

# 583C Lecture notes

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## 1 The Hodge index theorem (continued)

Let  $X$  be a smooth projective surface over  $k = \mathbb{C}$ . We prove the first statement in the Hodge index theorem: the wedge product is positive definite on  $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$ . (The proof of the second statement, that the signature of  $H_{\mathbb{R}}^{1,1}$  is  $(1, h^{1,1} - 1)$ , is more involved.) Let  $0 \neq [\omega] \in (H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$ . Then  $\omega = \alpha + \bar{\alpha}$  where  $\alpha$  is a  $(2, 0)$ -form, and

$$\omega^2 = \omega \wedge \omega = 2\alpha \wedge \bar{\alpha}$$

(Note  $\alpha^2 = \bar{\alpha}^2 = 0$  since  $\alpha$  is a  $(2, 0)$ -form.) Locally on  $X$ , write  $\alpha = f dz_1 \wedge dz_2$  where  $f$  is a smooth  $\mathbb{C}$ -valued function and  $z_1, z_2$  are complex coordinates on  $X$ . Then

$$\begin{aligned} \omega^2 &= 2\alpha \wedge \bar{\alpha} = 2|f|^2 dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \\ &= -2|f|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = -2|f|^2 (-2i)^2 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \\ &= 8|f|^2 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2. \end{aligned}$$

Here we wrote  $z_j = x_j + iy_j$  and used the identity

$$dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy.$$

We deduce that  $[\omega]^2 = \int_X \omega^2 > 0$  as required.

We now describe the algebraic consequences of the Hodge index theorem. Let  $X$  be a smooth projective surface over  $k = \mathbb{C}$  as above. We say two divisors  $D, D'$  on  $X$  are *numerically equivalent* and write  $D \equiv D'$  if  $D \cdot C = D' \cdot C$  for every curve  $C \subset X$ . We write  $\text{Num } X$  for the (free) abelian group of divisors modulo numerical equivalence, and  $\rho(X)$  for the rank of  $\text{Num } X$ .

**Corollary 1.1.**  $\text{Num } X = H^{1,1} \cap (H^2(X, \mathbb{Z})/\text{Tors})$  and the intersection product on  $\text{Num } X$  has signature  $(1, \rho - 1)$ .

*Proof.* Recall that the intersection product on  $\text{Cl}(X) = \text{Pic}(X)$  is induced from the wedge product via the first Chern class

$$c_1: \text{Pic } X \rightarrow H^2(X, \mathbb{Z}),$$

and the image of  $c_1$  is  $H^{1,1} \cap H^2(X, \mathbb{Z})$ . Let  $H$  be a hyperplane section of  $X$  in some embedding in projective space. Then  $H^2 > 0$ , so the wedge product on the orthogonal complement  $H^\perp \subset H_{\mathbb{R}}^{1,1}$  is negative definite by the Hodge index theorem. In particular, the form is negative definite on  $H^\perp \subset H^{1,1} \cap (H^2(X, \mathbb{Z})/\text{Tors})$ , and so is nondegenerate of hyperbolic signature  $(1, r-1)$  on  $H^{1,1} \cap (H^2(X, \mathbb{Z})/\text{Tors})$ . We deduce that  $\text{Num } X = H^{1,1} \cap (H^2(X, \mathbb{Z})/\text{Tors})$ , of signature  $(1, \rho-1)$ , as required.  $\square$

We give an algebraic proof that  $\text{Num } X$  has hyperbolic signature  $(1, \rho-1)$  using the Riemann–Roch formula. Let  $H$  be a hyperplane section of  $X$ . Then  $H^2 > 0$ . If the signature is not hyperbolic, there exists a divisor  $D$  such that  $D \cdot H = 0$  and  $D^2 > 0$ . (Note that the intersection product on  $\text{Num } X$  is nondegenerate by construction.) We show that such a divisor cannot exist. Consider the Riemann–Roch formula for  $nD$ ,  $n \in \mathbb{Z}$ :

$$h^0(\mathcal{O}_X(nD)) + h^0(\mathcal{O}_X(K_X - nD)) \geq \chi(\mathcal{O}_X(nD)) = \chi(\mathcal{O}_X) + \frac{1}{2}nD(nD - K_X).$$

In particular, as  $n \rightarrow \pm\infty$ ,  $\chi(\mathcal{O}_X(nD)) \sim \frac{1}{2}n^2D^2 \rightarrow \infty$ . We claim that  $h^0(\mathcal{O}_X(nD)) = 0$  for all  $n \in \mathbb{Z}$ . Indeed if  $0 \neq s \in H^0(\mathcal{O}_X(nD))$ , then  $D' = (s=0)$  is an effective divisor linearly equivalent to  $D$ . But then  $D' \cdot H = D \cdot H = 0$ , and  $H$  is a hyperplane section, so  $D' = 0$ . This contradicts  $D^2 > 0$ . We deduce that  $h^0(\mathcal{O}_X(K_X - nD)) \rightarrow \infty$  as  $n \rightarrow \pm\infty$ . For  $n \gg 0$ , pick  $0 \neq s \in H^0(K_X - nD)$ , then tensor product with  $s$  defines an inclusion  $H^0(K_X + nD) \subset H^0(2K_X)$ . So  $h^0(K_X + nD) \leq h^0(2K_X)$  for  $n \gg 0$ , a contradiction.

## 2 Birational geometry of surfaces

Let  $X$  be a smooth complex surface and  $P \in X$  a point. The *blowup* of  $P \in X$  is a birational morphism  $\pi: \tilde{X} \rightarrow X$  such that  $E := \pi^{-1}P$  is a copy of  $\mathbb{P}^1$ , the *exceptional curve*, and  $\pi$  restricts to an isomorphism

$$\tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{P\}.$$

If  $x, y$  are local coordinates at  $P \in X$ , then, working locally analytically at  $P \in X$ , we can identify  $P \in X$  with  $0 \in \mathbb{A}_{x,y}^2$ . Then the blowup  $\pi: \tilde{X} \rightarrow X$

is given by two charts as follows:

$$\mathbb{A}_{u,y'}^2 \rightarrow \mathbb{A}_{x,y}^2 \quad (u, y') \mapsto (x, y) = (u, uy')$$

$$\mathbb{A}_{x',v}^2 \rightarrow \mathbb{A}_{x,y}^2 \quad (x', v) \mapsto (x, y) = (vx', v)$$

Here the exceptional curve  $E$  is given by  $(u = 0)$  and  $(v = 0)$  in the two charts. We observed earlier that  $E^2 = -1$ .

**Proposition 2.1.** *We have an isomorphism*

$$\mathrm{Pic}(X) \oplus \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic} \tilde{X}, \quad (D, n) \mapsto \pi^* D + nE.$$

*Proof.* By the definition of the divisor class group, we have an exact sequence

$$\mathbb{Z}E \rightarrow \mathrm{Cl}(\tilde{X}) \rightarrow \mathrm{Cl}(\tilde{X} \setminus E) \rightarrow 0$$

where the second arrow is given by restriction to the open set  $\tilde{X} \setminus E \subset \tilde{X}$ . (See [Hartshorne, p. 133, II.6.5] for more details). Now  $\tilde{X} \setminus E \simeq X \setminus \{P\}$  and  $\mathrm{Cl}(X \setminus \{P\}) = \mathrm{Cl}(X)$  because  $P \in X$  has codimension 2. So we obtain an exact sequence

$$\mathbb{Z}E \rightarrow \mathrm{Pic}(\tilde{X}) \rightarrow \mathrm{Pic}(X) \rightarrow 0.$$

It remains to show that the first arrow is injective. This follows from  $E^2 = -1$ .  $\square$

Next we describe the blowup topologically. Let  $P \in B \subset X$  be a ball around  $P \in X$  (in the Euclidean topology) and  $N = \pi^{-1}B$ , a tubular neighbourhood of  $E \subset \tilde{X}$ . We consider the Mayer–Vietoris sequence in (integral) cohomology for  $\tilde{X} = (\tilde{X} \setminus E) \cup N$  and  $X = (X \setminus \{P\}) \cup B$ . Recall that the Mayer–Vietoris sequence for a union  $K \cup L$  is

$$\dots \rightarrow H^i(K \cup L) \rightarrow H^i(K) \oplus H^i(L) \rightarrow H^i(K \cap L) \rightarrow H^{i+1}(K \cup L) \rightarrow \dots$$

In our case we obtain a commutative diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & H^i(\tilde{X}) & \rightarrow & H^i(\tilde{X} \setminus E) \oplus H^i(N) & \rightarrow & H^i(\partial N) \rightarrow H^{i+1}(\tilde{X}) \rightarrow \dots \\ & & \uparrow & & \uparrow & & \uparrow & \uparrow \\ \dots & \rightarrow & H^i(X) & \rightarrow & H^i(X \setminus \{P\}) \oplus H^i(B) & \rightarrow & H^i(\partial B) \rightarrow H^{i+1}(X) \rightarrow \dots \end{array}$$

Now  $B$  is contractible, so  $H^i(B) = H^i(\mathrm{pt})$ , and  $N$  is homotopy equivalent to  $E \simeq \mathbb{P}^1$ , so  $H^i(N) \simeq H^i(\mathbb{P}^1)$ . Also  $\tilde{X} \setminus E \simeq X \setminus \{P\}$  and  $\partial N \simeq \partial B$ . We deduce that  $H^2(\tilde{X}, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \oplus \mathbb{Z}$  and  $H^i(\tilde{X}, \mathbb{Z}) \simeq H^i(X, \mathbb{Z})$  for  $i \neq 2$ . Note that the isomorphism  $H^2(\tilde{X}, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \oplus \mathbb{Z}$  is compatible with the isomorphism  $\mathrm{Pic}(\tilde{X}) \simeq \mathrm{Pic} X \oplus \mathbb{Z}$  obtained above under  $c_1$ .

## References

[Hartshorne] R. Hartshorne, Algebraic geometry.