583C Lecture notes

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1 Topology (continued)

1.1 Classification of quadratic forms (continued)

Remark 1.1. A free abelian group L together with a symmetric bilinear form $Q\colon L \times L \to \mathbb{Z}$ is sometimes called a *lattice*.

Example 1.2. Let $\pi\colon X \to \mathbb{P}^2$ be the blowup of n distinct points $P_1, \ldots, P_n$ in the complex projective plane $\mathbb{P}^2$. Let $H \subset \mathbb{P}^2$ be a hyperplane not containing any of the P_i and $E_1, \ldots, E_n$ the exceptional curves. Then $H_2(X, \mathbb{Z})$ is free with basis $\pi^{-1}H, E_1, \ldots, E_n$, and the intersection product Q has matrix $(1) \oplus (-1)^n$ with respect to this basis,

Example 1.3. Let X be a K3 surface (that is, a compact complex surface X such that the canonical divisor K_X is trivial and $\pi_1(X, x) = 0$) Then $H_2(X, \mathbb{Z})$ is free of rank 22 and the intersection form Q has type $H^3 \oplus (-E_8)^2$.

1.2 de Rham cohomology

Here we describe the de Rham approach to cohomology via differential forms.

We first note some elementary facts about homology and cohomology groups. Let X be a topological space. Then we can define (simplicial) homology and cohomology groups $H_i(X, A)$, $H^i(X, A)$ with coefficients in an abelian group A as before. If $\mathbb{F}$ is a field of characteristic 0 (for example $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) then $H_i(X, \mathbb{F}) = H_i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$. Recall that $H^i(X, \mathbb{Z}) / \operatorname{Tors} = H_i(X, \mathbb{Z})^*$. If $\mathbb{F}$ is a field then $H^i(X, \mathbb{F}) = H_i(X, \mathbb{F})^*$.

Now let X be a compact smooth manifold of (real) dimension d. Let $C^k_{\mathrm{dR}}(X, \mathbb{R})$ denote the $\mathbb{R}$-vector space of smooth $\mathbb{R}$-valued k-forms ω on X. That is, locally on X,

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots dx_{i_k} = \sum_{|I| = k} f_I dx_I$$

where $x_1, \ldots, x_d$ are local coordinates on X and the f_I are smooth $\mathbb{R}$-valued functions on X. Let

$$d\colon C^k_{\mathrm{dR}}(X,\mathbb{R}) \to C^{k+1}_{\mathrm{dR}}(X,\mathbb{R})$$

be the *exterior derivative*, that is (working locally),

$$d\omega = d(\sum f_I dx_I) = \sum df_I \wedge dx_I$$

where

$$df = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} dx_i$$

Then $d^2 = 0$, and we define the *(real) de Rham cohomology groups* $H^i_{\mathrm{dR}}(X,\mathbb{R})$ by

$$H^i_{\mathrm{dR}}(X,\mathbb{R}) = \frac{\ker(d\colon C^i(X,\mathbb{R}) \to C^{i+1}(X,\mathbb{R}))}{\mathrm{im}(d\colon C^{i-1}(X,\mathbb{R}) \to C^i(X,\mathbb{R}))}.$$

We say a smooth differential form ω is *closed* if $d\omega = 0$, and we say ω is *exact* if $\omega = d\eta$ for some η. So, $H^i_{\mathrm{dR}}(X,\mathbb{R})$ is the $\mathbb{R}$-vector space of closed i-forms modulo exact forms.

There is a natural $\mathbb{R}$-bilinear pairing

$$H^i_{\mathrm{dR}}(X,\mathbb{R}) \times H_i(X,\mathbb{R}) \to \mathbb{R}, \quad (\omega,\gamma) \mapsto \int_\gamma \omega. \tag{1}$$

Note that this is well defined by Stokes' theorem: if $\omega = d\eta$ then $\int_\gamma d\eta = \int_{d\gamma} \omega = 0$ because $d\gamma = 0$, and if $\gamma = d\beta$ then $\int_{d\beta} \omega = \int_\beta d\omega = 0$ because $d\omega = 0$.

Theorem 1.4. *(de Rham's theorem) [GH, p. 44] The map*

$$H^i_{\mathrm{dR}}(X,\mathbb{R}) \to H_i(X,\mathbb{R})^* = H^i(X,\mathbb{R})$$

induced by (1) is an isomorphism.

The *wedge product* or *exterior product* on de Rham cohomology is the product

$$H^i_{\mathrm{dR}}(X,\mathbb{R}) \times H^j_{\mathrm{dR}}(X,\mathbb{R}) \to H^{i+j}_{\mathrm{dR}}(X,\mathbb{R}), \quad (\omega,\eta) \mapsto \omega \wedge \eta$$

induced by wedge product of forms. Note that this is well defined because

$$d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^{\deg \omega} \omega \wedge d\xi$$

so, if ω is closed and $\eta = d\xi$ is exact, then

$$\omega \wedge d\xi = (-1)^{\deg \omega} d(\omega \wedge \xi)$$

is exact.

Now we discuss Poincaré duality from the de Rham point of view. Recall that the intersection product defines a non-degenerate bilinear form

$$\cap \colon H_i(X, \mathbb{R}) \times H_{d-i}(X, \mathbb{R}) \to \mathbb{R}.$$

Consider the identification

$$H_i(X, \mathbb{R}) \xrightarrow{\sim} H_{d-i}(X, \mathbb{R})^* = H^{d-i}(X, \mathbb{R}) = H^{d-i}_{\mathrm{dR}}(X, \mathbb{R})$$

induced by $\cap$ and the de Rham isomorphism. Here an i-cycle α maps to a $(d-i)$ form ω (determined up to an exact form) such that for any $(d-i)$-cycle $\beta \in H_{d-i}(X, \mathbb{R})$,

$$\alpha \cap \beta = \int_\beta \omega.$$

We identify the form ω explicitly. Assume for simplicity that the homology class α is represented by a smooth submanifold $A \subset X$ of dimension i. There exists a "tubular neighbourhood" N of A in X isomorphic to a neighbourhood of the zero section in a vector bundle over A (the normal bundle of A in X). We construct a form ω such that the support of ω is contained in N, ω is closed, and $\int_{N_a} \omega = 1$ for all $a \in A$, where N_a denotes the fibre of the bundle $N \to A$ over a. Explicitly, locally on X write $A = (x_1 = \cdots = x_{d-i} = 0) \subset X$, where $x_1, \ldots, x_d$ are local coordinates on X, and let $\omega = f dx_1 \wedge \cdots \wedge dx_{d-i}$ where $f = f(x_1, \ldots, x_{d-i})$ is a smooth bump function on $\mathbb{R}^{d-i}$ supported in a small neighbourhood of the origin, with integral 1. Globally, we can use a partition of unity to patch the local forms together. Now suppose that $\beta \in H_{d-i}(X, \mathbb{Z})$, and represent β by a piecewise smooth $(d-i)$-cycle B intersecting A transversely in a finite number of points. Then $\int_B \omega = A \cap B$ — at each intersection point, we get the integral of the bump function f (which equals 1 by construction), with a sign given by the orientations.

References

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry.