Compact complex surfaces

Paul Hacking

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Title: Compact complex surfaces (583)

Quarter: Spring

Preferred times: Tues/Thurs

Course description:

A complex manifold of dimension n is a space glued from open balls in $\mathbb{C}^n$ by holomorphic glueing maps. It can be viewed as the additional data of a "complex structure" on a real manifold of dimension $2n$.

For example consider the case $n = 1$. Recall that a compact orientable real manifold of dimension 2 is diffeomorphic to a sphere with g handles for some $g \geq 0$ (the genus = number of holes). The choice of complex structure on a surface of genus $g \geq 1$ is not unique, and depends on $3g - 3$ parameters for $g \geq 2$. Also, each complex manifold of dimension 1 is an *algebraic variety* — the zero locus in a complex projective space $\mathbb{P}^N$ of a set of polynomial equations. (If you would like to learn more about this, please consider taking 507/508).

The classification of compact complex surfaces ($n = 2$) is very rich and varied. The aim of this course is to give an overview of the classification with a strong emphasis on understanding the key examples. This is a large subject and I will not be able to prove every theorem we will use, rather we will try to understand what is going on in concrete examples.

"Most" but not all complex surfaces are algebraic varieties, and so our methods will be mostly algebraic in nature. However in certain cases (e.g. K3 surfaces) it is very important to allow non-algebraic examples to understand the true picture.

Here is a tentative syllabus. For more details, please check my website http://www.math.washington.edu/~hacking/

(1) Basic invariants of surfaces

   (a) Betti numbers $b_i = \dim H^i(X, \mathbb{R})$

   (b) The intersection form $\cup\colon H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}) = \mathbb{Z}$, results of Freedman and Donaldson on topology of 4-manifolds.

   (c) The Hodge decomposition $H^m(X,\mathbb{C}) = \oplus_{p+q=m} H^{p,q}$ (here $H^{p,q}$ denotes the space of de Rham cohomology classes representable by a form locally of type $\sum_{|I|=p,|J|=q} f_{I,J} dz_I \wedge d\bar{z}_J$, where $z_1, \ldots, z_n$ are holomorphic coordinates).

   (d) The canonical line bundle $\omega_X = \wedge^2 \Omega_X$ of holomorphic 2-forms.

   (e) Irregularity $q = \dim \Gamma(X, \Omega_X)$ (the dimension of the space of global holomorphic 1-forms), the Albanese map (a natural map from X to a complex torus of dimension q).

(2) Birational geometry of surfaces (blow ups).

(3) Ruled surfaces (surfaces birationally equivalent to $\mathbb{P}^1 \times C$).

(4) Complex tori ($X = \mathbb{C}^2/L$ where $L \cong \mathbb{Z}^4$ is a lattice).

(5) K3 surfaces (e.g. quartic hypersurface in $\mathbb{P}^3$).

(6) Elliptic fibrations (surfaces admitting a map to a curve with general fibre a smooth curve of genus 1).

(7) Non algebraic surfaces, e.g., Hopf surface.

(8) Surfaces of general type (surfaces such that $\omega_X^{\otimes N}$ defines a birational map for $N \gg 0$).

Prerequisites: Chapters I-III of Hartshorne OR Chapters 0-1 of Griffiths and Harris OR 507-508 (Algebraic geometry) and 544-545 (Topology and geometry of manifolds) with instructor's permission.