583C Lecture notes

Paul Hacking

May 9, 2008

1 Ruled surfaces (continued)

Proof of characterisation of ruled surfaces (continued). We give a more elementary proof of the two facts needed to show that the fibration $f \colon X \to C$ is locally trivial in the Euclidean topology, given that each fibre is a smooth rational curve of multiplicity one. Let $P \in C$ be a point and $F \simeq \mathbb{P}^1$ the fibre over P. We first show that there exists a line bundle $\mathcal{L}$ on X such that $\mathcal{L}|_F \simeq \mathcal{O}_F(1)$. We have $K_X \cdot F < 0$ and $F^2 = 0$, so K_X is not linearly equivalent to an effective divisor (because if D is an effective divisor, C is an irreducible curve, and $D \cdot C < 0$, then C is contained in the support of D and $C^2 < 0$). Equivalently, $h^0(K_X) = 0$. Thus $H^2(X, \mathbb{C}) = H^{1,1}$, and so $\operatorname{Num} X = H^2(X, \mathbb{Z})/\operatorname{Tors}$. Now, by Poincaré duality, the intersection product gives an isomorphism $\operatorname{Num} X \simeq (\operatorname{Num} X)^*$. The image of the map

$$\theta \colon \operatorname{Num} X \to \mathbb{Z}, \quad D \mapsto D \cdot F$$

is a subgroup $d\mathbb{Z} \subset \mathbb{Z}$, some $d \in \mathbb{N}$. Then $\frac{1}{d}\theta \in (\operatorname{Num} X)^*$ corresponds to some $G \in \operatorname{Num} X$, that is, $D \cdot F = d(D \cdot G)$ for all D. Thus $F \equiv dG$. Now $K_X \cdot F = -2$, so $d = 1$ or 2. Also, for X a smooth projective surface and D a divisor on X, $D^2 - D \cdot K_X$ is even. Indeed,

$$D^2 - D \cdot K_X = 2(\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X))$$

by the Riemann–Roch formula. We deduce that $d = 1$, that is, there exists a divisor H such that $H \cdot F = 1$. Let $\mathcal{L} = \mathcal{O}_X(H + rF)$ for some $r \in \mathbb{Z}$, then $\mathcal{L}|_F \simeq \mathcal{O}_F(1)$. We next show that, for $r \gg 0$, the map $\Gamma(X, \mathcal{L}) \to \Gamma(F, \mathcal{O}_F(1))$ on global sections is surjective. Consider the exact sequence of sheaves on X

$$0 \to \mathcal{O}_X(H + (r-1)F) \to \mathcal{O}_X(H + rF) \to \mathcal{O}_F(1) \to 0$$

and the associated long exact sequence of cohomology

$$\begin{array}{rcccl} & & H^0(\mathcal{O}_X(H+rF)) & \to & H^0(\mathcal{O}_F(1)) \quad \to \\ H^1(\mathcal{O}_X(H+(r-1)F)) & \to & H^1(\mathcal{O}_X(H+rF)) & \to & 0 \end{array} \tag{1}$$

(here we used $H^1(\mathcal{O}_F(1)) = 0$). Now, the space $H^1(\mathcal{O}_X(H))$ is finite dimensional, and the map

$$\theta_r \colon H^1(\mathcal{O}_X(H+(r-1)F)) \to H^1(\mathcal{O}_X(H+rF))$$

is surjective for each r by the exact sequence (1). Hence θ_r is an isomorphism for $r \gg 0$. So the restriction map $H^0(\mathcal{O}_X(H+rF)) \to H^0(\mathcal{O}_F(1))$ is surjective for $r \gg 0$ by (1).

Finally, we show that the fibration $f \colon X \to C$ is locally trivial in the Zariski topology. Recall that X is a $\mathbb{P}^1$-bundle over C for the Euclidean topology. We show that $X \to C$ is the projectivisation of a rank 2 (analytic) vector bundle E on C^{an}. Then by GAGA, E is actually an algebraic vector bundle on C. It follows that $f \colon X \to C$ is a $\mathbb{P}^1$-bundle for the Zariski topology. We have an exact sequence of sheaves of (non-abelian) groups on C^{an} (we omit the superscript "an" in what follows)

$$0 \to \mathcal{O}_C^\times \to \mathrm{GL}_2(\mathcal{O}_C) \to \mathrm{PGL}_2(\mathcal{O}_C) \to 0.$$

Note that this is a *central extension* of sheaves of groups: that is, the kernel is abelian and contained in the centre of the second term. In this situation, we obtain a long exact sequence of cohomology

$$\cdots \to H^1(\mathcal{O}_C^\times) \to H^1(\mathrm{GL}_2(\mathcal{O}_C)) \to H^1(\mathrm{PGL}_2(\mathcal{O}_C)) \to H^2(\mathcal{O}_C^\times). \tag{2}$$

This sequence does *not* continue further to the right (it is not possible to make sense of Cech cohomology H^i of a sheaf of nonabelian groups for $i \geq 2$). The set $H^1(\mathrm{GL}_2(\mathcal{O}_C))$ is the set of isomorphism classes of rank 2 vector bundles over C. This is analogous to the identification $\operatorname{Pic} C \simeq H^1(\mathcal{O}_C^\times)$. Explicitly, if E is a rank r vector bundle over a complex manifold X, let $\mathcal{U} = \{U_i\}$ be an open covering of X and

$$\phi_i \colon E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^r$$

local trivialisations. Then

$$\phi_j \circ \phi_i^{-1} \colon U_{ij} \times \mathbb{C}^r \xrightarrow{\sim} U_{ij} \times \mathbb{C}^r, \quad (x, v) \mapsto (x, g_{ij}(x) \cdot v),$$

where $g_{ij} \in \Gamma(U_{ij}, \mathrm{GL}_r(\mathcal{O}_X))$ are the *transition functions*. The g_{ij} satisfy the cocycle condition $g_{jk} g_{ij} = g_{ik}$, equivalently, $g_{jk} g_{ij} g_{ik}^{-1} = 1$ (note that the order of the factors is important here for $r > 1$). If we change the trivialisations ϕ_i by multiplication by $f_i \in \Gamma(U_i, \mathrm{GL}_r(\mathcal{O}_X))$, then the new transition functions are given by $g'_{ij} = f_j g_{ij} f_i^{-1}$. The Cech cohomology set $H^1(\mathcal{U}, \mathrm{GL}_r(\mathcal{O}_X))$ with respect to the open covering $\mathcal{U}$ is by definition the set of tuples $(g_{ij}) \in \bigoplus \Gamma(U_{ij}, \mathrm{GL}_r(\mathcal{O}_X))$ satisfying $g_{jk} g_{ij} g_{ik}^{-1} = 1$, modulo the equivalence relation $(g_{ij}) \sim (f_j g_{ij} f_i^{-1})$ for all $(f_i) \in \bigoplus \Gamma(U_i, \mathrm{GL}_r(\mathcal{O}_X))$. The Cech cohomology set $H^1(X, \mathrm{GL}_r(\mathcal{O}_X))$ is obtained by taking the direct limit $\varinjlim H^1(\mathcal{U}, \mathrm{GL}_r(\mathcal{O}_X))$ over all open coverings $\mathcal{U}$ as usual. By the previous discussion, this set is identified with the set of isomorphism classes of rank r vector bundles over X. Returning to our example, a similar analysis shows that the set $H^1(\mathrm{PGL}_2(\mathcal{O}_C))$ is the set of isomorphism classes of $\mathbb{P}^1$-bundles over C (because $\mathrm{Aut}\, \mathbb{P}^1_{\mathbb{C}} = \mathrm{PGL}_2(\mathbb{C})$). The map $H^1(\mathrm{GL}_2(\mathcal{O}_C)) \to H^1(\mathrm{PGL}_2(\mathcal{O}_C))$ sends a vector bundle to its projectivisation. We claim that this map is surjective. Equivalently, by the exact sequence (2), $H^2(\mathcal{O}_C^\times) = 0$. The exponential sequence on C

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_C \to \mathcal{O}_C^\times \to 0$$

yields the long exact sequence of cohomology

$$\cdots \to H^2(\mathcal{O}_C) \to H^2(\mathcal{O}_C^\times) \to H^3(C, \mathbb{Z}) \to \cdots$$

Now $H^2(\mathcal{O}_C) = 0$ because $\mathcal{O}_C$ is coherent and $\dim_{\mathbb{C}} C = 1 < 2$, and $H^3(C, \mathbb{Z}) = 0$, so $H^2(\mathcal{O}_C^\times) = 0$ as required. Thus the $\mathbb{P}^1$-bundle $f\colon X \to C$ is the projectivisation of a rank 2 vector bundle over C. This completes the proof. □

Remark 1.1. We can make the last step of the proof more explicit as follows. Let $\mathcal{U} = \{U_i\}$ be an open covering of C by small discs and let $h_{ij} \in \mathrm{PGL}_2(\mathcal{O}_X(U_{ij}))$ be transition functions for the $\mathbb{P}^1$-bundle $f\colon X \to C$ with respect to this covering. Lift h_{ij} to $g_{ij} \in \mathrm{GL}_2(\mathcal{O}_C(U_{ij}))$. Then $g_{jk} g_{ij} g_{ik}^{-1} = \alpha_{ijk} \in \mathcal{O}_C^\times(U_{ijk})$ (because $h_{jk} h_{ij} h_{ik}^{-1} = 1$ in $\mathrm{PGL}_2(\mathcal{O}_C(U_{ijk}))$). One checks that $\alpha = (\alpha_{ijk})$ is a Cech 2-cocycle for $\mathcal{O}_C^\times$. Since $H^2(\mathcal{O}_C^\times) = 0$ we can write $\alpha = d\beta$, that is, $\alpha_{ijk} = \beta_{jk} \beta_{ij} \beta_{ik}^{-1}$. Now define $g'_{ij} = g_{ij} \beta_{ij}^{-1}$, then g'_{ij} is a lift of h_{ij} and $g'_{jk} g'_{ij} {g'_{ik}}^{-1} = 1$. Thus the g'_{ij} define a vector bundle E over C such that the projectivisation of E is isomorphic to the $\mathbb{P}^1$-bundle $f\colon X \to C$.

2 Invariants of ruled surfaces

Let $f\colon X \to C$ be a ruled surface (a $\mathbb{P}^1$-bundle over a curve C).

We first observe that the map $f_*\colon \pi_1(X) \to \pi_1(C)$ is an isomorphism. In general, if $p\colon E \to B$ is a (topological) fibre bundle with fibre F, then there is a homotopy long exact sequence

$$\cdots \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \cdots$$

(recall that $\pi_0(X)$ is the set of connected components of X). See [Hatcher, p. 376, Thm. 4.41]. For example, given a loop γ in B based at a point $b \in B$, we can lift it to a path in E, whose end points lie in the fibre $F = p^{-1}b$. Then if F is connected we can join the end points by a path in F to obtain a loop $\tilde{\gamma}$ in E such that $p_*\tilde{\gamma} = \gamma$. This shows that $\pi_1(E) \to \pi_1(B)$ is surjective if $\pi_0(F)$ is trivial. In our situation, the map $f\colon X \to C$ is a fibre bundle with fibre $F \simeq \mathbb{P}^1$. In particular, F is connected and $\pi_1(F) = 0$. So $f_*\colon \pi_1(X) \to \pi_1(C)$ is an isomorphism as claimed. Passing to abelianisations, we deduce that $f_*\colon H_1(X,\mathbb{Z}) \to H_1(C,\mathbb{Z})$ is an isomorphism.

References

[Hatcher] A. Hatcher, Algebraic topology.