583C Lecture notes

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1 Topology (continued)

1.1 Poincaré duality and the universal coefficient theorem

Theorem 1.1. *(Poincaré duality I) [GH, p. 53] Let X be a compact oriented smooth manifold of dimension d. Then the intersection product*

$$\cap\colon H_i(X,\mathbb{Z}) \times H_{d-i}(X,\mathbb{Z}) \to \mathbb{Z}$$

is unimodular, that is, the induced map

$$H_i(X,\mathbb{Z})/\operatorname{Tors} \to H_{d-i}(X,\mathbb{Z})^* \quad \alpha \mapsto (\alpha \cap \cdot)$$

is an isomorphism.

Recall that, if L, M are abelian groups and

$$b\colon L \times M \to \mathbb{Z}$$

is a bilinear pairing, we say b is *unimodular* if the induced map

$$L/\operatorname{Tors} \to M^* := \operatorname{Hom}(M,\mathbb{Z}), \quad l \mapsto b(l,\cdot)$$

is an isomorphism. Equivalently, if we pick bases for L/Tors and M/Tors, the matrix of b with respect to these bases has determinant ± 1.

Theorem 1.2. *(Universal coefficient theorem)[Hatcher, p. 195, Thm. 3.2] Let X be a topological space. Then there are natural exact sequences*

$$0 \to \operatorname{Ext}^1(H_{i-1}(X,\mathbb{Z}),\mathbb{Z}) \to H^i(X,\mathbb{Z}) \to H_i(X,\mathbb{Z})^* \to 0.$$

In particular

$$H^i(X,\mathbb{Z})/\operatorname{Tors} \simeq H_i(X,\mathbb{Z})^*$$

and

$$\operatorname{Tors} H^i \simeq \operatorname{Tors} H_{i-1}$$

(this last isomorphism is not canonical).

Note: If you do not know what Ext^1 is, you can ignore the first statement in the theorem.

Proof. (Sketch) Recall that the homology $H_i(X, \mathbb{Z})$ is the homology of the complex of chains

$$\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots$$

for some triangulation of X, and the cohomology $H^i(X, \mathbb{Z})$ is the cohomology of the dual complex

$$\cdots \leftarrow C^*_{i+1} \leftarrow C^*_i \leftarrow C^*_{i-1} \leftarrow \cdots$$

Observe that there is a natural map $H^i(X, \mathbb{Z}) \to H_i(X, \mathbb{Z})^*$. Recall that each C_i is a free abelian group (generated by the simplices of dimension i). One shows that the complex $(C., d)$ splits as a direct sum of shifts of complexes of the following types:

$$0 \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0, \quad 1 \mapsto n$$

Dualising these complexes, we deduce that $H^i(X, \mathbb{Z})/\operatorname{Tors} \simeq H_i(X, \mathbb{Z})^*$ and $\operatorname{Tors} H^i(X, \mathbb{Z}) \simeq \operatorname{Tors} H_{i-1}(X, \mathbb{Z})$. The exact sequence in the statement is obtained by being careful about naturality. □

We can now give a slightly stronger form of Poincaré duality.

Theorem 1.3. *(Poincaré duality II) [GH, p. 53] Notation as in 1.1. There is a natural isomorphism*

$$H_i(X, \mathbb{Z}) \simeq H^{d-i}(X, \mathbb{Z})$$

which induces the isomorphism $H_i(X, \mathbb{Z})/\operatorname{Tors} \simeq H_{d-i}(X, \mathbb{Z})^*$ given by the intersection product. In particular, $\operatorname{Tors} H_i(X, \mathbb{Z}) \simeq \operatorname{Tors} H^{d-i}(X, \mathbb{Z})$.

Proof of 1.1, 1.3. (Sketch) Fix a triangulation $X \simeq |\Sigma|$. Consider the dual complex Σ' of Σ. This is a cell complex (not necessarily a simplicial complex) with support $|\Sigma'| = |\Sigma|$, and cells of dimension $d - i$ in bijection with simplices of Σ of dimension i. It is constructed as follows: consider the barycentric subdivision $\hat{\Sigma}$ of $|\Sigma|$. For $v \in \Sigma$ a vertex, the corresponding d-cell $v' \in \Sigma'$ is the union of all the simplices in $\hat{\Sigma}$ which contain v. For $\sigma \in \Sigma$ an i-simplex, the corresponding $(d-i)$-cell $\sigma' \in \Sigma'$ is the intersection of the d-cells v' corresponding to the vertices v of σ. (Please draw a picture for $d = 2$). We observe that, for an i-simplex $\sigma \in \Sigma$, the corresponding cell $\sigma' \in \Sigma'$ is the unique $(d-i)$-cell of Σ' meeting σ, and intersects it transversely in one point. Now let $(C.(\Sigma, \mathbb{Z}), d)$ be the chain complex for the triangulation $X \simeq |\Sigma|$ and $(C.(\Sigma', \mathbb{Z}), d)$ the chain complex for the cellular subdivision $X \simeq |\Sigma'|$. (Note that we can use cellular subdivisions to compute homology exactly as for triangulations.) One shows that the maps

$$C_i(\Sigma, \mathbb{Z}) \xrightarrow{\sim} C^{d-i}(\Sigma', \mathbb{Z}) = C_{d-i}(\Sigma', \mathbb{Z})^*, \quad \sigma \mapsto (\sigma')^*.$$

commute (up to sign) with the differentials d. Here $C_j(\Sigma', \mathbb{Z})$ is the free abelian group with basis given by the j-cells τ of Σ' and for such a τ we write τ^* for the corresponding element of the dual basis of $C^j(\Sigma', \mathbb{Z}) = C_j(\Sigma', \mathbb{Z})^*$. Passing to homology we obtain the isomorphism $H_i(X, \mathbb{Z}) \simeq H^{d-i}(X, \mathbb{Z})$. □

1.2 Topological invariants of surfaces

Let X be a compact oriented smooth 4-manifold. Then $H_0(X, \mathbb{Z}) = \mathbb{Z}$, $H_4(X, \mathbb{Z}) = \mathbb{Z}$, and $H_1(X, \mathbb{Z}) = \pi_1(X, x)^{\mathrm{ab}}$. The intersection form

$$Q := \cap \colon H_2(X, \mathbb{Z}) / \operatorname{Tors} \times H_2(X, \mathbb{Z}) / \operatorname{Tors} \to \mathbb{Z}$$

is symmetric and unimodular. We also have $\operatorname{Tors} H_2 \simeq \operatorname{Tors} H_1$, $\operatorname{Tors} H_3 = 0$, and $H_3(X, \mathbb{Z}) \simeq H_1(X, \mathbb{Z})^*$ by Poincaré duality and the universal coefficient theorem.

So, to recap, the topological invariants are the fundamental group $\pi_1(X, x)$ and the intersection form Q on $H_2(X, \mathbb{Z}) / \operatorname{Tors}$.

Remark 1.4. Assume X is a smooth projective complex surface. We can usually reduce to the simply connected case ($\pi_1(X, x) = 0$) as follows. If $\pi_1(X, x)$ is finite, let $p \colon \tilde{X} \to X$ be the universal cover of X. Then $\tilde{X}$ inherits the structure of a smooth complex surface from X, and $\tilde{X}$ is projective because p is a finite morphism. So X is the quotient of the smooth projective surface $\tilde{X}$ by the free action of the finite group $\pi_1(X, x)$. If $H_1(X, \mathbb{Z}) = \pi_1(X, x)^{\mathrm{ab}}$ is infinite, then the Albanese morphism is a non-trivial morphism from X to a complex torus, and we can use this to study X (more on this later).

1.3 Results of Freedman and Donaldson

We state without proof two results about the classification of smooth 4 manifolds. See [BHPV, Ch. IX] for more details (note: unfortunately, this material is only contained in the 2nd edition).

Theorem 1.5. *(Freedman '82) A simply connected compact oriented* 4*-manifold is determined up to (oriented) homeomorphism by its intersection form* Q*.*

Theorem 1.6. *(Donaldson '83) There exist infinitely many smooth complex projective surfaces which are homeomorphic but not diffeomorphic.*

1.4 Classification of quadratic forms

Let L be a free abelian group of finite rank. Let $Q\colon L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form.

We can pick a basis of the $\mathbb{R}$-vector space $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ such that the matrix of Q with respect to this basis is diagonal with diagonal entries n_+ 1's and n_- (-1)'s. The pair (n_+, n_-) is the *signature* of Q.

We say Q is *positive definite* if $Q(x,x) > 0$ for all $x \neq 0$, *negative definite* if $Q(x,x) < 0$ for all $x \neq 0$, and *indefinite* otherwise. In terms of the signature, Q is positive definite if $n_- = 0$, negative definite if $n_+ = 0$, and indefinite otherwise.

We say Q is *even* if $Q(x,x)$ is even for all $x \in L$, and Q is *odd* otherwise.

Theorem 1.7. *[Serre, Ch. V] An indefinite unimodular quadratic form is determined up to isomorphism by its signature and parity. (The same holds for definite forms if the rank is* ≤ 8*).*

The quadratic forms as in the theorem can be described explicitly as follows. If Q is odd, Q is of type

$$(1)^{n_+} \oplus (-1)^{n_-}.$$

(That is, with respect to some basis of L, the quadratic form Q has matrix the diagonal matrix with diagonal entries n_+ 1's and n_- (-1)'s.) If Q is even, Q is of type

$$H^a \oplus (\pm E_8)^b$$

for some $a > 0$ and $b \geq 0$, where H is the *hyperbolic plane* with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and E_8 is the positive definite quadratic form of rank 8 with matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Remark 1.8. The classification of definite quadratic forms is much more involved. However, if X is a smooth projective complex surface with definite quadratic form Q, then $H_2(X, \mathbb{Z})$ has rank 1 and $Q = (1)$ (more details later).

References

[BHPV] Barth, Hulek, Peters, Van de Ven, Compact complex surfaces, 2nd ed.

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry.

[Hatcher] A. Hatcher, Algebraic topology, available at www.math.cornell.edu/~hatcher/AT/ATpage.html

[Serre] J-P. Serre, A Course in arithmetic.