Name: KEY

Math 307F First Midterm Exam
October 17, 2007

Instructions: There are five problems, each worth 20 points, for a total of 100 points. You are allowed the use of a scientific calculator (but graphing calculators and other calculational devices are **NOT ALLOWED**). Your are also allowed the use of one page of handwritten notes, front and back, maximum size of 8.5 by 11 inches.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the **Box** provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

Problem 1	20 points	
Problem 2	20 points	
Problem 3	20 points	
Problem 4	20 points	
Problem 5	20 points	
Total	100 points	

1. (20 points). Solve the initial value problem

$$t\,y' + y = t\sin t, \quad y(\pi) = 2.$$

Answer: $\boxed{y = \dfrac{\sin t - t\cos t + \pi}{t}}$

1st order linear. Put into standard form

$$y' + \left(\frac{1}{t}\right)y = \sin t$$

Integrating factor is

$$\mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$

Multiplying by $\mu(t)$ gives

$$ty' + y = [ty]' = t\sin t$$

Integrating gives

$$t\,y = \int \underbrace{t}_{\substack{u \\ du = dt}} \underbrace{\sin t\, dt}_{\substack{dv \\ v = -\cos t}} \overset{\text{(parts)}}{=} -t\cos t - \int (-\cos t)\,dt$$

$$= -t\cos t + \sin t + C$$

Put $t = \pi$, $y = 2$ to solve for C:

$$2\pi = (-\pi)(-1) + 0 + C \Rightarrow C = \pi$$

$\therefore$

$$y = \frac{\sin t - t\cos t + \pi}{t}$$

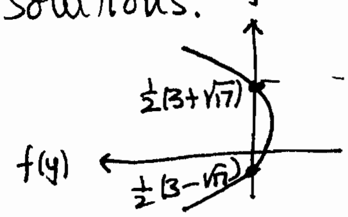
2. (20 points). (a) Find all equilibrium solutions to the autonomous differential equation

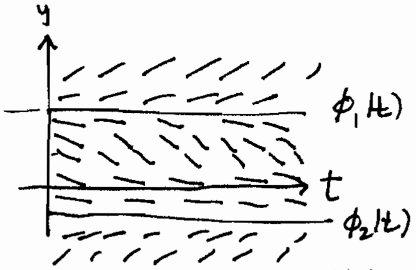
$$\frac{dy}{dt} = y^2 - 3y - 2,$$

and for each solution state whether it is stable, unstable, or semistable.

Answer: $\phi_1(t) \equiv \frac{1}{2}(3+\sqrt{17})$, unstable
$\phi_2(t) = \frac{1}{2}(3-\sqrt{17})$, stable.

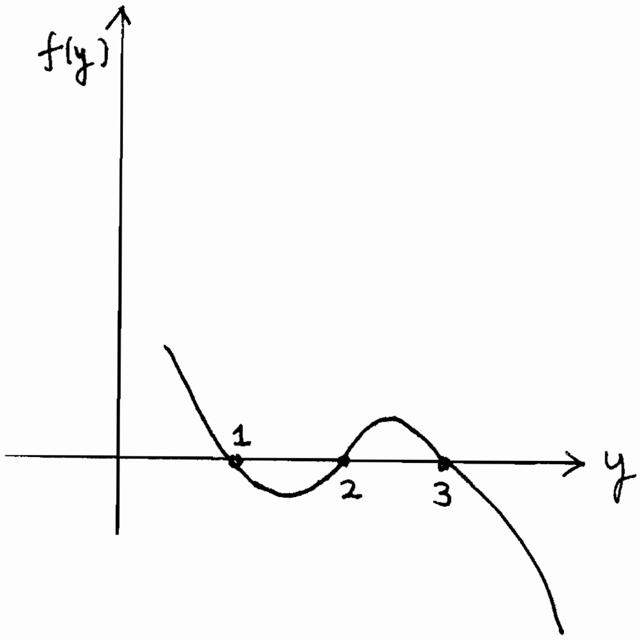
The critical points of $f(y) = y^2 - 3y - 2$ are the roots (by The quadratic formula) $\frac{1}{2}(3 \pm \sqrt{9+8}) = \frac{1}{2}(3 \pm \sqrt{17})$, so $\phi_1(t) \equiv \frac{1}{2}(3+\sqrt{17})$ and $\phi_2(t) \equiv \frac{1}{2}(3-\sqrt{17})$ are The equilibrium solutions.





$\phi_1(t)$ is unstable
$\phi_2(t)$ is stable

(b) On the axes below, sketch the graph of a function $f(y)$ so that the autonomous differential equation $y' = f(y)$ has exactly three equilibrium solutions $\phi_1(t) = 1$, $\phi_2(t) = 2$, and $\phi_3(t) = 3$, and also such that $\phi_1(t)$ is stable, $\phi_2(t)$ is unstable, and $\phi_3(t)$ is stable. Be sure to explain why your function satisfies the given conditions.



$f(y)$ needs to vanish at $y=1$, $y=2$, and $y=3$ only. Since $\phi_1(t) \equiv 1$ is stable, $f'(1) < 0$. Since $\phi_2(t) \equiv 2$ is unstable, $f'(2) > 0$. Since $\phi_3(t) \equiv 3$ is stable, $f'(3) < 0$. The graph to The left satisfies These conditions

3. (20 points). Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{x\,y^2}{y^2+1}, \quad y(0) = 1.$$

To obtain full credit, you need to get your solution for y as an *explicit* function of x.

Answer: $\boxed{y = \frac{x^2}{4} + \frac{1}{2}\sqrt{\frac{x^4}{4} + 4}}$

Separable: $\left(\frac{y^2+1}{y^2}\right)dy = x\,dx$

or

$$\left(1 + \frac{1}{y^2}\right)dy = x\,dx$$

so $\int \left(1 + \frac{1}{y^2}\right)dy = \int x\,dx \Rightarrow y - \frac{1}{y} = \frac{x^2}{2} + C$

Put $x=0$, $y=1$ to find C:

$$1 - \frac{1}{1} = 0 + C \Rightarrow C = 0.$$

Implicit solution is

$$y - \frac{1}{y} = \frac{x^2}{2}$$

Now solve for y: $\frac{y^2-1}{y} = \frac{x^2}{2} \Rightarrow y^2 - 1 = \left(\frac{x^2}{2}\right)y$

$$\Rightarrow y^2 - \left(\frac{x^2}{2}\right)y - 1 = 0$$

Quadratic formula $\Rightarrow$

$$y = \frac{\frac{x^2}{2} \pm \sqrt{\left(\frac{x^2}{2}\right)^2 + 4}}{2}$$

Choose + sign since $y(0)=$

$$y = \frac{x^2}{4} + \frac{1}{2}\sqrt{\frac{x^4}{4} + 4}$$

4. Consider the initial value problem

$$\frac{dy}{dt} = y, \quad y(0) = 2.$$

(a) Use Euler's method, with step size $h = 0.1$ and starting at $t_0 = 0$, to approximate the value of the solution at $t = 0.4$. Carry out your calculations to at least four decimal places.

Answer: 2.9282

(b) As a check, solve the equation to get the exact value of the solution at $t = 0.4$. In the box below put both the solution $y(t)$ and the exact value $y(0.4)$.

Answer: $y(t) = 2e^t, \quad y(0.4) = 2.9836$

(a)

n	t_n	y_n
0	0.0	2.0000
1	0.1	2.2000
2	0.2	2.4200
3	0.3	2.6620
4	0.4	2.9282

$y_1 = y_0 + f(t_0, y_0)\cdot h$

$= 2 + 2(0.1) = 2.2000$

$y_2 = 2.2 + (2.2)(0.1) = 2.4200$

$y_3 = 2.42 + (2.42)(0.1) = 2.662$

$y_4 = 2.662 + (2.662)(0.1) = 2.9282$

(b) $\frac{dy}{y} = dt \Rightarrow \ln y = t + C$

$\Rightarrow y = ce^t$

$2 = y(0) = c$ so $y = 2e^t$ is the exact solution

$y(0.4) = 2e^{0.4} = 2.9836$

5. (20 points). The temperature of a cup of coffee decreases at a rate proportional to the difference between the cup's temperature and the external room temperature (this is known as Newton's law of cooling). Suppose the room temperature is a constant 70° F. The coffee initially has a temperature of 200° F, and after 10 minutes it cools down to 170° F. What is the total time it takes the cup to cool down to 100° F?

Answer: $\boxed{55.89 \text{ min}}$

Let $T(t)$ = temperature at time t minutes.

We are given $T(0) = 200$, $T(10) = 170$

Newton's law of cooling $\Rightarrow$

$$\frac{dT}{dt} = -k(T-70) \qquad k>0 \text{ some constant}$$

$$\frac{dT}{T-70} = -k\,dt$$

$\int$ gives $\quad \ln(T-70) = -kt + C$

$$T - 70 = e^{-kt} \cdot e^{C} = ce^{-kt}$$

Let $t=0$: $\quad 200 - 70 = 130 = ce^{0} = c$

Let $t=10$: $\quad 170 - 70 = 100 = 130e^{-kt} = 130e^{-10k}$

$$\Rightarrow \ln\left(\frac{100}{130}\right) = -10k \Rightarrow k = 0.026236$$

Find t_0 so that $T(t_0) = 100$:

$$30 = 100 - 70 = 130e^{-(0.026236)t_0}$$

$$\Rightarrow \ln\left(\frac{30}{130}\right) = -(0.026236)t_0 \Rightarrow t_0 = 55.89 \text{ min}$$