583C Lecture notes

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1 Sheaves and cohomology

Here we give a quick introduction to sheaves and their cohomology. For more details, see [GH, p. 34–43].

1.1 Sheaves

Let X be a topological space. A *sheaf* $\mathcal{F}$ on X is the following data:

(1) For every open set $U \subset X$, an abelian group $\mathcal{F}(U)$, the *sections of* $\mathcal{F}$ *over* U, and

(2) for every inclusion of open sets $V \subset U$ a homomorphism $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$, the *restriction map*,

such that

(1) $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

(2) For $U \subset X$ open and $U = \bigcup_{i \in I} U_i$ an open covering of U, the sequence

$$\begin{array}{ccccccc} 0 & \to & \mathcal{F}(U) & \to & \bigoplus_{i\in I} \mathcal{F}(U_i) & \to & \bigoplus_{i \neq j \in I} \mathcal{F}(U_i \cap U_j) \\ & & s & \mapsto & (s|_{U_i}) & & \\ & & & & (s_i) & \mapsto & (s_i|_{U_{ij}} - s_j|_{U_{ij}}) \end{array}$$

is exact.

Here for $s \in \mathcal{F}(U)$ and $V \subset U$ we write $s|_V$ for $\rho_{UV}(s)$. We also sometimes write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$.

Example 1.1. (1) If X is a complex manifold or smooth algebraic variety of dimension n, the holomorphic or regular functions on X form a sheaf $\mathcal{O}_X$, the *structure sheaf*, and the holomorphic or regular p-forms form a sheaf Ω^p_X for $0 \leq p \leq n$ (by convention $\Omega^0_X = \mathcal{O}_X$). We write $\omega_X = \Omega^n_X$.

(2) Similarly, if X is a smooth manifold of dimension n, the smooth functions form a sheaf $\mathcal{A}_X$, and the smooth p-forms form a sheaf $\mathcal{A}^p_X$, $0 \leq p \leq n$. Note: The use of these sheaves is not essential in the theory of smooth manifolds, because we can always globalise local data using smooth bump functions. They are however sometimes used in an auxiliary role (e.g., proof of de Rham theorem [GH, p. 43–44]).

(3) If X is a topological space and A is an abelian group (for example, $A = \mathbb{Z}$), the locally constant A-valued functions on X form a sheaf $\underline{A}$, the *constant sheaf with stalk A*.

A morphism of sheaves $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a homomorphism $\alpha_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ for each open $U \subset X$, compatible with the restriction maps. The *kernel* of α is the subsheaf $\ker \alpha$ of $\mathcal{F}$ defined by $(\ker \alpha)(U) = \ker \alpha_U$. The *image* of α is the subsheaf $\operatorname{im} \alpha$ of $\mathcal{G}$ defined as follows: $s \in \mathcal{G}(U)$ lies in $(\operatorname{im} \alpha)(U)$ if there exists a open covering $U = \bigcup U_i$ of U such that $s|_{U_i} \in \operatorname{im} \alpha_{U_i}$ for all i. (Note: $(\operatorname{im} \alpha)(U) \neq \operatorname{im} \alpha_U$ in general.) The *cokernel* of α is the sheaf $\operatorname{coker} \alpha$ defined as follows: an element of $(\operatorname{coker} \alpha)(U)$ is given by an open covering $U = \bigcup U_i$ of U and sections $s_i \in \mathcal{G}(U_i)$ such that

$$s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \operatorname{im} \alpha_{U_i \cap U_j}$$

for all $i \neq j$. Two such data $(s_i \in \mathcal{F}(U_i))$, $(s'_j \in \mathcal{F}(U'_j))$ define the same element of $(\operatorname{coker} \alpha)(U)$ if for all $P \in U$ there exists an an open neighbourhood V of P and i, j such that $V \subset U_i \cap U'_j$ and $s_i|_V - s'_j|_V \in \operatorname{im}(\alpha_V)$. (Note: $(\operatorname{coker} \alpha)(U) \neq \operatorname{coker} \alpha_U$ in general.)

The *stalk* of a sheaf $\mathcal{F}$ at a point $P \in X$ is

$$\mathcal{F}_P = \varinjlim_{U \ni P} \mathcal{F}(U),$$

the direct limit over open neighbourhoods U of P of the abelian groups $\mathcal{F}(U)$. That is, an element of $\mathcal{F}_P$ is given by a section $s \in \mathcal{F}(U)$ for some open neighbourhood U of P, and $(s \in \mathcal{F}(U))$, $(t \in \mathcal{F}(V))$ define the same element of $\mathcal{F}_P$ if there exists an open neighbourhood W of P such that $W \subset U \cap V$ and $s|_W = t|_W$.

Example 1.2. (1) Let X be a complex manifold, $P \in X$ a point, and $z_1, \ldots, z_n$ local complex coordinates at P. Then an element of $\mathcal{O}_{X,P}$ is a power series in $z_1, \ldots, z_n$ with positive radius of convergence.

(2) If X is an algebraic variety then $\mathcal{O}_{X,P}$ is the local ring of regular functions at P.

We say a sequence of sheaves

$$\mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G}$$

is *exact* if the induced sequence of stalks at P is exact for all $P \in X$. Equivalently, $\operatorname{im} \alpha = \ker \beta$ (where $\operatorname{im} \alpha$ and $\ker \beta$ are the subsheaves of $\mathcal{F}$ defined above).

Example 1.3. Let X be a complex manifold. The *exponential sequence* is the exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \to & \underline{\mathbb{Z}} & \to & \mathcal{O}_X & \to & \mathcal{O}_X^{\times} & \to & 0 \\ & & & & f & \mapsto & \exp(2\pi i f) & & \end{array}$$

Here $\underline{\mathbb{Z}}$ denotes the constant sheaf with stalk $\mathbb{Z}$, and $\mathcal{O}_X^{\times}$ denotes the sheaf of holomorphic functions on X which are nowhere zero (with group law pointwise multiplication). Note that $\mathcal{O}_X(U) \to \mathcal{O}_X^{\times}(U)$ is *not* surjective in general (for example, if $X = U = \mathbb{C} \setminus \{0\}$).

We introduce some more terminology. Let X be a complex manifold or smooth algebraic variety of dimension n. Then the structure sheaf $\mathcal{O}_X$ is a *sheaf of rings*, that is, each $\mathcal{O}_X(U)$ is a ring and the restriction maps are ring homomorphisms. We say a sheaf $\mathcal{F}$ on $\mathcal{O}_X$ is an $\mathcal{O}_X$*-module* if $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for each U, and this structure is compatible with the restriction maps. An $\mathcal{O}_X$*-module* $\mathcal{F}$ is *locally free of rank* r if there is an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\mathcal{F}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ for each i. For example Ω_X^p is locally free of rank $\binom{n}{p}$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if there is an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X and an exact sequence

$$\mathcal{O}_{U_i}^{\oplus m_i} \to \mathcal{O}_{U_i}^{\oplus n_i} \to \mathcal{F}|_{U_i} \to 0$$

for each i (that is, $\mathcal{F}$ is locally a cokernel of a map of free sheaves of finite rank).

1.2 Cohomology of sheaves

Let $\mathcal{F}$ be a sheaf on a topological space X, and $\mathcal{U} = \{U_i\}_{i \in I}$ a finite open covering of X. We write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Define

$$C^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$$

and

$$d \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}), \quad (ds)_{i_0 \cdots i_p} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \cdots \hat{i_j} \cdots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}}}.$$

One checks that $d^2 = 0$. We define the *Cech cohomology* $H^p(\mathcal{U}, \mathcal{F})$ *of* $\mathcal{F}$ *relative to the open covering* $\mathcal{U}$ to be the cohomology of the complex $(C^{\cdot}(\mathcal{U}, \mathcal{F}), d)$, that is

$$H^p(\mathcal{U}, \mathcal{F}) = \frac{\ker(d \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}))}{\operatorname{im}(d \colon C^{p-1}(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F}))}$$

Note immediately that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ by the sheaf axioms.

Now let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of the open covering $\mathcal{U}$. That is, for all $j \in J$ there exists $i \in I$ such that $V_j \subset U_i$. Fix a map $\phi \colon J \to I$ such that $V_j \subset U_{\phi(j)}$ for all j. Then ϕ induces maps

$$\rho_\phi \colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F}) \quad (\rho_\phi s)_{j_0 \cdots j_p} = s_{\phi(j_0) \cdots \phi(j_p)}|_{V_{j_0 \cdots j_p}}$$

which are compatible with the differentials d, and so induce maps

$$\rho_\phi \colon H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{V}, \mathcal{F})$$

on cohomology. One shows that the maps ρ_ϕ on cohomology do not depend on the choice of ϕ (because the maps ρ_ϕ on complexes for different choices of ϕ are chain homotopic). We define the *Cech cohomology* $H^p(X, \mathcal{F})$ of $\mathcal{F}$ by

$$H^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F}),$$

the direct limit over open coverings $\mathcal{U}$ of X of the $H^q(\mathcal{U}, \mathcal{F})$.

Example 1.4. Let X be a topological space. Then $H^p(X, \underline{\mathbb{Z}}) \simeq H^p(X, \mathbb{Z})$, that is, the Cech cohomology of the constant sheaf with stalk $\mathbb{Z}$ is isomorphic to the (simplicial) integral cohomology of X. To see this, let $X \simeq |\Sigma|$ be a triangulation, with vertices $\{v_i\}_{i \in I}$. Let $U_i = \operatorname{Star}(v_i)$, the union of the interiors of simplices containing v_i. Then $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of X, and $U_{i_0 \cdots i_p}$ is non-empty iff $\langle v_{i_0} \cdots v_{i_p} \rangle$ is a simplex of Σ and is connected when nonempty. We obtain an isomorphism

$$C^p(\mathcal{U}, \underline{\mathbb{Z}}) \xrightarrow{\sim} C^p(X, \mathbb{Z}) = C_p(X, \mathbb{Z})^* \quad s = (s_{i_0 \cdots i_p}) \mapsto (\langle v_{i_0} \cdots v_{i_p} \rangle \mapsto s_{i_0 \cdots i_p})$$

which is compatible with the differentials, and so an isomorphism

$$H^p(\mathcal{U}, \underline{\mathbb{Z}}) \xrightarrow{\sim} H^p(X, \mathbb{Z}).$$

We can make the open covering $\mathcal{U}$ arbitrarily fine by subdividing Σ. So, passing to the limit over all coverings, we deduce

$$H^p(X, \underline{\mathbb{Z}}) \xrightarrow{\sim} H^p(X, \mathbb{Z})$$

as claimed.

Cech cohomology groups can be computing using the following theorem.

Theorem 1.5. *(Leray theorem) Let X be a topological space, $\mathcal{F}$ a sheaf on X, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X such that*

$$H^q(U_{i_0 \cdots i_p}, \mathcal{F}) = 0 \text{ for all } q > 0 \text{ and } i_0, \ldots i_p. \tag{1}$$

Then $H^p(X, \mathcal{F}) = H^p(\mathcal{U}, \mathcal{F})$.

Example 1.6. The hypothesis (1) is satisfied in the following cases:

(1) X an algebraic variety, $\mathcal{F}$ a coherent sheaf, and $\mathcal{U}$ an open covering of X by affine open sets.

(2) X a complex manifold of dimension n, $\mathcal{F}$ a coherent sheaf, and $\mathcal{U}$ a covering of X by polydiscs (that is, open sets of the form Δ^n for $\Delta = (|z| < r) \subset \mathbb{C}$, some $r > 0$).

(3) X a topological space, $\mathcal{F} = \underline{A}$ a constant sheaf, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X such that $U_{i_0 \cdots i_p}$ is contractible for all $i_0, \ldots, i_p$ (cf. Ex. 1.4).

Here is one of the main applications of sheaf cohomology. Let

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of sheaves. Then there is an associated long exact sequence of cohomology

$$\begin{array}{ccccccc} 0 & \to & H^0(X,\mathcal{E}) & \to & H^0(X,\mathcal{F}) & \to & H^0(X,\mathcal{G}) \\ & \xrightarrow{\delta} & H^1(X,\mathcal{E}) & \to & H^1(X,\mathcal{F}) & \to & H^1(X,\mathcal{G}) \\ & \xrightarrow{\delta} & H^2(X,\mathcal{E}) & \to & \cdots & & \end{array}$$

To see this, assume for simplicity that there exist arbitrarily fine open coverings $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that the sequence

$$0 \to \mathcal{E}(U_{i_0\cdots i_p}) \to \mathcal{F}(U_{i_0\cdots i_p}) \to \mathcal{G}(U_{i_0\cdots i_p}) \to 0$$

is exact for all $i_0,\ldots,i_p$. (This is always satisfied in practice.) Then we have an exact sequence of complexes

$$0 \to C^{\cdot}(\mathcal{U},\mathcal{E}) \to C^{\cdot}(\mathcal{U},\mathcal{F}) \to C^{\cdot}(\mathcal{U},\mathcal{G}) \to 0$$

which (as usual, see [Hatcher, p. 116-7]) induces a long exact sequence of cohomology

$$\cdots \to H^i(\mathcal{U},\mathcal{E}) \to H^i(\mathcal{U},\mathcal{F}) \to H^i(\mathcal{U},\mathcal{G}) \xrightarrow{\delta} H^{i+1}(\mathcal{U},\mathcal{E}) \to \cdots$$

Taking the limit over open coverings $\mathcal{U}$ gives the result.

In particular, given a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

we have an exact sequence of abelian groups

$$0 \to \mathcal{E}(X) \to \mathcal{F}(X) \to \mathcal{G}(X) \xrightarrow{\delta} H^1(X,\mathcal{E})$$

So, a section $t \in \mathcal{G}(X)$ is the image of a section $s \in \mathcal{F}(X)$ iff $\delta(t) = 0$ in $H^1(X,\mathcal{E})$.

1.3 Analytic and algebraic approaches

If X is a smooth complex variety then there is an associated complex manifold, denoted X^{an}. Note that the topology on X is the Zariski topology (the open sets are complements of finite unions of closed subvarieties) and the topology on X^{an} is the usual Euclidean topology.

We state some results from [Serre56]. If $\mathcal{F}$ is a coherent sheaf on X there is an naturally associated coherent sheaf $\mathcal{F}^{\mathrm{an}}$ on X^{an}. Now suppose X is projective, so X^{an} is compact. Then, for $\mathcal{F}$ a coherent sheaf on X, there are natural isomorphisms

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

for each i . In particular, $\mathcal{F}(X) \simeq \mathcal{F}^{\mathrm{an}}(X)$. Note that this is *not* true in general if X^{an} is not compact. For example, if $X = \mathbb{A}^1_x$ and $\mathcal{F} = \mathcal{O}_X$, then $\mathcal{F}^{\mathrm{an}} = \mathcal{O}_{X^{\mathrm{an}}}$, the sheaf of holomorphic functions on X^{an}, and $\exp(x) \in \mathcal{O}_{X^{an}}(X) \setminus \mathcal{O}_X(X)$.

References

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[Hatcher] A. Hatcher, Algebraic topology, available at www.math.cornell.edu/~hatcher/AT/ATpage.html

[Serre56] J-P. Serre, Géométrie algébrique et géométrie analytique, available at www.numdam.org