583C Lecture notes

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1 Birational geometry of surfaces (continued)

Proposition 1.1. *Let X be a smooth surface, $\pi\colon \tilde{X} \to X$ the blowup of a point $P \in X$, and $E = \pi^{-1}P$ the exceptional curve. Then $K_{\tilde{X}} = \pi^*K_X + E$.*

Proof. Let x, y be local coordinates at $P \in X$ and $\omega = dx \wedge dy$, a rational 2-form on X. Then $K_X = (\omega)$, the divisor of zeroes and poles of ω. (Here $(\omega) := \sum_{C \subset X} \nu_C(\omega) C$ where the sum is over irreducible divisors $C \subset X$ and $\nu_C(\omega)$ is the order of vanishing of ω along C defined as follows: let $Q \in C$ be a general point, z, w local coordinates at Q, and write $\omega = f dz \wedge dw$ where $f \in k(X)$, then $\nu_C(\omega) := \nu_C(f)$. Equivalently, (ω) is the divisor of zeroes and poles of ω regarded as a rational section of the canonical line bundle ω_X). Also $K_{\tilde{X}} = (\pi^*\omega) = (d(\pi^*x) \wedge d(\pi^*y))$. (If $f\colon X \to Y$ is a morphism of smooth varieties and ω is a k-form on Y then the *pullback* $f^*\omega$ of ω is defined in the obvious way: locally on Y, $\omega = \sum g_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $x_1, \ldots, x_n$ are local coordinates and $f^*\omega := \sum f^*g_{i_1 \cdots i_k} d(f^*x_{i_1}) \wedge \cdots \wedge d(f^*x_{i_k})$, where $f^*g = g \circ f$ denotes pullback of functions.)

We need to compare $\pi^*K_X = \pi^*(\omega)$ and $K_{\tilde{X}} = (\pi^*\omega)$. Clearly these divisors coincide over $\tilde{X} \setminus E$ (because π restricts to an isomorphism $\tilde{X} \setminus E \simeq X \setminus \{P\}$.) The divisor $\pi^*(\omega)$ does not contain $E = \pi^{-1}P$ because ω is regular and nonzero at P. We compute the coefficent of E in $(\pi^*\omega)$ by a local calculation: one chart of the blowup is

$$\mathbb{A}^2_{u,y'} \to \mathbb{A}^2_{x,y}, \quad (u, y') \mapsto (x, y) = (u, uy').$$

So, on this chart,

$$\pi^*\omega = d(\pi^*x) \wedge d(\pi^*y) = du \wedge d(uy') = du \wedge (y'du + udy') = udu \wedge dy'.$$

Thus $(\pi^*\omega) = (u = 0) = E$ in this chart. Combining, we deduce that $K_{\tilde{X}} = \pi^*K_X + E$. ☐

2 Elimination of indeterminacy of rational maps

Let X and Y be varieties and $f\colon X \dashrightarrow Y$ a rational map. (Recall that a *rational map* $f\colon X \to Y$ is a morphism $f\colon U \to Y$ from a Zariski open subset $U \subset X$, and we regard two rational maps $f_1, f_2\colon X \dashrightarrow Y$ as equivalent if they agree on $U_1 \cap U_2$.) Assume that X is smooth. Then f restricts to a morphism $f\colon U \to Y$ where $U = X \setminus Z$ and $Z \subset X$ is closed of codimension ≥ 2. To see this, assume (for simplicity) Y is projective, and write $Y \subset \mathbb{P}^N$, where Y is not contained in a hyperplane. Then $f\colon X \dashrightarrow Y \subset \mathbb{P}^N$ is given by $f = (f_0\colon \cdots f_N)$ where $f_i \in k(X)^\times$ are nonzero rational functions on X. In particular, f is well defined at $P \in X$ if each f_i is regular at P and some f_j is nonzero at P. Moreover, the $(N+1)$-tuples $(f_0\colon \cdots : f_N)$ and $(gf_0\colon \cdots : gf_N)$ define the same rational map for any $g \in k(X)^\times$. So, locally near a given point $P \in X$, we can clear denominators so that each f_i is regular, and cancel common factors so that the locus of common zeroes has codimension ≥ 2. Thus f is well defined outside a locus $Z \subset X$ of codimension ≥ 2 as claimed. In particular, if X is a smooth surface, a rational map $f\colon X \dashrightarrow Y$ is well defined outside a finite set of points $Z \subset X$.

Recall that for X a smooth variety and D a divisor on X the *complete linear system* $|D|$ on X is the set of effective divisors D' linearly equivalent to D. It is identified with the projectivisation $\mathbb{P}\Gamma(\mathcal{O}_X(D))$ of the space of global sections s of the associated line bundle $\mathcal{O}_X(D)$ via $s \mapsto D' = (s = 0)$. A *linear system* is a projective subspace of a complete linear system. We say an irreducible divisor $F \subset X$ is a *fixed component* of a linear system δ if every $D \in \delta$ contains F.

A rational map $f\colon X \dashrightarrow Y \subset \mathbb{P}^N$ as above corresponds to the linear system δ on X without fixed components given by

$$\delta := \{f^*H \mid H \subset \mathbb{P}^N \text{a hyperplane}\}.$$

Note that f^*H makes sense because f is well defined outside a codimension 2 locus $Z \subset X$. Explicitly, write $f = (f_0\colon \cdots : f_N)$, $D_i = (f_i)$, and $D_i' = D_i - \min_j D_j$. (Here by $\min_j D_j$ we mean: write $D_j = \sum_k n_{jk} Y_k$, then $\min_j D_j := \sum_k (\min_j n_{jk}) Y_k$.) Then $D_i' = f^*(X_i = 0)$, and δ is the linear system generated (as a projective space) by the D_i'. Equivalently, in terms of line bundles, let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^N(1)}$ be the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$, and $s_i = f^*X_i \in \Gamma(X, \mathcal{L})$ the global sections of $\mathcal{L}$ given by the pullback of the global sections X_i of $\mathcal{O}_{\mathbb{P}^N}(1)$. Then $f = (s_0\colon \cdots : s_N)$, and $D_i' = (s_i = 0)$.

The *base locus* of δ is

$$\operatorname{Bs}\delta := \{P \in X \mid P \in D \text{ for all } D \in \delta\} \subset X,$$

It is the locus $Z \subset X$ where f is undefined. In the above notation

$$Z = \operatorname{Bs}\delta = \bigcap_{i=0}^{N} D_i' = (s_0 = \cdots = s_N = 0) \subset X.$$

Proposition 2.1. *Let X be a smooth surface and $f\colon X \dashrightarrow Y$ a rational map to a variety Y. Then there exists a sequence of blowups*

$$W = X_n \xrightarrow{\pi_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

such that the induced map $g = f \circ \pi_1 \circ \cdots \pi_n$ is a morphism. That is, we have a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

where $p = \pi_1 \circ \cdots \circ \pi_n$ is the composite of a sequence of blowups and g is a morphism.

Proof. Let δ be the linear system defining the rational map f and suppose $P \in X$ is a basepoint of δ. Let $D \in \delta$ be a general element. Let x, y be local coordinates at P and write $D = (f(x,y) = 0)$ near P. Let

$$f(x,y) = f_m(x,y) + f_{m+1}(x,y) + \cdots$$

where $f_k(x,y)$ is homogeneous of degree k. So m is the *multiplicity* of D at $P \in X$. We compute that $\pi^*D = D' + mE$ where D' is the strict transform of D. (Recall that the *strict transform* D' of D is defined as follows: D' is the closure in $\tilde{X}$ of the inverse image of the restriction of D to $X \setminus \{P\}$ under the isomorphism $\tilde{X} \setminus E \simeq X \setminus \{P\}$.) We use the chart

$$\mathbb{A}^2_{u,y'} \to \mathbb{A}^2_{x,y}, \quad (u, y') \mapsto (x,y) = (u, uy').$$

In this chart, $\pi^*D = (\pi^*f) = (f(u, uy') = 0)$, and

$$f(u, uy') = (f_m(1, y') + uf_{m+1}(u, y') + \cdots)u^m,$$

so $\pi^*D = D' + mE$ as claimed. The composite rational map $\tilde{f} = f \circ \pi$ is defined by the linear system

$$\tilde{\delta} = \{\tilde{D} := \pi^*D - mE \mid D \in \delta\}.$$

Note that

$$\tilde{D}^2 = (\pi^*D - mE)^2 = D^2 - m < D^2.$$

(Here we used $E^2 = -1$, and $\pi^*A \cdot E = 0$, $\pi^*A \cdot \pi^*B = A \cdot B$ for A, B divisors on X.) If $\tilde{f}$ is not a morphism we repeat this process.

If δ is a linear system without fixed components then $D^2 \geq 0$ for $D \in \delta$ (because there exist effective divisors $D_1, D_2 \in \delta$ with no common components, so $D^2 = D_1 \cdot D_2 \geq 0$). Hence the above procedure stops after a finite number of blowups. ☐

Example 2.2. Consider the rational map

$$f \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1, \quad (X_0 : X_1 : X_2) \mapsto (X_0 : X_1).$$

The map f corresponds to the linear system

$$\delta = \{(a_0X_0 + a_1X_1 = 0) \mid (a_0 : a_1) \in \mathbb{P}^1\}$$

on $\mathbb{P}^2$ which has a single basepoint $P = (0 : 0 : 1) \in \mathbb{P}^2$. Let $\pi \colon \tilde{X} \to \mathbb{P}^2$ be the blowup of $P \in \mathbb{P}^2$. Then the composite $\tilde{f} := f \circ \pi \colon \tilde{X} \to \mathbb{P}^1$ is a morphism.