/4\rho + 1/4\rho^2.$$

The mean number of progeny is $\mu = 1 \times 1/4 + 2 \times 1/4 = 3/4 < 1$. In this case extinction is certain.

(b) In this case

$$\rho = 1/4 + 1/4\rho + 1/2\rho^2.$$

The mean number of progeny is $\mu = 1 \times 1/4 + 2 \times 1/2 = 5/4 > 1$. In this case there is a positive probability of survival. The probability of extinction is the smallest root of the equation

$$\rho = 1/4 + 1/4\rho + 1/2\rho^2.$$

This quadratic has the roots 1/2 and 1. So the probability of extinction is 1/2.