583C Example sheet 1

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(1) Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree d which contains a line L. Compute the self-intersection L^2 of the divisor L on X in terms of d. (Hint: Let H be a hyperplane section of X. What is $H \cdot L$? Now think about what happens if we take a hyperplane containing L.) Explain your answer geometrically for $d = 1, 2, 3$ (and $d = 4$ if you know about K3 surfaces).

(2) The divisor class group of projective space $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}$, generated by the class of a hyperplane H. (If you have never seen this before, prove it!) We write $\mathcal{O}_{\mathbb{P}^n}(d)$ for the line bundle corresponding to dH, for $d \in \mathbb{Z}$. Write down transition functions for $\mathcal{O}_{\mathbb{P}^n}(1)$ with respect to the standard affine open covering $\mathcal{U} = \{U_i\}_{i=0}^n$ of $\mathbb{P}^n$ given by $U_i = (X_i \neq 0)$.

(3) Recall that complex projective space $\mathbb{P}^n$ is the space of lines in $\mathbb{C}^{n+1}$. The *tautological line bundle* on projective space is the line bundle $L \to \mathbb{P}^n$ whose fibre over a point is the corresponding line in $\mathbb{C}^{n+1}$. Thus L is a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ of rank $n+1$ over $\mathbb{P}^n$. Show that the sheaf of sections $\mathcal{L}$ of L is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

(4) Let ω be the rational n-form on $\mathbb{P}^n$ defined by

$$\omega = dx_1 \wedge \cdots \wedge dx_n$$

where $x_1 = X_1/X_0, \ldots, x_n = X_n/X_0$ are the affine coordinates on $U_0 = (X_0 \neq 0) \simeq \mathbb{A}^n$. Show directly that the canonical divisor class $K_{\mathbb{P}^n}$ of projective space is linearly equivalent to $-(n+1)H$ (where H is the hyperplane class) by computing the divisor (ω) of zeroes and poles of ω.

(5) Recall that $\operatorname{Pic} X \simeq H^1(X, \mathcal{O}_X^\times)$. Compute directly that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^\times) \simeq \mathbb{Z}$ using the open covering $\mathcal{U} = \{U_i\}$, $U_i = (X_i \neq 0)$. (Warning: You

need to check that this open covering does compute the Cech cohomology using the Leray theorem. Note that $\mathcal{O}_X^\times$ is *not* a coherent sheaf.)

(6) The cohomology of line bundles on $\mathbb{P}^n$ can be completely described as follows. Let $S = k[X_0, \ldots, X_n]$ and write S_d for the space of homogeneous polynomials of degree d, so $S = \bigoplus_{d \geq 0} S_d$. Then

  (a) $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = S_d$,
  (b) $H^i(\mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for $0 < i < n$, and
  (c) $H^n(\mathcal{O}_{\mathbb{P}^n}(d)) \simeq H^0(\mathcal{O}_{\mathbb{P}^n}(-n-1-d))^*$.

Note that (c) is an instance of Serre duality. Check this explicitly for $n = 1$ by using the Cech complex for the affine open covering $\mathcal{U} = \{U_i\}$, $U_i = (X_i \neq 0)$. (Note: If X is an algebraic variety and $\mathcal{F}$ is a coherent sheaf on X then $H^i(X, \mathcal{F}) = H^i(\mathcal{U}, \mathcal{F})$ for any affine open covering $\mathcal{U}$ of X.) The same approach works for arbitrary n, see [Hartshorne, III.5.1, p. 225].

(7) Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree d. We compute the invariants of X.

  (a) Use the adjunction formula for $X \subset \mathbb{P}^3$ to show that

$$K_X \sim (d-4)H$$

  where H is a hyperplane section. In particular $K_X^2 = (d-4)^2 d$. Equivalently (in terms of line bundles), $\omega_X \simeq \mathcal{O}_X(d-4)$, where we write $\mathcal{O}_X(k)$ for the restriction $\mathcal{O}_{\mathbb{P}^n}(k)|_X$.

  (b) Use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

  together with the description of cohomology of line bundles on projective space to show that $h^1(\mathcal{O}_X) = 0$ and $h^2(\mathcal{O}_X) = \binom{d-1}{3}$. (There are many alternative ways to prove this. For example, X is simply connected by the Lefschetz hyperplane theorem [GH, p. 156–158], so $h^1(\mathcal{O}_X) = 0$ by the Hodge decomposition. Also, $h^2(\mathcal{O}_X) = h^0(\omega_X)$ by Hodge theory or Serre duality, and as noted above $\omega_X \simeq \mathcal{O}_X(d-4)$. Now by tensoring the above exact sequence by $\mathcal{O}_{\mathbb{P}^3}(d-4)$ and inspecting the long exact sequence of cohomology we find that the map $H^0(\mathcal{O}_{\mathbb{P}^n}(d-4)) \to H^0(\omega_X)$ is an isomorphism.)

(c) Use Noether's formula to compute the Euler number $e(X)$ of X.

Use the above results to write down the Hodge numbers $h^{p,q}$ of X. Now explain your results geometrically for $d = 1, 2, 3$.

(8) Let X be an algebraic variety (with its Zariski topology), A an abelian group, and $\underline{A}$ the constant sheaf on X with stalk A. Show directly (by considering the Cech complex) that $H^p(X, \underline{A}) = 0$ for $p > 0$. Note however that if X is a smooth algebraic variety over $k = \mathbb{C}$, and X^{an} is the associated complex manifold (with its Euclidean topology), then $H^p(X^{\mathrm{an}}, \underline{A}) = H^p(X^{\mathrm{an}}, A)$, the simplicial cohomology of X^{an} with coefficients in A.

(9) Let X be a smooth projective curve and D a divisor on X. Let $\mathcal{K}$ be the sheaf of rational functions on X, that is, $\Gamma(U, \mathcal{K}) = k(X)$ for $\emptyset \neq U \subset X$. (Equivalently, $\mathcal{K}$ is the constant sheaf on X with stalk $k(X)$.)

(a) We have an inclusion $\mathcal{O}_X(D) \subset \mathcal{K}$. Show that the quotient sheaf is

$$\bigoplus_{P \in X} \frac{k(X)}{\mathcal{O}_X(D)_P},$$

the direct sum over $P \in X$ of the skyscraper sheaf at P with stalk $k(X)/\mathcal{O}_X(D)_P$. (If X is a variety, $P \in X$ is a point, and A an abelian group, the *skyscraper sheaf at P with stalk A* is the sheaf on X defined by $\mathcal{F}(U) = A$ if $P \in U$ and $\mathcal{F}(U) = 0$ otherwise).

(b) Let t be a local coordinate at a point $P \in X$ and n_P the coefficient of P in D. Show that $k(X)/\mathcal{O}_X(D)_P$ can be identified with the k-vector space of Laurent polynomials

$$a_\nu t^\nu + a_{\nu+1} t^{\nu+1} + \cdots + a_{-n_P-1} t^{-n_P-1}$$

where $\nu \leq -n_P - 1$ and $a_i \in k$.

(c) Use the exact sequence of sheaves

$$0 \to \mathcal{O}_X(D) \to \mathcal{K} \to \bigoplus_{P \in X} \frac{k(X)}{\mathcal{O}_X(D)_P} \to 0$$

to obtain an exact sequence of abelian groups

$$0 \to H^0(\mathcal{O}_X(D)) \to k(X) \to \bigoplus_{P \in X} \frac{k(X)}{\mathcal{O}_X(D)_P} \to H^1(\mathcal{O}_X(D)) \to 0$$

(d) An *adèle* is a family $(r_P)_{P\in X}$ of elements of $k(X)$ such that $r_P \in \mathcal{O}_{X,P}$ for all but finitely many $P \in X$. The set R of adéles is a k-algebra, and there is an obvious inclusion $k(X) \subset R$. For $D = \sum n_P P$ a divisor on X we define $R(D) \subset R$ by

$$R(D) = \{(r_P)_{P\in X} \mid \nu_P(r_P) \geq -n_P\}.$$

(This is analogous to $H^0(\mathcal{O}_X(D)) \subset k(X)$.) Use (c) above to show that

$$H^1(\mathcal{O}_X(D)) = \frac{R}{R(D)+k(X)}.$$

This description of $H^1(\mathcal{O}_X(D))$ was used in 508A.

(10) Let X be a smooth manifold and $\mathcal{A}$ the sheaf of smooth $\mathbb{R}$-valued functions on X. Suppose $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules on X. Show that $H^p(X,\mathcal{F}) = 0$ for $p > 0$ as follows. Let $\mathcal{U} = \{U_i\}$ be an open covering of X. Then there is a *partition of unity subordinate to* $\mathcal{U}$, that is, smooth functions $\rho_i \colon X \to \mathbb{R}$ such that $\sum_{i\in I} \rho_i = 1$, $0 \leq \rho_i \leq 1$, and $\rho_i(x) = 0$ for $x \in X \setminus U_i$. Now suppose $s = (s_{i_0\cdots i_p}) \in C^p(\mathcal{U},\mathcal{F})$ satisfies $ds = 0$. Define $t \in C^{p-1}(\mathcal{U},\mathcal{F})$ by $t_{i_0\cdots i_{p-1}} = \sum_{i\in I} \rho_i s_{ii_0\cdots i_{p-1}}$. Show that $dt = s$. (Note that the following result can be proved by the same method: If X is an affine variety and $\mathcal{F}$ is a coherent sheaf on X then $H^p(X,\mathcal{F}) = 0$ for $p > 0$. See [Serre55, p. 236-7].)

References

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry.

[Hartshorne] R. Hartshorne, Algebraic geometry.

[Serre55] J-P. Serre, Faisceaux algébriques cohérents, available at www.jstor.org