

# 583C Lecture notes

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## 1 Topology (continued)

### 1.1 de Rham cohomology (continued)

We show that the pairing

$$\wedge: H_{\text{dR}}^i(X, \mathbb{R}) \times H_{\text{dR}}^{d-i}(X, \mathbb{R}) \rightarrow \mathbb{R}, \quad (\omega, \eta) \mapsto \int_X \omega \wedge \eta$$

given by the wedge product is identified with the pairing

$$\cap: H_{d-i}(X, \mathbb{R}) \times H_i(X, \mathbb{R}) \rightarrow \mathbb{R}$$

given by the intersection product via the isomorphisms

$$H_i(X, \mathbb{R}) \simeq H_{\text{dR}}^{d-i}(X, \mathbb{R}), \quad H_{d-i}(X, \mathbb{R}) \simeq H_{\text{dR}}^i(X, \mathbb{R})$$

described above. In particular, the wedge pairing is nondegenerate. Assume for simplicity that  $\alpha \in H_i(X, \mathbb{R})$ ,  $\beta \in H_{d-i}(X, \mathbb{R})$  are represented by closed submanifolds  $A, B \subset X$  of dimensions  $i, d-i$ . Let  $\omega, \eta$  be representatives of the corresponding deRham cohomology classes which are supported in a small tubular neighbourhood of  $A, B$ . We may assume that  $A, B$  intersect transversely in a finite number of points. Let  $P$  be an intersection point of  $A, B$ , then we can choose local coordinates  $x_1, \dots, x_d$  at  $P$  such that

$$A = (x_1 = \dots = x_{d-i} = 0) \subset X, \quad B = (x_{d-i+1} = \dots = x_d = 0) \subset X.$$

As above we may assume that, working locally at  $P \in X$ ,

$$\omega = f(x_1, \dots, x_{d-i}) dx_1 \wedge \dots \wedge dx_{d-i}, \quad \eta = g(x_{d-i+1}, \dots, x_d) dx_{d-i+1} \wedge \dots \wedge dx_d,$$

where  $f, g$  are bump functions at  $0 \in \mathbb{R}^{d-i}$  and  $0 \in \mathbb{R}^i$  with integral 1. We find that the contribution to  $\int_X \omega \wedge \eta$  from this chart is  $i_P(A, B) = \pm 1$ , the

intersection number of  $A, B$  at  $P$  (where the sign comes from the orientations — see the description of the intersection product). Adding together the local contributions we deduce  $\int_X \omega \wedge \eta = \alpha \cap \beta$ , as required. (See [GH, p. 58–59] for an alternative argument.)

*Remark 1.1.* The cup product

$$\cup: H^i(X, \mathbb{R}) \times H^j(X, \mathbb{R}) \rightarrow H^{i+j}(X, \mathbb{R})$$

on (simplicial) cohomology corresponds to the wedge product on de Rham cohomology via the de Rham isomorphism [GH, p. 60]. (The cup product can be defined as follows: Let  $\langle v_0, \dots, v_n \rangle$  denote the simplex with vertices  $v_0, \dots, v_n$ . For  $\phi \in C^i(X, \mathbb{Z}), \psi \in C^j(X, \mathbb{Z})$  simplicial cochains, define  $\phi \cup \psi \in C^{i+j}(X, \mathbb{Z})$  by

$$\phi \cup \psi(\langle v_0 \cdots v_{i+j} \rangle) = \phi(\langle v_0 \cdots v_i \rangle) \psi(\langle v_i \cdots v_{i+j} \rangle).$$

This induces a well defined product on cohomology. See [Hatcher, p. 206, Sec. 3.2] for more details.)

## 2 Hodge theory

Let  $X$  be a complex manifold of complex dimension  $n$ . Consider  $\mathbb{C}$ -valued  $k$ -forms  $\omega$  on  $X$ . That is, locally  $\omega = \sum_{|I|=k} f_I dx_I$  where  $x_1, \dots, x_{2n}$  are local real coordinates on  $X$ , and  $f_I$  is a smooth  $\mathbb{C}$ -valued function on  $X$ . We define the complex de Rham cohomology group  $H_{\text{dR}}^k(X, \mathbb{C})$  as the space of closed  $\mathbb{C}$ -valued  $k$ -forms modulo exact forms (as in the real case). Note of course that  $H_{\text{dR}}^k(X, \mathbb{C}) = H_{\text{dR}}^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ .

Now let  $z_1, \dots, z_n$  be local complex coordinates on  $X$ , and write  $z_i = x_i + iy_i$  for each  $i = 1, \dots, n$ . So  $x_1, y_1, \dots, x_n, y_n$  are local real coordinates on  $X$ . We can write a  $\mathbb{C}$ -valued  $k$ -form in terms of the  $dz_i = dx_i + idy_i, d\bar{z}_i = dx_i - idy_i$  (instead of the  $dx_i, dy_i$ ). A  $(p, q)$ -form is a  $\mathbb{C}$ -valued  $(p+q)$ -form which is locally of the form

$$\sum_{|I|=p, |J|=q} f_{I,J} dz_I \wedge d\bar{z}_J$$

for some  $\mathbb{C}$ -valued functions  $f_{I,J}$  on  $X$ . Let

$$H^{p,q}(X) \subset H_{\text{dR}}^{p+q}(X, \mathbb{C})$$

denote the complex subspace of de Rham cohomology classes represented by a closed  $(p, q)$ -form. Note immediately that  $H^{q,p} = \bar{H}^{p,q}$ , that is,  $H^{q,p} \subset H^k(X, \mathbb{C})$  is the complex conjugate of the subspace  $H^{p,q} \subset H^k(X, \mathbb{C})$ .

**Theorem 2.1.** (Hodge decomposition) [GH, p. 116] Let  $X$  be a smooth complex projective variety (or, more generally, a compact complex Kähler manifold). Then

$$H_{\text{dR}}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Moreover, there is a natural isomorphism  $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$ . That is,  $H^{p,q}(X)$  is isomorphic to the  $q$ th cohomology group of the sheaf  $\Omega_X^p$  of holomorphic  $p$ -forms on  $X$ .

A holomorphic  $p$ -form  $\omega$  on  $X$  is a  $\mathbb{C}$ -valued form which is locally of the form  $\omega = \sum_{|I|=p} f_I dz_I$  where  $z_1, \dots, z_n$  are local complex coordinates on  $X$  and  $f_I$  is a holomorphic  $\mathbb{C}$ -valued function on  $X$ . The sheaf  $\Omega_X^p$  of holomorphic  $p$ -forms is the data of the spaces of holomorphic  $p$ -forms  $\Omega_X(U)$  on  $U$  for each open set  $U \subset X$  together with the restriction maps  $\Omega_X^p(U) \rightarrow \Omega_X^p(V)$  for  $V \subset U$ . (We will discuss sheaves and cohomology of sheaves in more detail shortly).

*Remark 2.2.* Recall that the wedge product defines a nondegenerate pairing

$$H_{\text{dR}}^k(X, \mathbb{C}) \times H_{\text{dR}}^{2n-k}(X, \mathbb{C}) \rightarrow H^n(X, \mathbb{R}) \simeq \mathbb{C}, \quad (\omega, \eta) \mapsto \int_X \omega \wedge \eta.$$

By the Hodge decomposition, this pairing decomposes into a direct sum of nondegenerate pairings

$$H^{p,q} \times H^{n-p,n-q} \rightarrow H^{n,n} \simeq \mathbb{C}.$$

(Note that the wedge product of a  $(p, q)$ -form and an  $(r, s)$ -form is a  $(p+r, q+s)$ -form, so can only be non-zero if  $p+r, q+s \leq n$ .) In terms of sheaf cohomology, we have a nondegenerate pairing

$$H^q(\Omega_X^p) \times H^{n-q}(\Omega_X^{n-p}) \rightarrow H^n(\Omega_X^n) \simeq \mathbb{C}.$$

We observe that this is an instance of Serre duality [Hartshorne, p. 244, III.7.7]. Indeed, the pairing of sheaves

$$\wedge: \Omega_X^p \times \Omega_X^{n-p} \rightarrow \Omega_X^n =: \omega_X$$

determines an identification

$$\Omega_X^{n-p} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^p, \omega_X) = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^p, \mathcal{O}_X) \otimes \omega_X = (\Omega_X^p)^\vee \otimes \omega_X.$$

So the pairing above can be rewritten as

$$H^q(\Omega_X^p) \times H^{n-q}((\Omega_X^p)^\vee \otimes \omega_X) \rightarrow H^n(\omega_X) \simeq \mathbb{C}.$$

This is the Serre duality pairing for the sheaf  $\Omega_X^p$ .

We define some notation. Let  $b_i(X) = \dim_{\mathbb{R}} H^i(X, \mathbb{R}) = \dim_{\mathbb{C}} H^i(X, \mathbb{C})$ , the  $i$ th Betti number, and  $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$ , the Hodge numbers. For  $\mathcal{F}$  a coherent sheaf on  $X$  (for example,  $\mathcal{O}_X$ ,  $\Omega_X^p$ ), let  $h^i(X, \mathcal{F}) = \dim_{\mathbb{C}} H^i(X, \mathcal{F})$ .

Now let  $X$  be a smooth complex projective surface. Then

$$H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1} = H^0(\Omega_X) \oplus H^1(\mathcal{O}_X)$$

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} = H^0(\omega_X) \oplus H^1(\Omega_X) \oplus H^2(\mathcal{O}_X)$$

The irregularity of  $X$  is  $q := h^0(\Omega_X) = h^1(\mathcal{O}_X)$ . The geometric genus of  $X$  is  $p_g := h^0(\omega_X) = h^2(\mathcal{O}_X)$ . We have

$$b_1 = b_3 = 2q, \quad b_2 = 2p_g + h^{1,1}$$

by the Hodge decomposition and Poincaré duality.

## References

- [GH] P. Griffiths, J. Harris, Principles of algebraic geometry.
- [Hartshorne] R. Hartshorne, Algebraic geometry.
- [Hatcher] A. Hatcher, Algebraic topology, available at [www.math.cornell.edu/~hatcher/AT/ATpage.html](http://www.math.cornell.edu/~hatcher/AT/ATpage.html)