583C Lecture notes

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1 Divisors and Line bundles

Let X be a smooth algebraic variety over $k = \mathbb{C}$.

A *divisor* on X is a finite formal $\mathbb{Z}$-linear combination of irreducible codimension one closed subvarieties

$$D = \sum n_i Y_i, \quad n_i \in \mathbb{Z}$$

We say D is *effective* and write $D \geq 0$ if $n_i \geq 0$ for all i. For $0 \neq f \in k(X)$ a nonzero rational function on X, the *principal divisor* associated to f is

$$(f) := \sum_{Y \subset X} \nu_Y(f) \cdot Y.$$

Here the sum is over codimension one subvarieties $Y \subset X$ and $\nu_Y(f)$ is the order of vanishing of f along Y. That is, locally at a general point $P \in Y$ we can write $Y = (g = 0)$ and $f = g^\nu h$, where h is regular at P and not divisible by g, and $\nu = \nu_Y(f) \in \mathbb{Z}$. So, (f) is the divisor of zeroes and poles of f counted with multiplicities.

A line bundle L over X is a morphism $p\colon L \to X$ with each fibre L_x a complex vector space of dimension 1, which is locally trivial in the following sense: there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, and local trivialisations

$$\phi_i\colon L|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}$$

compatible with the vector space structure on the fibres and the projection maps p, pr_1. Then

$$\phi_j \circ \phi_i^{-1}\colon U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}, \quad (x, v) \mapsto (x, g_{ij}(x) \cdot v),$$

where the *transition functions* $g_{ij}\colon U_{ij} \to \mathbb{C}^\times$ are nowhere zero regular functions, that is, $g_{ij} \in \mathcal{O}_X^\times(U_{ij})$. The (g_{ij}) define a Cech cocycle in $C^1(\mathcal{U}, \mathcal{O}_X^\times)$.

Indeed, we have $g_{jk}g_{ij} = g_{ik}$ so $(dg)_{ijk} = g_{jk}g_{ik}^{-1}g_{ij} = 1$. If we change the trivialisations ϕ_i by composing with multiplication on fibres by $f_i \in \mathcal{O}_X^\times(U_i)$, then (g_{ij}) is replaced by $(f_j g_{ij} f_i^{-1}) = (g_{ij} f_j f_i^{-1}) = g \cdot df$. Let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of line bundles. The set $\mathrm{Pic}(X)$ is an abelian group with group law (fibrewise) tensor product: $(L \otimes M)_x := L_x \otimes_{\mathbb{C}} M_x$. This corresponds to multiplication of transition functions. We deduce that

$$\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times).$$

Let $p\colon L \to X$ be a line bundle. We can consider the sheaf $\mathcal{L}$ of regular sections of L, that is, for $U \subset X$ open,

$$\mathcal{L}(U) = \{s \mid s\colon U \to L|_U \text{ regular}, \; p \circ s = \mathrm{id}_U\}.$$

In terms of local trivialisations,

$$\mathcal{L}(X) = \{(s_i) \mid s_i \in \mathcal{O}_X(U_i), \; s_j = g_{ij}s_j\}.$$

Conversely, given $\mathcal{L}$ we can reconstruct $p\colon L \to X$.

A *rational section* s of L is a section over some (Zariski) open subset U. In terms of local trivialisations, s is given by $s_i \in k(X)$ such that $s_j = g_{ij}s_i$, cf. above. For s a nonzero rational section we can define the divisor $D = (s) = \sum_{Y \subset X} \nu_Y(s) \cdot Y$ of zeroes and poles of s exactly as for rational functions, using local trivialisations of L. D is determined by L modulo prinicipal divisors (f) (because if s, t are two nonzero rational sections then $f = t/s$ is a rational function). We say D_1, D_2 are *linearly equivalent* if they differ by a principal divisor, write $\mathrm{Div}(X)$ for the group of divisors, and $\mathrm{Cl}(X)$ for the *divisor class group* of divisors modulo linear equivalence. Then the above construction defines an isomorphism

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X), \quad L \mapsto (s). \tag{1}$$

The inverse of this isomorphism can be described as follows. Given D a divisor, define a sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X) \mid f = 0 \text{ or } (D + (f))|_U \geq 0\}$$

Then $\mathcal{O}_X(D)$ is the sheaf of sections of a line bundle L, and $L \mapsto D$ under the isomorphism (1).

2 The Picard group

Let X be a smooth complex projective variety of dimension n. Let X^{an} denote the associated compact complex manifold. Consider the exact sequence of sheaves (the *exponential sequence*)

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}}^{\times} \to 0,$$

where the second arrow is given by $f \mapsto \exp(2\pi i f)$. The induced long exact sequence of cohomology gives

$$0 \to H^1(X, \mathbb{Z}) \to H^1(\mathcal{O}_X) \to \operatorname{Pic} X \to H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X) \to \cdots$$

Here we used the following facts:

(1) The sequence of global sections is

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^{\times} \to 0$$

(a global regular function on a projective variety is constant), in particular it is exact,

(2) $H^i(\mathcal{O}_{X^{\mathrm{an}}}) = H^i(\mathcal{O}_X)$ by GAGA [Serre56],

(3) $H^1(X, \mathcal{O}_{X^{\mathrm{an}}}^{\times}) = \operatorname{Pic} X^{\mathrm{an}}$ (we can define the Picard group for a complex manifold in the same way as above), and

(4) $\operatorname{Pic} X^{\mathrm{an}} = \operatorname{Pic} X$ by GAGA.

The map $\operatorname{Pic} X \to H^2(X, \mathbb{Z})$ in the exact sequence above is called the *first Chern class* and denoted c_1. We will describe it explicitly shortly.

The maps $H^i(X, \mathbb{Z}) \to H^i(\mathcal{O}_X)$ in the long exact sequence are the following composition (see [GH, p. 163]):

$$H^i(X, \mathbb{Z}) \to H^i(X, \mathbb{C}) \simeq H^i_{\mathrm{dR}}(X, \mathbb{C}) \twoheadrightarrow H^{0,i} \simeq H^i(\mathcal{O}_X).$$

Here the first map is given by extension of scalars from $\mathbb{Z}$ to $\mathbb{C}$, the second is the de Rham isomorphism, the third is the projection onto the factor $H^{0,i}$ of the Hodge decomposition of $H^i_{\mathrm{dR}}(X, \mathbb{C})$, and the fourth is the Dolbeault isomorphism of the Hodge summand $H^{0,i}$ with $H^i(\mathcal{O}_X)$ (stated earlier as part of Hodge decomposition theorem). In particular, we obtain an exact sequence

$$0 \to H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \to \operatorname{Pic} X \xrightarrow{c_1} H^{1,1} \cap H^2(X, \mathbb{Z}) \to 0$$

Indeed, by the long exact sequence and the above description of the map $H^2(X,\mathbb{Z}) \to H^2(\mathcal{O}_X)$, the image of $c_1\colon \operatorname{Pic}(X) \to H^2(X,\mathbb{Z})$ is the subgroup of classes $\omega \in H^2(X,\mathbb{Z})$ such that, writing $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$ using the de Rham isomorphism and Hodge decomposition, we have $\omega^{0,2} = 0$. But since ω is a real class, $\omega^{2,0} = \overline{\omega^{0,2}}$, so this is equivalent to $\omega \in H^{1,1}$. (This is the *Lefschetz theorem on* $(1,1)$*-classes*). The kernel of c_1 is denoted $\operatorname{Pic}^0(X)$ and called the *Picard variety.* It is a complex torus of dimension $q = h^1(\mathcal{O}_X)$. Indeed, we have $\operatorname{Pic}^0(X) = H^1(\mathcal{O}_X)/H^1(X,\mathbb{Z})$ by the exact sequence, so it remains to show that $H^1(X,\mathbb{Z}) \subset H^1(\mathcal{O}_X)$ is a lattice, that is, the map of $\mathbb{R}$-vector spaces

$$H^1(X,\mathbb{Z}) \otimes \mathbb{R} \to H^1(\mathcal{O}_X)$$

is an isomorphism. (Then the quotient $H^1(\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is diffeomorphic as a smooth manifold to the real torus $(S^1)^{2q}$.) This fact follows from the Hodge decomposition and the description of the map $H^1(X,\mathbb{Z}) \to H^1(\mathcal{O}_X)$ above: $H^1(\mathcal{O}_X)$ has complex dimension q, and $H^1(X,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$ has complex dimension $2q$, equivalently, $H^1(X,\mathbb{Z})$ has rank $2q$. So $H^1(X,\mathbb{Z}) \otimes \mathbb{R}$ and $H^1(\mathcal{O}_X)$ have the same real dimension $2q$, and it suffices to show that the above map is injective. If ω is in the kernel then $\omega = \omega^{1,0} + \omega^{0,1}$ where $\omega^{0,1} = 0$, and $\omega^{1,0} = \overline{\omega^{0,1}}$ because ω is a real class, so $\omega = 0$ as required.

We can describe the first Chern class $c_1\colon \operatorname{Pic} X \to H^2(X,\mathbb{Z})$ explicitly as follows. If L is a line bundle on X, let D be an element of the associated divisor class (the locus of zeroes and poles of a rational section of L). Then D defines a cycle $[D] \in H_{2n-2}(X,\mathbb{Z})$ of real codimension 2. Let

$$\mathrm{PD}\colon H_{2n-2}(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$$

denote the Poincaré duality isomorphism. Then $c_1(L) = \mathrm{PD}([D])$. See [GH, p. 141–143].

To recap, the Picard group $\operatorname{Pic}(X)$ is an extension

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \xrightarrow{c_1} H^{1,1} \cap H^2(X,\mathbb{Z}) \to 0$$

of the discrete group $H^{1,1} \cap H^2(X,\mathbb{Z})$ by the continuous group $\operatorname{Pic}^0(X)$. Moreover $\operatorname{Pic}^0(X) = H^1(\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is a complex torus of dimension q.

Example 2.1. If $H^1(\mathcal{O}_X) = 0$ then $\operatorname{Pic}(X)$ is discrete, and if in addition $H^2(\mathcal{O}_X) = 0$ then $\operatorname{Pic} X \simeq H^2(X,\mathbb{Z})$. This is the case for rational surfaces.

Example 2.2. For a K3 surface X we have $H^1(\mathcal{O}_X) = 0$ and $H^2(\mathcal{O}_X) \simeq \mathbb{C}$. Thus $\operatorname{Pic} X \simeq H^{1,1} \cap H^2(X,\mathbb{Z})$. The abelian group $H^2(X,\mathbb{Z})$ has rank 22, and $H^{1,1}$ has complex dimension 20. The complex subspace $H^{1,1} \subset$

$H^2(X,\mathbb{C})$ is preserved by complex conjugation, so corresponds to a real subspace $H^{1,1}_{\mathbb{R}} \subset H^2(X,\mathbb{R})$ of the same dimension. The intersection $\operatorname{Pic} X = H^{1,1}_{\mathbb{R}} \cap H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{R})$ has rank $0 \le \rho(X) \le 20$, and all these values occur. (Note however that the examples with $\rho(X) = 0$ are non algebraic complex manifolds.) A general projective K3 surface has $\operatorname{Pic} X \simeq \mathbb{Z}$ generated by an ample line bundle (that is, some multiple of the generator corresponds to the divisor class given by a hyperplane section in an embedding $X \subset \mathbb{P}^N$).

References

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry.

[Serre56] J-P. Serre, Géométrie algébrique et géométrie analytique, available at www.numdam.org