

Name: KEY

Math 307F Final Exam
December 13, 2007

Instructions: There are ten problems, with the value of each problem indicated, for a total of 160 points. You are allowed the use of one page of handwritten notes, front and back, on standard size paper. You are also allowed use of a scientific calculator (but graphing calculators and other calculational devices are not allowed). The next page of the exam contains a table of Laplace transforms, which you may use in your solutions.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the **Box** provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

Problem 1	15 points	
Problem 2	15 points	
Problem 3	15 points	
Problem 4	15 points	
Problem 5	20 points	
Problem 6	15 points	
Problem 7	15 points	
Problem 8	15 points	
Problem 9	15 points	
Problem 10	20 points	
Total	160 points	

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > a $	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > a $	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

1. (15 points). Solve the initial value problem

$$y' = x^2y^2 + e^xy^2, \quad y(0) = 1.$$

Your solution should give y explicitly as a function of x .

Answer:

$$y = -\frac{1}{\frac{x^3}{3} + e^x - 2}$$

$$\frac{dy}{dx} = x^2y^2 + e^xy^2 = (x^2 + e^x)y^2$$

Separable: $\frac{dy}{y^2} = (x^2 + e^x)dx$

Integrate: $\int \frac{dy}{y^2} = \int (x^2 + e^x)dx$

$$\Rightarrow -\frac{1}{y} = \frac{x^3}{3} + e^x + C$$

$$y(0) = 1 \Rightarrow -\frac{1}{1} = \frac{0}{3} + 1 + C \Rightarrow C = -2$$

Solving for y :

$$y = -\frac{1}{\frac{x^3}{3} + e^x - 2}$$

2. (15 points) Find the general solution to

$$\sin(t^2)y' + [2t \cos(t^2)]y = e^t + t.$$

Answer:

$$y = \frac{e^t + \frac{t^2}{2} + C}{\sin(t^2)}$$

First put into standard form:

$$y' + \left[\frac{2t \cos(t^2)}{\sin(t^2)} \right] y = \frac{e^t + t}{\sin(t^2)}$$

Find The integrating factor

$$\mu(t) = e^{\int \frac{2t \cos(t^2)}{\sin(t^2)} dt}$$

Use The substitution $u = \sin(t^2)$, $du = 2t \cos(t^2) dt$

So

$$\int \frac{2t \cos(t^2)}{\sin(t^2)} dt = \int \frac{du}{u} = \ln u = \ln(\sin(t^2))$$

$$\mu(t) = e^{\ln(\sin(t^2))} = \sin(t^2)$$

Get

$$[\sin(t^2)y]' = \frac{e^t + t}{\sin(t^2)} \cdot \sin(t^2) = e^t + t$$

$$\Rightarrow (\sin(t^2))y = e^t + \frac{t^2}{2} + C$$

$$\Rightarrow y = \frac{e^t + \frac{t^2}{2} + C}{\sin(t^2)}$$

3. (15 points) A tank initially contains 100 liters of pure water. A mixture containing salt in the concentration of 5 grams/liter enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Let $S(t)$ denote the amount (in grams) of salt in the tank at time t .

(a) Find a formula for $S(t)$.

(b) What is the limiting amount of salt in the tank as $t \rightarrow \infty$?

Answers:

$$(a) S(t) = 500[1 - e^{-t/50}] ; (b) 500 \text{ grams}$$

$$(a) \frac{dS}{dt} = \text{rate in} - \text{rate out}$$

$$= (2 \text{ liter/min}) \times (5 \text{ gram/liter}) - (2 \text{ liter/min}) \left(\frac{S(t)}{100} \text{ gm/liter} \right)$$

$$= 10 - \frac{1}{50} S(t). \text{ (gram/min).}$$

$$\frac{dS}{dt} + \frac{1}{50} S = 10 \quad (\text{1st order linear, } \mu(t) = e^{\frac{1}{50}t})$$

$$\left[e^{\frac{1}{50}t} S \right]' = 10 e^{\frac{1}{50}t}$$

$$\Rightarrow \left[e^{\frac{1}{50}t} S \right] = 10 \frac{1}{\frac{1}{50}} e^{\frac{1}{50}t} + C = 500 e^{\frac{1}{50}t} + C$$

No salt initially, so $S(0) = 0 \Rightarrow$

$$0 = S(0) = e^{\frac{1}{50} \cdot 0} S(0) = 500 e^{\frac{0}{50}} + C \Rightarrow C = -500$$

Solving,

$$S(t) = 500 - 500 e^{-\frac{t}{50}}$$

$$\Rightarrow S(t) = 500[1 - e^{-t/50}]$$

(b) Since $e^{-t/50} \rightarrow 0$ as $t \rightarrow \infty$, $S(t) \rightarrow 500$

as $t \rightarrow \infty$. This is reasonable since it equals

The concentration of the inflow times the volume of the tank = $(5 \text{ gm/liter}) \times (100 \text{ liters})$.

4. (15 points) Solve the initial value problem

$$y'' + 6y' + 10y = 0, \quad y(0) = 1, y'(0) = 0.$$

Answer:

$$y(t) = e^{-3t} \cos t + 3e^{-3t} \sin t$$

Char eqn: $r^2 + 6r + 10 = 0$

Roots : $r = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i, \quad \lambda = -3, \mu = 1$

General solution is Therefore

$$y(t) = c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t$$

$$1 = y(0) = c_1$$

$$y'(t) = -e^{-3t} \sin t - 3e^{-3t} \cos t + c_2 e^{-3t} \cos t - 3e^{-3t} \sin t$$

$$0 = y'(0) = -3 + c_2 \Rightarrow c_2 = 3.$$

$$y(t) = e^{-3t} \cos t + 3e^{-3t} \sin t$$

5. (20 points) Find the general solution to

$$y'' - 4y' + 4y = \sin(2t) + e^{2t}.$$

Answer:

$$y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{8} \cos(2t) + \frac{1}{2} t^2 e^{2t}$$

Char eqn: $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r=2$ is a repeated root.

So $y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$. Use undet'd coeff for $g_1(t) = \sin(2t)$, $g_2(t) = e^{2t}$ separately.

$$y_{p_1} = A \cos(2t) + B \sin(2t)$$

$$y'_{p_1} = -2A \sin(2t) + 2B \cos(2t)$$

$$y''_{p_1} = -4A \cos(2t) - 4B \sin(2t)$$

$$y''_{p_1} - 4y'_{p_1} + 4y_{p_1} = [-4A - 8B + 4A] \cos(2t) + [-4B + 8A + 4B] \sin(2t)$$

$$= -8B \cos(2t) + 8A \sin(2t) = \sin(2t)$$

$$\text{so } B=0, A = \frac{1}{8}, \text{ and } y_{p_1} = \frac{1}{8} \cos(2t)$$

Since both e^{2t} and $t e^{2t}$ are solutions to the homogeneous eqn, we use $y_{p_2} = C t^2 e^{2t}$, so $y'_{p_2} = C[2t^2 e^{2t} + 2t e^{2t}]$, $y''_{p_2} = C[2(2t^2 e^{2t} + 2t e^{2t}) + 2(2t e^{2t} + e^{2t})]$. \Rightarrow

$$y''_{p_2} - 4y'_{p_2} + 4y_{p_2} = [4C - 8C + 4C] t^2 e^{2t} + [8C - 8C] t e^{2t} + 2C e^{2t} = e^{2t}$$

$$\text{so } C = \frac{1}{2}, \text{ and } y_{p_2} = \frac{1}{2} t^2 e^{2t}$$

6. (15 points) A mass weighing 5 lb stretches a spring 6 inches. Suppose that the weight is pulled down 1 ft and given a downward velocity of 8 ft/sec. There is no damping, nor are there external forces. Determine the subsequent motion $u(t)$, and also the amplitude R of this motion.

Answers:

$$u(t) = \cos(8t) + \sin(8t); \quad R = \sqrt{2}$$

$$mg = 5 \text{ lb} \Rightarrow m = \frac{5}{32}$$

$$mg = 5 = kL = \frac{1}{2}k \quad (6 \text{ in} = \frac{1}{2} \text{ foot}) \Rightarrow k = 10$$

Using the usual convention of downward being positive,

$$\frac{5}{32}u'' + 10u = 0, \quad u(0) = 1, \quad u'(0) = 8$$

or $u'' + 64u = 0$. Char eqn is $r^2 + 64 = 0 \Rightarrow$

$r = \pm 8i$, so the general solution is

$$u(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

$$1 = u(0) = c_1$$

$$u'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$8 = u'(0) = 8c_2 \Rightarrow c_2 = 1$$

$$\Rightarrow u(t) = \cos(8t) + \sin(8t)$$

$$R = \sqrt{1^2 + 1^2} = \sqrt{2}$$

7. (15 points) Let u_c be the standard unit step function defined by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

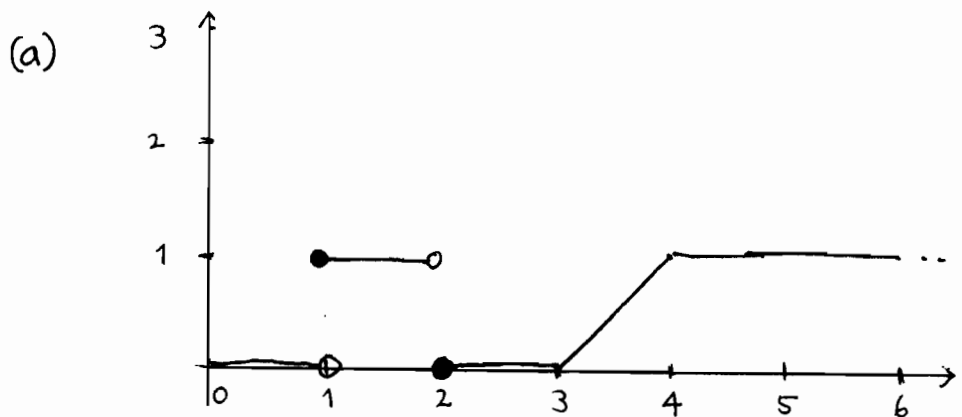
Let

$$f(t) = u_1(t) - u_2(t) + (t-3)u_3(t) - (t-4)u_4(t).$$

- (a) On the axes below sketch the graph of $f(t)$.
 (b) Compute the Laplace transform $F(s)$ of $f(t)$.

Answer for (b):

$$F(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s^2} - \frac{e^{-4s}}{s^2}$$



(b)

$$u_1(t) \xrightarrow{\mathcal{L}} \frac{e^{-s}}{s}, \quad u_2(t) \xrightarrow{\mathcal{L}} \frac{e^{-2s}}{s}$$

$$(t-3)u_3(t) \xrightarrow{\mathcal{L}} \frac{e^{-3s}}{s^2}, \quad (t-4)u_4(t) \xrightarrow{\mathcal{L}} \frac{e^{-4s}}{s^2}$$

So using linearity of \mathcal{L} , get

$$F(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s^2} - \frac{e^{-4s}}{s^2}$$

8. (15 points) Find the inverse Laplace transform of

$$F(s) = \frac{s^2 + s + 3}{(s-1)(s^2+4)}$$

Answer:

$$f(t) = e^t + \frac{1}{2} \sin(2t)$$

Use partial fractions:

$$\frac{s^2 + s + 3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

$$\begin{aligned} \Rightarrow s^2 + s + 3 &= A(s^2+4) + (Bs+C)(s-1) \\ &= [A+B]s^2 + [C-B]s + [4A-C] \end{aligned}$$

So

$$A + B = 1$$

$$-B + C = 1$$

$$4A - C = 3$$

$$5A$$

$$= 5 \Rightarrow A = 1 \Rightarrow B = 0$$

$$\Rightarrow C = 1.$$

$$F(s) = \frac{1}{s-1} + \frac{1}{s^2+4}$$

$\downarrow \mathcal{L}^{-1}$ $\downarrow \mathcal{L}^{-1}$

$$f(t) = e^t + \frac{1}{2} \sin(2t)$$

9. (15 points). Use Laplace transforms to solve the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

You can check your answer!

Answer:

$$y(t) = e^{2t} + e^{-t}$$

Apply \mathcal{L} :

$$[s^2 Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 2Y(s) = 0$$

Using initial conditions,

$$(s^2 - s - 2)Y(s) = 2s - 1$$

$$Y(s) = \frac{2s-1}{s^2-s-2} = \frac{2s-1}{(s-2)(s+1)}$$

Use partial fractions:

$$\frac{2s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \quad (\text{"cover-up"}) \Rightarrow A=1, B=1$$

$$\Rightarrow Y(s) = \frac{1}{s-2} + \frac{1}{s+1}$$

$$\begin{array}{ccc} \downarrow \mathcal{L}^{-1} & & \downarrow \mathcal{L}^{-1} \\ y(t) = e^{2t} + e^{-t} \end{array}$$

$$\text{Check: } y'' - y' - 2y = [4 - 2 - 2]e^{2t} + [1 + 1 - 2]e^{-t} = 0 \quad \checkmark$$

$$y(0) = 1 + 1 = 2 \quad \checkmark$$

$$y' = 2e^{2t} - e^{-t}, \quad y'(0) = 2 - 1 = 1 \quad \checkmark$$

10. (20 points) Let $f(t)$ be the forcing function defined by $f(t) = 0$ if $0 \leq t < 2$, and $f(t) = 2$ if $t \geq 2$. Solve the initial value problem

$$y'' + 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Answer:

$$y(t) = u_2(t) [1 - 2e^{-(t-2)} + e^{-2(t-2)}]$$

Apply \mathcal{L} : $f(t) = 2u_2(t)$

$$\begin{aligned} s^2 Y(s) + 3s Y(s) + 2Y(s) &= \mathcal{L}\{f(t)\} \\ &= 2 \frac{e^{-2s}}{s} \end{aligned}$$

$$\Rightarrow Y(s) = e^{-2s} \underbrace{\frac{2}{s(s^2+3s+2)}}_{H(s)}$$

First compute $h(t) = \mathcal{L}^{-1}\{H(s)\}$, use partial fractions

$$H(s) = \frac{2}{s(s^2+3s+2)} = \frac{2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

using "cover-up" or linear eqns, get $A=1$, $B=-2$, $C=1$.

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s} - 2\frac{1}{s+1} + \frac{1}{s+2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= 1 - 2e^{-t} + e^{-2t} \end{aligned}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{e^{-2s} H(s)\} = u_2(t) h(t-2) \\ &= u_2(t) [1 - 2e^{-(t-2)} + e^{-2(t-2)}] \end{aligned}$$