583C Example sheet 2

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(1) Let X be a smooth projective complex surface and $E \subset X$ a (-2)-curve, that is, $E \simeq \mathbb{P}^1$ and $E^2 = -2$.

(a) Show that there exists a morphism $f\colon X \to Y$ to a normal surface Y such that $f(E)$ is a point $P \in Y$ and f restricts to an isomorphism
$$X \setminus E \xrightarrow{\sim} Y \setminus \{P\}.$$
[Hint: Follow the proof of Castelnuovo's contractibility criterion]

(b) Let $Z = (x^2 + y^2 + z^2 = 0) \subset \mathbb{A}^3$ be an ordinary double point singularity. Consider the blowup $\pi\colon \tilde{Z} \to Z$ of $0 \in Z$. Show that $\tilde{Z}$ is smooth and $F := \pi^{-1}(0)$ is a (-2)-curve.

(c) (Harder) Show that there exist Euclidean neighbourhoods $P \in U \subset Y$ and $Q \in V \subset Z$ and a commutative diagram
$$\begin{array}{ccc} f^{-1}U & \xrightarrow{\sim} & \pi^{-1}V \\ \downarrow f & & \downarrow \pi \\ U & \xrightarrow{\sim} & V \end{array}$$

See [Reid, p. 93] for a more general result.

(2) Let X be a smooth projective complex surface such that $\rho(X) = 2$, $-K_X$ is ample, and the intersection form on $\operatorname{Pic} X$ is even (that is, D^2 is even for every divisor D). We show that $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, following the proof of the characterisation of $\mathbb{P}^2$.

(a) Compute $K_X^2 = 8$.

(b) Show that $K_X \cdot D$ is even for every divisor D, and deduce that $-K_X$ is numerically equivalent to $2H$ for some ample divisor H. (Here you need to show $c_1\colon \operatorname{Pic} X \to H^2(X, \mathbb{Z})$ is an isomorphism and use Poincaré duality.)

(c) Compute $h^0(\mathcal{O}_X(H)) = 4$.

(d) Show that the line bundle $\mathcal{O}_X(H)$ is generated by global sections, and so defines a finite morphism $\phi\colon X \to \mathbb{P}^3$.

(e) Show that $\phi(X) \subset \mathbb{P}^3$ is a smooth quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and that $\phi\colon X \to Q$ is an isomorphism. (Hint: compute $h^0(\mathcal{O}_X(2H)) = 9 = h^0(\mathcal{O}_{\mathbb{P}^3}(2)) - 1$.)

(3) Let $f\colon X \to C$ be a morphism from a smooth projective complex surface X to a smooth curve C with connected fibres, such that $K_X \cdot F < 0$ for F a fibre. Prove that if a fibre $F = \sum m_i E_i$ is reducible, then there exists a component E_i such that $K_X \cdot E_i < 0$, and that E_i is a (-1)-curve. Now show by induction that $X \to C$ is obtained from a ruled surface $Y \to C$ by a sequence of blowups.

(4) Let X, Y be smooth projective surfaces and $f\colon X \dashrightarrow Y$ a birational map. Suppose that K_X and K_Y are both nef. (We say a divisor D on a variety X is *nef* if $D \cdot C \geq 0$ for every curve $C \subset X$.) Show that f is an isomorphism as follows.

(a) Use the decomposition theorem for birational maps to write $f = h \circ g^{-1}$ where $g\colon Z \to X$, $h\colon Z \to Y$ are compositions of sequences of blowups. Let $\{E_i\}$ and $\{F_j\}$ be the exceptional curves for g and h respectively.

(b) Show that $K_Z = g^*K_X + \sum a_i E_i$ where $a_i > 0$ for all i, and similarly $K_Z = h^*K_Y + \sum b_j F_j$ with $b_j > 0$ for all j.

(c) Show that if E_i is a (-1)-curve then $E_i = F_j$ for some j. So we can contract E_i, and the result follows by induction.

(5) For $t \in \mathbb{C}$, consider the vector bundle $\mathcal{E}_t$ on $\mathbb{P}^1$ defined by the transition matrix

$$g_{01} = \begin{pmatrix} 1 & tx^{-1} \\ 0 & x^{-2} \end{pmatrix}.$$

Here we use the standard affine open covering $U_0 = (X_0 \neq 0)$, $U_1 = (X_1 \neq 0)$ of $\mathbb{P}^1_{(X_0:X_1)}$, and $x = X_1/X_0$.

(a) Show that $\mathcal{E}_t \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for $t \neq 0$ and $\mathcal{E}_0 \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$.

(b) If you know about Ext groups, show that there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}_t \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

and interpret (a) in terms of $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}) = H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{C}$.

(c) Now consider the family of ruled surfaces $X \to \mathbb{P}^1 \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ with fibres $X_t \to \mathbb{P}^1$ the projectivisation of the bundles $\mathcal{E}_t$ over $\mathbb{P}^1$. Show that $X_t \simeq \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$, and X_0 is a rational ruled surface with a section S of self-intersection $S^2 = -2$.

(d) Let S, F be the above section and a fibre of the ruled surface $X_0 \to \mathbb{P}^1$. Let F_1, F_2 be fibres of the two projections $X_t \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Because the fibres of the family $X \to \mathbb{A}^1_t$ are smooth, the cohomology of nearby fibres is canonically identified. Show that under this identification (and the isomorphisms $c_1\colon \operatorname{Pic} X_t \to H^2(X_t, \mathbb{Z})$), F_1, F_2 correspond to $F, S+F$.

References

[Reid] Miles Reid, Chapters on algebraic surfaces, arXiv:alg-geom/9602006v1