Name: KEY

Math 307G Final Exam

December 11, 2006

Instructions: There are ten problems, with the value of each problem indicated, for a total of 160 points. You are allowed the use of one page of handwritten notes, front and back, on standard size paper. You are also allowed use of a scientific calculator (but graphing calculators and other calculational devices are not allowed). The next page of the exam contains a table of Laplace transforms, which you may use in your solutions.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the **Box** provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

Problem 1	15 points	
Problem 2	15 points	
Problem 3	15 points	
Problem 4	15 points	
Problem 5	15 points	
Problem 6	20 points	
Problem 7	15 points	
Problem 8	15 points	
Problem 9	15 points	
Problem 10	20 points	
Total	160 points	

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

1. (15 points). Solve the initial value problem

$$y' = \frac{x+1}{x^2(2y+1)}, \quad y(1) = 0.$$

Your solution should give y *explicitly* as a function of x.

Answer: $\boxed{y = \frac{1}{2}\left[-1 + \sqrt{5 - \frac{4}{x} + 4\ln x}\right]}$

Separable equation: $\frac{dy}{dx} = \frac{1}{(2y+1)} \frac{x+1}{x^2}$

or $(2y+1)\,dy = \left(\frac{x+1}{x^2}\right)dx = \left(\frac{1}{x} + \frac{1}{x^2}\right)dx$

Integrating gives

$$y^2 + y = \ln x - \frac{1}{x} + C$$

$y(1) = 0 \Rightarrow \quad 0 = 0 - 1 + C \Rightarrow C = 1.$

$$y^2 + y = \ln x - \frac{1}{x} + 1$$

Solving for y using quadratic formula:

$$y = \frac{1}{2}\left[-1 \pm \sqrt{1 - 4\left(\frac{1}{x} - \ln x - 1\right)}\right]$$
$$= \frac{1}{2}\left[-1 \pm \sqrt{5 - \frac{4}{x} + 4\ln x}\right]$$

The initial condition shows we need to choose the "+" sign.

2. (15 points) Solve the initial value problem

$$(t^2+1)y' + (2t)y = t\,e^t, \quad y(0) = 2.$$

Answer: $\boxed{y = \frac{1}{t^2+1}\left[te^t - e^t + 3\right]}$

First order linear, so put into standard form:

$$y' + \underbrace{\left(\frac{2t}{t^2+1}\right)}_{p(t)} y = \underbrace{\frac{t}{t^2+1}\, e^t}_{g(t)}$$

The integrating factor is

$$\mu(t) = e^{\int \frac{2t}{t^2+1}dt} = e^{\int \frac{d(t^2+1)}{t^2+1}} = e^{\ln(t^2+1)} = t^2+1$$

So

$$\left[(t^2+1)y\right]' = te^t$$

Integrate te^t by parts, to get

$$(t^2+1)y = te^t - e^t + C_1$$

$y(0)=2 \Rightarrow$ $\quad 2 = 0 - 1 + C_1 \Rightarrow C_1 = 3$

So

$$y = \frac{1}{t^2+1}\left[te^t - e^t + 3\right]$$

3. (15 points) A population of bacteria increases at a rate proportional to the square root of the current population. At time $t = 0$ days the population is 100, and at time $t = 4$ the population is 900.

(a) Find a formula for the population $P(t)$ at time t days.

(b) At what time does the population reach 3600?

Answer: (a) $P(t) = (5t+10)^2$, (b) 10 days

(a) $\frac{dP}{dt} = k\sqrt{P}$, separable equation

$P^{-1/2}\,dP = k\,dt$, so $\int P^{-1/2}\,dP = \int k\,dt$, or

$$2\sqrt{P} = kt + C$$

$P(0) = 100 \Rightarrow 2\times 10 = C \Rightarrow C = 20$

$P(4) = 900 \Rightarrow 2\times 30 = 4k + 20 \Rightarrow k = 10$

$$2\sqrt{P(t)} = 10t + 20$$

so $\quad P(t) = (5t+10)^2$

(b) $3600 = P(t) = (5t+10)^2$

$\Rightarrow 60 = 5t + 10$

$\Rightarrow t = 10$ days

4. (15 points) Solve the initial value problem

$$y'' + 2y' + 5y = 0, \quad y(0) = 2,\ y'(0) = 4.$$

Answer: $\boxed{y(t) = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)}$

Char eqn: $r^2 + 2r + 5 = 0$

$\Rightarrow \quad r = \frac{1}{2}\left[-2 \pm \sqrt{4-20}\right]$

$\Rightarrow \quad r = -1 \pm 2i \qquad \lambda = -1,\ \mu = 2$

So. The general solution is

$$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

$$2 = y(0) = c_1 \Rightarrow c_1 = 2$$

$$y'(t) = 2\left[e^{-t}(-2\sin(2t)) - e^{-t}\cos(2t)\right] + c_2\left[e^{-t}\, 2\cos(2t) - e^{-t}\sin(2t)\right]$$

$$4 = y'(0) = -2 + 2c_2 \Rightarrow c_2 = 3$$

So

$$y(t) = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$$

5. (15 points) Find the general solution to

$$y'' + y' - 2y = e^t + \sin t$$

Answer: $\boxed{y = c_1 e^t + c_2 e^{-2t} + \frac{1}{3} t e^t - \frac{1}{10}\cos t - \frac{3}{10}\sin t}$

Char eqn: $r^2 + r - 2 = 0$

$(r+2)(r-1) = 0 \Rightarrow r = 1,\ r = -2$

So $y_c = c_1 e^t + c_2 e^{-2t}$

Use undetermined coefficients for $g_1(t) = e^t$ and $g_2(t) = \sin t$ separately.

$\underline{g_1(t)}$: e^t is a solution to homog eqn, so need to multiply by t: $y_{P_1} = Ate^t$, so

$y_{P_1}' = A(t+1)e^t$, $y_{P_1}'' = A(t+2)e^t$. Then

$$e^t = A(t+2)e^t + A(t+1)e^t - 2Ate^t$$

$$= 3Ae^t \Rightarrow A = \frac{1}{3}, \quad y_{P_1} = \frac{1}{3}te^t$$

$\underline{g_2(t)}$: Can use standard form

$y_{P_2} = B\cos t + C\sin t$, $y_{P_2}' = -B\sin t + C\cos t$

$y_{P_2}'' = -B\cos t - C\sin t$.

$$\Rightarrow \sin t = y_{P_2}'' + y_{P_2}' - 2y_{P_2} = [-B + C - 2B]\cos t + [-C - B - 2C] \cdot \sin t$$

$$\Rightarrow \begin{cases} -3B + C = 0 \\ -B - 3C = 1 \end{cases} \Rightarrow -10B = 1 \Rightarrow B = -\frac{1}{10}$$

$$\Rightarrow C = -\frac{3}{10}$$

6. (20 points) A 10 lb weight stretches a spring 2 ft. Suppose the weight is pulled down an additional foot and given a downward velocity of 2 ft/sec. There is no damping, nor are there external forces. Determine the amplitude of the subsequent motion.

Answer: $R = \frac{1}{2}\sqrt{5}$

$mg = kL \Rightarrow 10 \text{ lb} = k \cdot 2 \Rightarrow k = 5 \text{ lb/ft}$

$mg = 10 \Rightarrow m = \frac{10}{g} = \frac{10}{32}$

No damping $\Rightarrow \gamma = 0$, so eqn is

$$\frac{10}{32}u'' + 5u = 0, \quad u(0) = 1, \quad u'(0) = 2$$

[we use our standard convention that distance downward is positive]

$$u'' + 16u = 0, \qquad r^2 + 16 = 0 \Rightarrow r = \pm 4i$$

$$\Rightarrow u(t) = c_1 \cos(4t) + c_2 \sin(4t)$$

$1 = u(0) = c_1$

$$u'(t) = -4\sin(4t) + 4c_2\cos(4t)$$

$2 = u'(0) = 4c_2 \Rightarrow c_2 = \frac{1}{2}$

$$u(t) = \cos(4t) + \frac{1}{2}\sin(4t)$$

amplitude $= R = \sqrt{1^2 + (\frac{1}{2})^2} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$

7. (15 points) Let $f(t)$ be a function whose Laplace transform is $F(s)$. Define a new function

$$g(t) = e^{-2t} f(3t).$$

Determine the Laplace transform $G(s)$ of $g(t)$ in terms of $F(s)$.

Answer: $G(s) = \frac{1}{3} F\left(\frac{s+2}{3}\right)$

Using the definition of Laplace transform:

$$G(s) = \int_0^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-st} e^{-2t} f(3t)\,dt$$

Use the substitution $u = 3t$, $du = 3dt$ so $\frac{1}{3}du = dt$ (and limits $0, \infty$ are unchanged)

$$G(s) = \frac{1}{3}\int_0^\infty e^{-s\left(\frac{u}{3}\right)} e^{-2\left(\frac{u}{3}\right)} f(u)\,du$$

$$= \frac{1}{3}\int_0^\infty e^{\left(-\frac{s}{3} - \frac{2}{3}\right)u} f(u)\,du$$

$$= \frac{1}{3}\int_0^\infty e^{-\left(\frac{s+2}{3}\right)u} f(u)\,dy$$

$$= \frac{1}{3} F\left(\frac{s+2}{3}\right).$$

You can also work this problem using two table entries.

8. (15 points) Find the inverse Laplace transform of

$$F(s) = \frac{2s-3}{s^2+2s+10}.$$

Answer: $\boxed{f(t) = 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)}$

The denominator is irreducible, so complete the square:

$$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2+9}$$

$$= \frac{2(s+1)}{(s+1)^2+3^2} - \frac{5}{(s+1)^2+3^2}$$

Use tables: $\downarrow \mathcal{L}^{-1}$ $\downarrow \mathcal{L}^{-1}$

$$2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$$

9. (15 points). Use Laplace transforms to solve the initial value problem

$$y'' - y' - 6y = 0, \quad y(0) = 1,\ y'(0) = -1.$$

You can *check your answer*!

Answer: $\boxed{y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}}$

Apply $\mathcal{L}$: $[s^2 Y(s) - s y(0) - y'(0)] - [s Y(s) - y(0)] - 6Y(s) = 0$

Using $y(0)=1$, $y'(0) = -1$, get

$$(s^2 - s - 6)Y(s) = s - 2$$

$$Y(s) = \frac{s-2}{s^2-s-6} = \frac{s-2}{(s-3)(s+2)}.$$

Use partial fractions on this:

$$\frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} \Rightarrow s - 2 = A(s+2) + B(s-3)$$
$$= (A+B)s + (2A - 3B)$$

$$\Rightarrow \begin{cases} A + B = 1 \\ 2A - 3B = -2 \end{cases} \Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5} \Rightarrow B = \frac{4}{5}$$

$$Y(s) = \frac{1}{5}\,\frac{1}{s-3} + \frac{4}{5}\,\frac{1}{s+2}$$

Use tables: $\downarrow \mathcal{L}^{-1}$ $\downarrow \mathcal{L}^{-1}$

$$y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

Check: e^{3t}, e^{-2t} are solns to homog eqn,

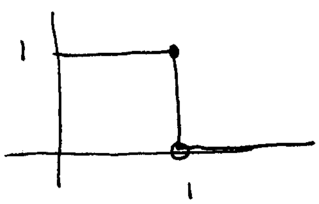
$y(0) = \frac{1}{5} + \frac{4}{5} = 1$ ✓, $y'(0) = \frac{3}{5} - \frac{8}{5} = -1$ ✓

10. (20 points) Let $f(t)$ be the forcing function defined by $f(t) = 1$ if $0 \le t \le 1$, and $f(t) = 0$ if $t > 1$. Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \ y'(0) = 0.$$

Answer: $\boxed{y(t) = 1 - \cos t - u_1(t)\left[1 - \cos(t-1)\right]}$

$f(t)$:



$f(t) = 1 - u_1(t)$

So $F(s) = \frac{1}{s} - \frac{e^{-s}}{s}$ (from table)

Apply $\mathcal{L}$ to equation (and use $y(0)=0$, $y'(0)=0$)

$$s^2 Y(s) + Y(s) = F(s) = \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$\Rightarrow \quad Y(s) = \frac{1}{s(s^2+1)} - \frac{1}{s(s^2+1)}e^{-s}$$

Let $H(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

$$1 = A(s^2+1) + Bs^2 + Cs = (A+B)s^2 + Cs + A$$

$$\Rightarrow A = 1, \ B = -1, \ C = 0$$

$$H(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

tables: $\mathcal{L}^{-1}\downarrow \qquad \mathcal{L}^{-1}\downarrow \qquad \downarrow\mathcal{L}^{-1}$

$$h(t) = 1 - \cos t$$

So $\quad y(t) = h(t) - u_1(t)\,h(t-1)$

$$= 1 - \cos t - u_1(t)\left[1 - \cos(t-1)\right]$$