583C Example sheet 3

Paul Hacking

June 9, 2008

(1) Let $X = V/L$ be a complex torus of dimension n and $\mathcal{L}$ a line bundle on X. Consider $\phi := c_1(\mathcal{L}) \in H^2(X,\mathbb{Z}) = \wedge^2 L^*$.

(a) Show that there exists a basis of the free abelian group L such that the skew bilinear form

$$\phi\colon L \times L \to \mathbb{Z}$$

has matrix

$$\begin{pmatrix} 0 & \delta_1 & & & & \\ -\delta_1 & 0 & & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & \delta_n \\ & & & & -\delta_n & 0 \end{pmatrix}$$

where $\delta_i \in \mathbb{Z}_{\geq 0}$ and δ_i divides δ_{i+1} for all i. (This is a purely algebraic fact, cf. [GH, p. 304, Lem.].)

(b) Show that

$$\mathcal{L}^n := c_1(\mathcal{L})^n = (\pm 1) \cdot n! \cdot \delta_1 \cdots \delta_n.$$

(Hint: by (a), there exists an identification $X_{\mathbb{R}} = (\mathbb{R}/\mathbb{Z})^{2n}$ with real coordinates $x_1, \ldots, x_{2n}$ such that $c_1(\mathcal{L}) \in H^2_{\text{dR}}(X,\mathbb{R})$ is represented by the form $\delta_1 dx_1 \wedge dx_2 + \cdots + \delta_n dx_{2n-1} \wedge dx_{2n}$.)

(2) Let X be a smooth complex projective variety (or compact Kähler manifold). Consider the Albanese morphism $\alpha\colon X \to \operatorname{Alb} X$.

(a) Show that α induces an isomorphism $\alpha^*\colon \Gamma(\Omega_{\operatorname{Alb} X}) \xrightarrow{\sim} \Gamma(\Omega_X)$.

(b) Show that if X is a complex torus then α is an isomorphism.

(c) For $k \in \mathbb{N}$, consider the morphism

$$\alpha^k \colon X^k \to \operatorname{Alb} X, \quad (x_1, \ldots, x_k) \mapsto \alpha(x_1) + \cdots + \alpha(x_k).$$

Show that the derivative of α^k at a point $(P_1, \ldots, P_k) \in X^k$ is dual to the restriction map

$$\Gamma(\Omega_X) \to (\Omega_X)_{P_1} \oplus \cdots (\Omega_X)_{P_k}$$

(Here $(\Omega_X)_P$ denotes the fibre of the cotangent bundle at $P \in X$.) Using this, show that for $k \geq h^0(\Omega_X)$ there exist $P_1, \ldots, P_k$ such that the derivative of α^k at $(P_1, \ldots, P_k)$ is surjective. Deduce that α^k is surjective for $k \geq h^0(\Omega_X)$. (Cf. [GH, p. 237].)

(3) Let X be a K3 surface and D an effective divisor such that $D^2 = -2$. Show that D is linearly equivalent to a positive linear combination of (-2)-curves. (Hint: follow the construction of an elliptic fibration from D with $D^2 = 0$ in the proof of Thm 12.6(3).)

(4) Use the Torelli theorem for K3 surfaces to show that there exists a 19 dimensional family of K3 surfaces admitting an elliptic fibration. Is an elliptic fibred K3 necessarily projective?

(5) Show that a projective K3 surface X is a Kummer surface iff there exist 16 disjoint (-2)-curves $C_1, \ldots, C_{16}$ such that $\sum C_i$ is divisible by 2 in $\operatorname{Pic} X$. (Hint: use the commutative diagram of blowups and double covers from p. 72 of the notes, and recall that a projective surface Y with $K_Y = 0$ is either a complex torus or a K3 surface. If you are stuck, see [BHPV, VIII.6.1].)

(6) Give an alternative proof that the Godeaux surface X constructed in Sec. 14 satisfies $h^0(K_X) = 0$ using Noether's formula.

(7) Show that the Hopf surface X described in Sec. 15 admits an elliptic fibration $X \to \mathbb{P}^1$.

References

[BHPV] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact complex surfaces, 2nd ed.

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry.