583C Lecture notes

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1 Invariants of ruled surfaces (continued)

Let $f\colon X \to C$ be a ruled surface over $k = \mathbb{C}$. We show that the pullback map on 1-forms

$$f^*\colon \Gamma(\Omega_C) \to \Gamma(\Omega_X)$$

is an isomorphism. We give two proofs. The first proof uses analytic methods. Let $\eta \in \Gamma(\Omega_X)$ be a global 1-form on X. Then the restriction $\eta|_F$ of η to a fibre F equals zero, because $F \simeq \mathbb{P}^1$ and $\Gamma(\Omega_{\mathbb{P}^1}) = 0$. Now let $P \in C$ be a point and $V \subset C^{\text{an}}$ a small disc about P. Let $F = f^{-1}P$ be the fibre over P and $U = f^{-1}V$ the tubular neighbourhood of F over V. Then U is homotopy equivalent to F, so $H_1(U, \mathbb{Z}) = 0$. Also the 1-form η is closed by Prop. 1.1 below. So we can define a holomorphic function $g\colon U \to \mathbb{C}$ such that $dg = \eta|_U$ by

$$g(P) = \int_{P_0}^{P} \eta$$

where $P_0 \in F$ is a fixed basepoint and the notation $\int_{P_0}^{P}$ means the integral over some path from P_0 to P in U. (The choice of path γ is irrelevant. Indeed if γ, γ' are two such paths then $\gamma - \gamma' = \partial\beta$ for some 2-cycle β because $H^1(U, \mathbb{Z}) = 0$. So

$$\int_\gamma \eta - \int_{\gamma'} \eta = \int_{\partial\beta} \eta = \int_\beta d\eta = 0$$

using $d\eta = 0$.) Now g is constant along fibres because (as noted above) $\eta|_G = 0$ for any fibre G. So $g = f^*h$ is the pullback of a holomorphic function h on $V \subset C$, and $\eta|_U = dg = f^*dh$ is the pullback of a holomorphic 1-form $\omega = dh$ on V. Now, since ω is uniquely determined by η, these local 1-forms patch to give a global 1-form ω on C such that $\eta = f^*\omega$, as required. The second proof is algebraic. In general, suppose given smooth varieties X,

Y, and a submersion (or smooth morphism) $f\colon X \to Y$, that is, a morphism such that the derivative

$$df_P\colon T_{X,P} \to T_{Y,f(P)}$$

of f at P is surjective for all $P \in X$. (Here $T_{X,P}$ denotes the tangent space to X at P, or, equivalently, the fibre of the tangent bundle T_X over $P \in X$.) Then we have an exact sequence of vector bundles on X

$$0 \to T_{X/Y} \to T_X \to f^*T_Y \to 0$$

where the map $T_X \to f^*T_Y$ is the derivative of f and the kernel $T_{X/Y}$ is the bundle of tangent vectors on X which are parallel to the fibres of f. Dualising we obtain an exact sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0,$$

and so an exact sequence of k-vector spaces

$$0 \to \Gamma(f^*\Omega_Y) \to \Gamma(\Omega_X) \to \Gamma(\Omega_{X/Y})$$

Note that $\Omega_{X/Y}|_F = \Omega_F$ for F a fibre. So, in our example $f\colon X \to C$, we find $\Gamma(\Omega_{X/C}) = 0$ because $\Gamma(\Omega_F) = 0$ (cf. the first proof). Thus $\Gamma(f^*\Omega_Y) \to \Gamma(\Omega_X)$ is an isomorphism. Now $\Gamma(f^*\Omega_Y) = \Gamma(f_*f^*\Omega_Y)$ by the definition of f_* for sheaves, $f_*f^*\Omega_Y = \Omega_Y \otimes f_*\mathcal{O}_C$ by the projection formula [Hartshorne, p. 124, Ex. II.5.1(d)], and $f_*\mathcal{O}_X = \mathcal{O}_C$ by Stein factorisation [Hartshorne, p. 280, III.11.5]. So $\Gamma(\Omega_C) \to \Gamma(\Omega_X)$ is an isomorphism as required.

Proposition 1.1. *Let X be a smooth complex projective variety (or, more generally, a compact Kähler manifold). Then a holomorphic k-form η on X is closed. Moreover, the map*

$$H^0(\Omega^k_X) \to H^k_{\mathrm{dR}}(X, \mathbb{C}), \quad \eta \mapsto [\eta]$$

is the natural inclusion

$$H^0(\Omega^k_X) = H^{k,0} \subset H^k_{\mathrm{dR}}(X, \mathbb{C}).$$

Proof. This follows from the description of the Hodge decomposition in terms of harmonic forms. See [CMSP, p. 95, Prop. 3.1.1]. □

Remark 1.2. Note that it is essential that X is compact. For example, the holomorphic 1-form $z_1 dz_2$ on $\mathbb{C}^2_{z_1,z_2}$ is not closed.

In general, if $f\colon X \to B$ is a (topological) fibre bundle with fibre F, then

$$e(X) = e(B)e(F)$$

where as usual $e(X)$ denotes the Euler number. (Proof: By decomposing B into pieces and using the Mayer–Vietoris sequence we can assume that f is a trivial bundle, so $X \simeq B \times F$. Recall that the Euler number can be computed from a cellular subdivision of X as $e(X) = \sum(-1)^i N_i$ where N_i is the number of cells of dimension i. Now taking triangulations of B and F we obtain a cellular subdivision of $X = B \times T$ with cells $\sigma \times \tau$ where σ and τ are simplices in the triangulations of B and F respectively. We deduce that $e(X) = e(B)e(F)$.) In our example $f\colon X \to C$ we obtain

$$e(X) = e(C)e(F) = (2-2g)(2) = 4-4g \tag{1}$$

where $g = g(C)$ is the genus of C. Now we can compute all the Betti numbers $b_i(X) = \dim_{\mathbb{R}} H^i(X,\mathbb{R})$. Recall that $H_1(X,\mathbb{Z}) \to H_1(C,\mathbb{Z})$ is an isomorphism, so $b_1(X) = b_1(C) = 2g$. Also, we always have $b_0 = 1$, and $b_i = b_{4-i}$ by Poincaré duality. So

$$e(X) := \sum(-1)^i b_i = 2 - 2b_1 + b_2 = 2 - 4g + b_2$$

and combining with (1) we obtain $b_2 = 2$. We also note that the integral homology and cohomology of X has no torsion. Indeed, $H_1(X,\mathbb{Z}) \simeq H_1(C,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$ is torsion free, and it follows that $H_i(X,\mathbb{Z})$ and $H^i(X,\mathbb{Z})$ are torsion free for each i by the universal coefficient theorem and Poincaré duality.

As noted earlier, since $K_X \cdot F < 0$ and $F^2 = 0$, we have $h^0(K_X) = 0$. (More generally, $h^0(nK_X) = 0$ for all $n > 0$.) In particular, $H^{2,0} = h^0(K_X) = 0$, so $H^2(X,\mathbb{C}) = H^{1,1}$ and

$$\operatorname{Num} X = H^{1,1} \cap H^2(X,\mathbb{Z})/\operatorname{Tors} = H^2(X,\mathbb{Z}).$$

Thus $\operatorname{Num} X \simeq \mathbb{Z}^2$, that is, $\rho(X) = 2$. We describe a basis of $\operatorname{Num} X$. We observe that, since $f\colon X \to C$ is a $\mathbb{P}^1$-bundle for the Zariski topology on C, there exists a section S of f. (Strictly speaking a *section* of $f\colon X \to C$ is a morphism $s\colon C \to X$ such that $f \circ s = \mathrm{id}_C$. Here we identify a section with its image $S = s(C) \subset X$.) Indeed, let $U \subset C$ be a Zariski open subset of C such that the restriction $X|_U$ is a trivial $\mathbb{P}^1$-bundle, that is, there is an isomorphism

$$\phi\colon X|_U \xrightarrow{\sim} U \times \mathbb{P}^1$$

over U. Now let S_U be a section of $X|_U$ (for example, $S_U = \phi^{-1}(U \times \{P\})$ for some fixed $P \in \mathbb{P}^1$), and define $S = \overline{S_U} \subset X$, the closure of S_U in X. Then S is a section of f. Let F be a fibre of f. We claim that S, F is a basis of $\operatorname{Num} X$. Indeed, it suffices to observe that the determinant of the matrix

$$\begin{pmatrix} S^2 & S \cdot F \\ S \cdot F & F^2 \end{pmatrix} = \begin{pmatrix} ? & 1 \\ 1 & 0 \end{pmatrix}$$

equals -1.

Remark 1.3. Note that $S^2 \mod 2$ is a topological invariant of F. Indeed S^2 is even iff the intersection form on $\operatorname{Num} X = H^2(X, \mathbb{Z})$ is even, that is, x^2 is even for all $x \in H^2(X, \mathbb{Z})$.

For example, the (holomorphic or algebraic) classification of ruled surfaces $f : X \to C$ over $C = \mathbb{P}^1$ can be described as follows. For each n there exists a ruled surface $\mathbb{F}_n \to \mathbb{P}^1$ with a section $S \subset \mathbb{F}_n$ such that $S^2 = -n$. If $n > 0$ then the section S is the unique section with negative self-intersection. (If $n = 0$ then $f : \mathbb{F}_0 \to \mathbb{P}^1$ is given by $\mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, any fibre S of the other projection $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is a section of f with $S^2 = 0$, and any other section has strictly positive self-intersection.) Every ruled surface over $\mathbb{P}^1$ is isomorphic to $\mathbb{F}_n$ for some n. Now we describe the topological classification: The ruled surfaces $\mathbb{F}_n$, $\mathbb{F}_m$ are homeomorphic iff $n \equiv m \mod 2$. In fact, if $n \equiv m \mod 2$, and $n > m$, there is a family $\mathcal{X} \to \Delta$ of smooth surfaces over the disc $\Delta = (|t| < 1) \subset \mathbb{C}$ such that $\mathcal{X}_t \simeq \mathbb{F}_m$ for $t \neq 0$ and $\mathcal{X}_0 \simeq \mathbb{F}_n$.

References

[CMSP] J. Carlson, S. Müller-Stach, C. Peters, Period mappings and period domains.

[Hartshorne] R. Hartshorne, Algebraic geometry.