583C Lecture notes

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April 28, 2008

1 Negativity of contracted locus

Before stating the result we quickly review the notion of a normal variety. A variety X is *normal* if for every point $Q \in X$ the local ring $\mathcal{O}_{X,Q}$ is integrally closed. Equivalently, X satisfies the following two conditions:

(R_1) The singular locus of X has codimension ≥ 2.

(S_2) If $U \subset X$ is open, $Z \subset X$ is closed of codimension ≥ 2, and f is a rational function which is regular on $U \setminus Z$, then f is regular on U.

See [Matsumura, p. 183, Thm. 23.8]. In particular, a normal surface has a finite number of singular points, and (for example) a surface obtained from a smooth surface by glueing two points together is not normal (since it does not satisfy the condition S_2). Smooth varieties are normal, and a curve is normal iff it is smooth. If X is a variety then the *normalisation* of X is a normal variety X^ν together with a finite birational morphism $\nu\colon X^\nu \to X$. Any map $f\colon Y \to X$ from a normal variety Y factors uniquely through the normalisation of X.

Theorem 1.1. *Let $f\colon X \to Y$ be a birational morphism from a smooth projective surface X to a normal projective surface Y. Let $E_1, \ldots, E_r$ be the exceptional curves over a point $Q \in Y$ (the curves contracted by f to Q). Then $(\sum a_i E_i)^2 < 0$ for $(a_1, \ldots, a_r) \neq 0$.*

Proof. We work locally over $Q \in Y$. Let $C \subset X$ be a irreducible curve such that $C \cap E_i \neq \emptyset$ (and $C \neq E_i$) for all i. For example, we can take C a general hyperplane section of X. Choose $g \in \mathcal{O}_{Y,Q}$ such that $f(C) \subset F := (g = 0)$. Then $(f^*g) = f^*F = F' + \sum \mu_i E_i$ where F' denotes the strict transform of F and $\mu_i > 0$ for each i. Write $D = \sum \mu_i E_i$. Then $D \cdot E_i = -F' \cdot E_i < 0$ for all i because $D + F' = (f^*g) \sim 0$ and F' contains C (which intersects each E_i) by construction. Now the result follows from the algebraic lemma below. $\square$

Lemma 1.2. *Let M be a free abelian group spanned by elements $e_1, \dots, e_r$ and*

$$M \times M \to \mathbb{Z}, \quad (a, b) \mapsto a \cdot b$$

a symmetric bilinear form such that

- *$e_i \cdot e_j \geq 0$ for all $i \neq j$, and*
- *there exists a linear combination $d = \sum \mu_j e_j$, with $\mu_j > 0$ for all j, such that $d \cdot e_i < 0$ for all i.*

Then $e_1, \dots, e_r$ are linearly independent and $m^2 < 0$ for $0 \neq m \in M$.

Proof. Write $a_{ij} = e_i \cdot e_j$ and let

$$\phi \colon \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}, \quad \phi(x, y) = \sum a_{ij} x_i y_j$$

be the symmetric bilinear form with matrix $A = (a_{ij})$. We need to show that ϕ is negative definite. The symmetric matrix A can be diagonalised by an orthogonal matrix P. That is, there exists P such that $P^T A P$ is diagonal and $P^T P = I$. Let $\lambda_1 \leq \ldots \leq \lambda_r$ be the diagonal entries of $P^T A P$ (the eigenvalues of A). Then, writing $x = Px'$, we have $\phi(x, x) = \sum \lambda_i {x'_i}^2$ and $\|x'\| = \|x\|$. We deduce that, for $0 \neq x \in \mathbb{R}^r$, we have

$$\frac{\phi(x, x)}{\|x\|^2} \leq \lambda_r,$$

with equality iff x is an eigenvector of A with eigenvalue λ_r.

Let $0 \neq x \in \mathbb{R}^r$ be a vector such that $\phi(x, x)/\|x\|^2$ is maximal. Note that replacing $x = (x_1, \dots, x_r)$ by $(|x_1|, \dots, |x_r|)$ does not decrease $\phi(x, x)/|x|^2$. Indeed

$$\sum a_{ij} x_i x_j \leq \sum a_{ij} |x_i| |x_j|$$

because $a_{ij} = e_i \cdot e_j \geq 0$ for $i \neq j$ by assumption. So, we may assume $x_i \geq 0$ for all i. For $m \in \mathbb{R}^r$ we have

$$\phi(m, x) = \sum_{i,j} a_{ij} m_i x_j = \sum_i (\sum_j a_{ij} x_j) m_i = \lambda_r \sum_i x_i m_i \tag{1}$$

because $Ax = \lambda_r x$ as noted above. Now set $m = d$, where $d = (\mu_1, \dots, \mu_r)$ is as in the statement. Then $\phi(d, x) < 0$ because $d \cdot e_i < 0$ and $x_i \geq 0$ for all i, and $\sum x_i \mu_i > 0$, so $\lambda_r < 0$ by (1). Thus ϕ is negative definite as required. $\square$

Corollary 1.3. *Let $f\colon X \to Y$ be a birational morphism from a smooth projective surface X to a normal projective surface Y. Let $\{E_i\}$ be the exceptional curves. Then we have an exact sequence*

$$0 \to \oplus\mathbb{Z}E_i \to \operatorname{Cl} X \to \operatorname{Cl} Y \to 0.$$

Proof. We have an exact sequence

$$\oplus\mathbb{Z}E_i \to \operatorname{Cl} X \to \operatorname{Cl}(X \setminus \cup E_i) \to 0$$

by the definition of the divisor class group (see [Hartshorne, p. 133, II.6.5]). Let $f(\cup E_i) = \{Q_1, \dots, Q_s\}$, then f restricts to an isomorphism

$$X \setminus \cup E_i \xrightarrow{\sim} Y \setminus \{Q_1, \dots, Q_s\}.$$

So

$$\operatorname{Cl}(X \setminus \cup E_i) \simeq \operatorname{Cl}(Y \setminus \{Q_1, \dots, Q_s\}) \simeq \operatorname{Cl}(Y).$$

Finally, injectivity of the map $\oplus\mathbb{Z}E_i \to \operatorname{Cl} X$ follows from the theorem. □

2 Factorisation of birational morphisms

Lemma 2.1. *Let $f\colon X \to Y$ be a birational morphism of smooth projective surface. Let E_i be the exceptional curves of f. Then*

$$K_X = f^*K_Y + \sum a_i E_i$$

where $a_i \in \mathbb{Z}$ and $a_i > 0$ for all i.

Proof. By Cor. 1.3 we have an exact sequence

$$0 \to \oplus\mathbb{Z}E_i \to \operatorname{Cl} X \to \operatorname{Cl} Y \to 0.$$

So $K_X = f^*K_Y + \sum a_i E_i$ for some $a_i \in \mathbb{Z}$. It remains to show that $a_i > 0$ for each i. Let $f(\cup E_i) = \{Q_1, \dots, Q_s\}$, so f restricts to an isomorphism

$$X \setminus \cup E_i \xrightarrow{\sim} Y \setminus \{Q_1, \dots, Q_s\},$$

and let ω be a rational 2-form on Y which is regular and nonzero at each Q_j. Then

$$(\pi^*\omega) = \pi^*(\omega) + \sum a_i E_i$$

and the divisor $\pi^*(\omega)$ is disjoint from the E_i by construction, so a_i is the order of vanishing of $\pi^*\omega$ along E_i. In particular $a_i \geq 0$ because $\pi^*\omega$ is regular near E_i. We show $a_i > 0$. Let $f(E_i) = Q_j$, let $P \in E_i$ be a general point, and choose local coordinates u, v at $P \in X$ and x, y at $Q_j \in Y$. Then $\omega = g dx \wedge dy$ where $g \in \mathcal{O}_{X,P}$ and $g(P) \neq 0$. Now $\pi^* \omega = \pi^* g \cdot J du \wedge dv$ where

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of the map f with respect to these coordinates. But $J(P) = 0$ by the inverse function theorem (if $J(P) \neq 0$, then f is a local isomorphism near $P \in X$). Thus $\pi^* \omega$ vanishes along E_i, so $a_i > 0$. ☐

Corollary 2.2. $K_X \cdot E_j < 0$ *for some* j*.*

Proof. We have $K_X \cdot (\sum a_i E_i) = (\sum a_i E_i)^2 < 0$ by Thm. 1.1 and $a_i > 0$ for all i. So $K_X \cdot E_j < 0$ for some j. ☐

References

[Hartshorne] R. Hartshorne, Algebraic geometry.

[Matsumura] H. Matsumura, Commutative ring theory.