Stat-491-Fall2014-Assignment-III

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1. (4 points). 3 white balls and 3 black balls are distributed in two urns in such a way that each urn contains 3 balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. (a) Compute the transition probability matrix and its stationary distribution.

   (b) Verify whether the Markov chain is reversible.

   Solution

   The transition matrix is

$$\begin{pmatrix} & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1/9 & 4/9 & 4/9 & 0 \\ 2 & 0 & 4/9 & 4/9 & 1/9 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (1)$$

   The Markov chain is reversible because the graph of it has $p(j,i) \neq 0$ whenever $p(i,j) \neq 0$ and there are no loops other than the parallel edges. Therefore one could start from state 0 assigning it some value, then assigning some value to 1 etc till state 3. There can be no contradiction because there are no other edges where we could have $\pi(i)p(i,j)$ different from $\pi(j)p(j,i)$.

2. (4 points). A certain Markov chain has transition matrix

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \quad (2)$$

   (a) Compute the stationary distribution.

   (b) Verify whether the Markov chain is reversible.

   Solution

   (a) Let π^T, the stationary distribution be the vector (a, b, c). Since $\pi^T P = \pi^T$, we have the following linear equations, after simplification:

$$-2a + c = 0; a - 2b + c = 0; a + 2b - 2c = 0.$$

   This is a set of dependent equations and the solution has the form λx, for arbitrary λ. This Markov chain is irreducible. So every entry of π^T is non zero. So we could set c say, to 1 and find a, b by solving the equations. We then get $a = 1/2, b = 3/4, c = 1$. The sum is 9/4. Normalizing we get $\pi^T = (2/9, 1/3, 4/9)$.

It can be verified that $\pi^T P = \pi^T$.

(b) We have the entries $p(1,2) \neq 0$ and $p(2,1) = 0$. So $\pi(1) \times p(1,2) \neq \pi(2) \times p(2,1)$. Therefore the Markov chain is not reversible.

3. (4 points). The transition matrix of a certain Markov chain on states A, B, C, D, is given below.

$$\begin{pmatrix} & A & B & C & D \\ A & 1/2 & 1/3 & 1/6 & 0 \\ B & 1/3 & 1/3 & 1/3 & 0 \\ C & 1/3 & 1/3 & 1/3 & 0 \\ D & 0 & 0 & 1 & 0 \end{pmatrix} \tag{3}$$

(a) Which states are recurrent and which, transient? (b) Compute the stationary distribution for the Markov chain.

Solution

(a)The state D is transient since we cannot return to it from any other state. So $\pi(D) = 0$. All the other states are recurrent since from them we can only reach nodes from which we can return to the starting point and therefore these states have positive π value.

(b) Let π^T, the stationary distribution be the vector (e, f, g, h). Since $\pi^T P = \pi^T$, we have the following linear equations, after simplification:

$$-3e + 2f + 2g = 0; e - f + g = 0; e + 4f - 4g = 0.$$

This is a set of dependent equations and the solution has the form λx, for arbitrary λ. We set $g = 1$ to begin with. Solving the linear equations we get $e = 3/2, f = 5/4, g = 1$. Normalizing to make the sum equal to 1, we get $\pi(A) = 2/5, \pi(B) = 1/3, \pi(C) = 4/15$.

4. (4 points) Consider the Markov chains in Figure 1

(a) compute stationary distributions for the first two chains on A, B, C and A', B', C', D'

(b) For the third Markov chain on $A, B, C, D, A', B', C', D'$ find a stationary distribution $\pi(\cdot)$ which takes value 1/12 on A. Which states are recurrent and which transient?

Solution

The first Markov chain has transition matrix shown below:

$$\begin{pmatrix} & A & B & C \\ A & 0 & 1/3 & 2/3 \\ B & 1/3 & 0 & 2/3 \\ C & 2/3 & 1/3 & 0 \end{pmatrix} \tag{4}$$

This Markov chain is irreducible since we can reach from any state to any other state. So all states are recurrent and their π value positive. As in previous problems we solve the equation

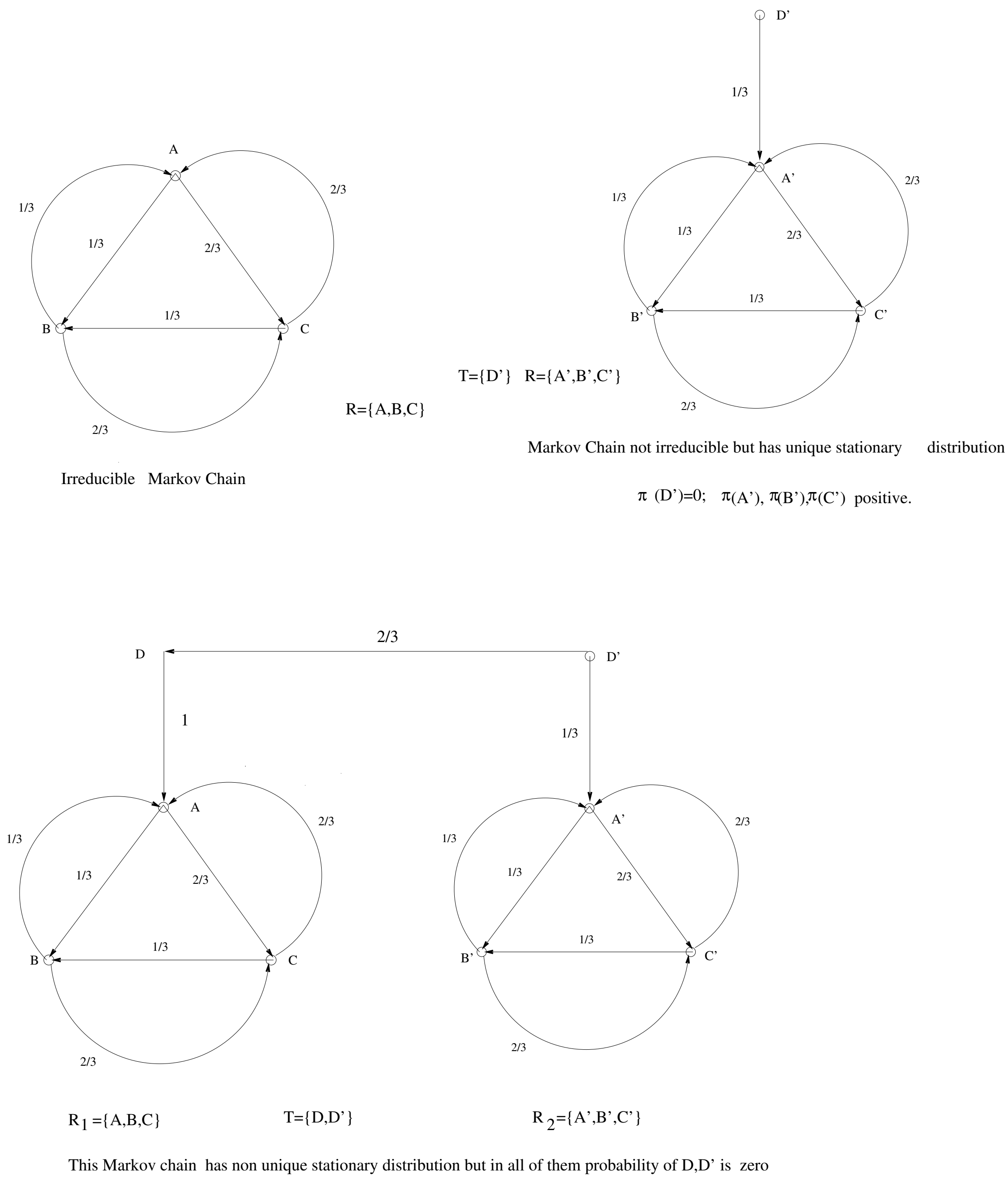

Figure 1: Markov Chains

$\pi^T P = \pi^T$, setting say $\pi(A) = 1$. Taking columns in the order A, B, C, we get $\pi'^T = (1, 5/7, 8/7)$. After normalizing this becomes $\pi^T = (7/20, 1/4, 2/5)$.

The second Markov chain has, as far as A', B', C' are concerned, the same transition matrix as A, B, C in the previous case. The state D' is transient since you can go from D' to A' but not return. So $\pi(D') = 0$. As in the previous case replacing A by A' etc taking columns in the order A', B', C', D', we get $\pi^T = (7/20, 1/4, 2/5, 0)$.

The third Markov chain has two transient states D, D' from which we can reach A, A' respectively

but not return. For the rest, going as in the previous chains, there are two primitive stationary distributions (column order $A, B, C, D, A', B', C', D'$):
$\pi^T = (7/20, 1/4, 2/5, 0, 0, 0, 0, 0)$ and $\pi^T = (0, 0, 0, 0, 7/20, 1/4, 2/5, 0)$. Every stationary distribution is a convex combination of these two primitive distributions. We need $\pi(A) = 1/12$. So we multiply the first distribution by $1/12 \times 1/(7/20) = 5/21$. Since we have to perform a convex combination this means the second distribution should be multiplied by $(1-5/21) = 16/21$. The resulting convex combination is

$$\pi^T = (1/12, 5/84, 2/21, 0, 4/15, 4/21, 32/105, 0).$$

This is the desired stationary distribution.

5. (4 points) In each of the following cases examine whether the random variable T is a 'stopping time' by
(i) precisely describing the set of sequences of states which determines whether $T = n$ or not and
(ii) justifying your conclusion about T being a stopping time.
(iii) Given that the initial state is chosen according to a probability distribution $\pi(\cdot)$, how would you determine $Pr\{T = n\}$?

   (a) $T = n$ if $X_n = y$;
   (b) $T = n$ if $X_n = y$ and for $0 \leq i \leq n, X_i = y$ exactly five times;
   (c) $T = n$ if for $n \leq i \leq n + 10, X_i = y$ exactly five times;
   (d) $T = n$ if $X_n =$ *a state* y such that $Pr\{X_{n+10} = z \mid X_n = y\}$ is 1/3. Here z is defined but y is specified only through the probability condition.

Solution

(a) Here we will assume that $X_n = y$ for the first time. We could also have taken it for the k^{th} time, for a fixed k and the solution is similar. Otherwise the condition is ambiguous. Set of all sequences of states $X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n = y$, where $p(x_i, x_i + 1) \neq 0$ for $0 \leq i \leq (n-1)$. T is a stopping time because whether $T = n$ or not requires us to check only values of X_i upto $i = n$.

(b) Set of all sequences of states $X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n = y$, where $p(x_i, x_i + 1) \neq 0$ for $0 \leq i \leq (n-1)$ and for exactly five of the is we have $x_i = y$. T is a stopping time because whether $T = n$ or not requires us to check only values of X_i upto $i = n$.

(c) Set of all sequences of states $X_0 = x_0, X_1 = x_1, \cdots, X_{n+10} = x_{n+10}$, where $p(x_i, x_i + 1) \neq 0$ for $0 \leq i \leq (n+9)$ and for exactly five of the i's upto $n + 10$ we have $x_i = y$. T is not a stopping time because by time n we do not know if we are going to get five y's upto $n + 10$.

(d) Let $y_1, \cdots, y_k$ be the states for which $Pr\{X_{n+10} = z \mid X_n = y_i\}$ is 1/3. The set that determines whether $T = n$ is the set of all sequences $X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n = y_i$, where $i = 1, \cdots k$. Here T is a stopping time, because at time n the permitted sequences tell us whether we should stop or not.

To compute the probability of $T = n$ when T is a stopping time, we compute for each permissible sequence (i.e., a sequence in the set we defined above) the product $p(x_0) \times p(x_0, x_1) \cdots \times p(x_{n-1}, x_n)$ and sum over all such sequences.