# 583C Lecture notes

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## 1 The intersection product on the Picard group

Let $X$ be a smooth projective surface over $k = \mathbb{C}$. Recall that the first Chern class
$$c_1 \colon \operatorname{Pic} X \to H^2(X, \mathbb{Z})$$
is identified via Poincaré duality with the map

$$\operatorname{Cl} X \to H_2(X, \mathbb{Z}), \quad D \mapsto [D]$$

from the divisor class group to homology given by regarding a divisor as a 2-cycle. We have the (topological) intersection product

$$\cap \colon H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$$

and we define the intersection product on $\operatorname{Pic} X = \operatorname{Cl} X$ as the induced product,
$$D_1 \cdot D_2 := [D_1] \cap [D_2].$$

Algebraically, if $C_1, C_2 \subset X$ are irreducible curves,

$$C_1 \cdot C_2 = \sum_{P \in C_1 \cap C_2} (C_1 \cdot C_2)_P$$

where the *intersection multiplicity* $(C_1 \cdot C_2)_P$ of $C_1, C_2$ at $P \in X$ is defined as follows: locally at $P \in X$ write $C_i = (f_i = 0) \subset X$, then

$$(C_1 \cdot C_2)_P := \dim_k \mathcal{O}_{X,P}/(f_1, f_2) \tag{1}$$

One can check this agrees with the topological intersection product by a $C^\infty$ perturbation argument, see [GH, p. 62].

*Remark* 1.1. If $X$ is a complex manifold and $Z, W \subset X$ are complex submanifolds meeting transversely, then, since the orientations of $Z, W, X$ are induced by the complex structure, at each point $P \in Z \cap W$ the intersection index $i_P(Z, W) = +1$. That is, there are no signs in the topological intersection product $[Z] \cap [W]$. (This easy observation is extremely important in algebraic combinatorics.)

*Remark* 1.2. If $X$ is defined over an arbitrary algebraically closed field $k$ one can define the intersection product using the equation (1). One then needs to show that it is well defined modulo linear equivalence.

Suppose $C \subset X$ is an irreducible curve. We describe two ways to compute the self-intersection $C^2$. First, we can find a rational function $f$ such that the principal divisor $(f)$ contains $C$ with multiplicity 1 (just take $f$ a local equation of $C$ at a point of $X$). Then $D = C - (f)$ is linearly equivalent to $C$ and does not contain $C$ as a component, so $C^2 = D \cdot C$ and we can compute $D \cdot C$ as above. Alternatively, we have the general formula

$$D \cdot C = \deg \mathcal{L}|_C$$

where $\mathcal{L} = \mathcal{O}_X(D)$ is (the sheaf of sections of) the line bundle associated to $D$. Setting $D = C$ we obtain

$$C^2 = \deg \mathcal{O}_X(C)|_C = \deg \mathcal{N}_{C/X}$$

where

$$\mathcal{N}_{C/X} = \mathcal{O}_X(C)|_C \tag{2}$$

is the normal bundle of $C \subset X$.

We explain the equality (2). We assume $C \subset X$ is smooth for simplicity. Recall that the *normal bundle* $\mathcal{N}_{C/X}$ of $C \subset X$ is the line bundle defined by the exact sequence of vector bundles on $C$

$$0 \to T_C \to T_X|_C \to \mathcal{N}_{C/X} \to 0$$

where $T_C$, $T_X$ denote the tangent bundles of $C$, $X$. Dually,

$$0 \to \mathcal{N}^*_{C/X} \to \Omega_X|_C \to \Omega_C \to 0.$$

Locally, let $C = (x = 0) \subset X$, then $\mathcal{N}^*_{C/X}$ is generated by $dx$. We have an exact sequence of sheaves

$$0 \to \mathcal{I}_{C/X} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

where $\mathcal{I}_{C/X} \subset \mathcal{O}_X$ is the ideal sheaf of regular functions vanishing on $C$. There is a natural isomorphism

$$\mathcal{I}_{C/X}|_C \xrightarrow{\sim} \mathcal{N}^*_{C/X}$$

given locally by $x \mapsto dx$. Finally, observe that $\mathcal{I}_{C/X} = \mathcal{O}_X(-C)$ — a section of $\mathcal{O}_X(-C)$ over $U \subset X$ is a rational function $f$ such that $(f) - C \geq 0$ on $U$, equivalently, $f$ is regular on $U$ and vanishes on $C$. So $\mathcal{O}_X(-C)|_C \simeq \mathcal{N}^*_{C/X}$, and dualising gives (2).

For $X$ a smooth variety, the *canonical line bundle* $\omega_X = \wedge^{\dim X} \Omega_X$ is the top exterior power of the cotangent bundle. The *canonical divisor class* $K_X$ is the associated divisor class. Now let $X$ be a smooth projective surface and $C \subset X$ a smooth curve. Then we have the *adjunction formula*

$$K_C = (K_X + C)|_C. \tag{3}$$

Taking degrees,

$$2g - 2 = \deg K_C = (K_X + C) \cdot C$$

where $g$ is the genus of $C$. The adjunction formula is deduced from the exact sequence

$$0 \to \mathcal{N}^*_{C/X} \to \Omega_X|_C \to \Omega_C \to 0 \tag{4}$$

as follows. If

$$0 \to U \to V \to W \to 0$$

is an exact sequence of vector spaces of dimensions $r$, $s$, $t$, we have a natural isomorphism

$$\wedge^s V \simeq \wedge^r U \otimes \wedge^t W.$$

This induces a corresponding isomorphism for an exact sequence of vector bundles. In particular, from (4) we obtain

$$\wedge^2 \Omega_X|_C \simeq \mathcal{N}^*_{C/X} \otimes \Omega_C.$$

Rearranging and using $\mathcal{N}_{C/X} = \mathcal{O}_X(C)|_C$ gives

$$\omega_C = \omega_X \otimes \mathcal{O}_X(C)|_C$$

and passing to the associated divisors we obtain the adjunction formula (3).

*Example* 1.3. Let $X$ be a smooth surface, $P \in X$ a point, and $\pi\colon \tilde{X} \to X$ the blowup of $P \in X$. So $\pi^{-1}P = E$ is a copy of $\mathbb{P}^1$, the *exceptional curve*, and $\pi$ restricts to an isomorphism

$$\pi\colon \tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{P\}.$$

We compute that the self intersection $E^2 = -1$. Let $D \subset X$ be a smooth curve through $P$. Let $D' \subset \tilde{X}$ be the *strict transform* of $D$, that is, $D'$ is the closure of the preimage of $D \setminus \{P\}$. Then $D'$ is a smooth curve which intersects $E$ transversely in one point. In particular, $D' \cdot E = 1$. Now consider the pullback $\pi^*D$ of $D$. (In general, if $f\colon X \to Y$ is a morphism of smooth varieties and $D$ is a divisor on $Y$, then locally on $Y$ the divisor $D$ is principal, say $D|_{U_i} = (g_i)$ for some open cover $\mathcal{U} = \{U_i\}$, and we define $f^*D$ by $f^*D|_{f^{-1}U_i} = (g_i \circ f)$.) Let $x, y$ be local coordinates at $P \in X$ such that $D = (y = 0)$. Over $P \in X$ we have a chart of the blowup $\pi$ of the form

$$\mathbb{A}^2_{u,y'} \to \mathbb{A}^2_{x,y}, \quad (u, y') \mapsto (u, uy').$$

We deduce that $\pi^*D = D' + E$ (because in this chart $\pi^*D = \pi^*(y = 0) = (uy' = 0) = E + D'$). Now $\pi^*D \cdot E = 0$ because we can write $D \sim B$, where $B$ is a divisor not containing $P$, then $\pi^*D \cdot E = \pi^*B \cdot E = 0$ since $\pi^*B$ is disjoint from $E$. Combining we deduce that $E^2 = -1$ as claimed.

We make a few more comments about the self-intersection $C^2$ of a smooth curve $C \subset X$. If $C^2 < 0$ then $C$ cannot move in a family, that is, there does not exist a non-trivial family $\{C_t\}$ of curves with $C_0 = C$. Indeed, given such a family we have $C^2 = C \cdot C_t \geq 0$, a contradiction. Now suppose $C^2 \geq 0$ and $\{C_t\}$ is a family with $C_0 = C$. The family determines a section $s \in \Gamma(C, \mathcal{N}_{C/X})$ of the normal bundle of $C$ in $X$ as follows. Locally, write $C = (f = 0) \subset X$, and $C_t = (f + tg + \cdots = 0)$ where $\cdots$ denotes higher order terms in $t$. Then the section $s$ of $\mathcal{N}_{C/X} = \mathcal{I}^*_{C/X}|_C$ is locally given by $f \mapsto \bar{g}$, where $\bar{g} \in \mathcal{O}_C$ is the image of $g \in \mathcal{O}_X$. (So, $s$ corresponds to the *first order deformation of* $C$ given by the $\{C_t\}$.) This gives a geometric explanation for the formula $C^2 = \deg \mathcal{N}_{C/X}$ in this case — the zero locus of $s$ is approximately equal to $C \cap C_t$ for small $t$.

## References

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry.