Optimal landing location on flat, uniform terrain

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A general solution procedure for the continuous demand variant of the pure location problem of Fermat–Weber is given. The solution of the Fermat–Weber problem is a specific case of the more general landing-location problem addressed in this paper. The location of a single landing site is optimized with respect to the total harvesting cost for a setting that includes the costs of roading, yarding, and hauling.

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Un processus de solution général est décrit pour les variants de demande continus pour le problème de localisation pure de Fermat–Weber. La solution au problème Fermat–Weber est un cas spécifique d'un problème plus général de localisation de jetée décrit dans cet article. La localisation d'une seule jetée est optimisée en relation avec le coût total d'exploitation pour une installation comprenant les coûts de voirie, de téléphérage et de débusquage.

[Traduit par la rédaction]

Introduction

A classic problem in location theory is that of finding the site for a service facility that minimizes the sum of the weighted Euclidean travel distances between that single service facility and multiple service demand points. Demand occurrence may be defined using discrete and (or) continuous probability distributions. As noted by Francis and White (1974), the two-dimensional discrete demand location problem, studied in its various forms since the 17th century, has only recently been completely solved. The continuous distribution variant of this basic problem has not been studied to a comparable extent nor has a general solution procedure been developed.

The usual approach to continuously distributed demand points is to aggregate the continuous distribution (Ghosh and Rushton 1987). The probabilities associated with arbitrarily delineated subregions are concentrated at specific points within the corresponding subregions, typically the centroids. Solution procedures developed for the case of discrete distributions are then applied. A serious shortcoming of this approach is the uncertain magnitude of the bias associated with the aggregation procedure. One example of the potentially large magnitude of this error is given by Drezner and Weslowsky (1980). More recent research has produced models that are not subject to aggregation error. Unfortunately these latter models suffer from other limitations that seriously limit their applicability. An accurate and easily implemented solution to the general problem of continuously distributed demand points across a planar region of arbitrary shape is still needed for many applications of a practical nature, including those of the forest engineer. It is the purpose of this paper to provide such a solution.

The single facility optimal location model is presented in the body of the paper. Here the objective is to summarize the results of extensive mathematical development in a fashion conducive to general understanding as well as further developmental work.

The paper concludes with some observations on future research in location theory, as it relates to forest engineering.

Background

The economic design and conduct of harvesting operations have always been of fundamental interest to the profession of forest engineering (Fernow 1901). In a 1922 presentation to the Pacific Logging Congress, James Girard, one of the first logging engineers with the United States Forest Service, observed the following:

> The skidding distance that gives a skidding cost which, when added to the cost of railroad construction results in the lowest total cost per thousand for both skidding and railroad construction, shows the distance apart that the railroad spurs should be.

Research in optimal transportation planning continued through the 1930s (Bradner et al. 1933), and in 1942 Matthews' text, *Cost Control in the Logging Industry*, was published. This widely read text helped establish and maintain research interest in the optimization of forest transportation systems.

Among the traditional layout optimization problems examined by forest engineers is one that involves the location of roads and centralized landings. In this analysis, roads and landings are simultaneously located in a predetermined systematic pattern across flat, uniformly forested terrain of effectively infinite extent. The optimization process determines pattern scale, which then serves as a rough guide to road and landing spacing (Silen 1955). In more recent work Peters (1978) re-examined this traditional optimal location problem of forest engineering. In his analysis, Peters employed modern analytical procedures and a new average yarding distance formula.

Models directed at optimizing the location of a finite number of central landings across a well-defined region of limited area have also been examined (Dykstra and Riggs 1977; Gibson and Rodenberg 1975). Landing-location constraints are a fundamental characteristic of these models, as is the centroidal aggregation of turns distributed across specified subregions.

Only limited research has been done on the optimal, unrestricted two-dimensional location of service facilities under unaggregated continuous demand-point distribution. Love (1972) developed and successfully applied an optimization algorithm that avoids demand aggregation and its associated computational bias. A serious limitation of his procedure, however, is its restriction to demand sources that are continuously distributed across rectangular regions on a plane. Drezner and Weslowsky (1980) extended the work done by Love to include regions of more general shape. Unfor-

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tunately there is no assurance that their algorithm will always converge to an optimum. A second serious difficulty with this latter model is that no generalized data entry procedure is available. Cavalier and Sherali (1986) suggest an alternative algorithm for optimal location of a single facility. Their algorithm will converge on the global optimum, and data entry may be easily accomplished. The Cavalier and Sherali model, however, has sacrificed generality. They restrict their attention to one level of demand intensity, and that demand is uniformly distributed over the region enclosed by a single convex polygon. For many practical applications the ability to examine a general, nonconvex demand region that has subregions each of a different demand intensity is an important, sometimes essential, capability.

Model specification

A forest harvesting operation is being planned, and the boundary of the harvest area has been laid out. The harvest area is partitioned into subareas, each of which exhibits homogeneous turn-making conditions. Yarding systems have been selected from among those available and may include a mix of tractive, cable, and aerial systems. Within a yarding system class and type there may be different productivity levels; e.g., within the tractive yarder class the articulated rubber-tired skidder type is often divided for purposes of production analysis into small, medium, and large horsepower productivity levels. Different yarding systems may be used in different partitions of the harvest area, but within any single partition only equipment of one productivity level of the assigned yarding system may be used. The entire area is on relatively flat ground. There are no significant barriers to yarding activity, nor are there restrictions on locating a spur road and landing. All harvested logs from all partitions will be brought into a single centralized landing where they will be loaded onto vehicles for further transport. Either this landing will be located on an existing road or it will be accessed by a terminating spur road that takes off from an existing road. The problem then is this: Given these design conditions, where should the landing be located so that the combined costs of spur-road construction, yarding activity, and subsequent truck hauling are minimized for the harvesting operation? Some minor definitional refinement of this basic problem will occur during the development of the mathematical model that follows.

Within any partition area, k, there are m_k turns distributed over a horizontal area, A_k. For any turn within this partition, the only information about its spatial location is that of the limits defined by the encompassing partition boundary. Under these conditions the uniform distribution is an appropriate turn location density function by the criterion of maximum entropy (Shannon 1948; Greulich 1989).

Given an arbitrarily established coordinate system, the estimated cost of yarding a single turn a distance ρ from its location at (x, y) within partition k to a landing located at (x_0, y_0) is given by

[1] $$YC_k = \beta_{0k} + \beta_{1k}\rho + \beta_{2k}\rho^2$$

where

[2] $$\rho = [(x - x_0)^2 + (y - y_0)^2]^{\frac{1}{2}}$$

Equation 1 may be thought of as the first three terms of a power series approximation to a more complex functional relationship between cost and yarding distance. For convexity in (x_0, y_0), it is required that $\beta_{1,k}$ and $\beta_{2,k}$ be non-negative; any of the three β coefficients may be zero. These coefficients are determined by equipment cost and productivity. Also however, as explicitly indicated by the subscript k, these coefficients may be determined in part by conditions unique to any partition; e.g., average turn volume and (or) average turn piece count.

In the case of ground skidding, when a cost-distance relationship of this type is used for the evaluation of a specific turn, there is an adjustment made to the straight-line yarding distance. Based on equipment characteristics and specific ground conditions, the winding course followed by the equipment on its route from the landing to the location of the turn and back again is accounted for with a wander factor, w_k (Hughes 1930; Segebaden 1964). For most cable and aerial yarding system applications this adjustment of the yarding distance is not necessary, in which case the wander factor is set equal to one.

Under the aforementioned assumptions the expected cost of yarding one turn, which is randomly located according to a continuous uniform probability distribution across the area of partition k, is given by

[3] $$E\{YC_k\} = \iint_{A_k} (\beta_{0k} + \beta_{1k}w_k\rho + \beta_{2k}w_k^2\rho^2)\,\frac{dx\,dy}{A_k}$$

which may be written

[4] $$E\{YC_k\} = \beta_{0k} + \beta_{1k}w_k\,E_k\{\rho\} + \beta_{2k}w_k^2\,E_k\{\rho^2\}$$

Assuming that each of the m_k turns is independently distributed, the expected total yarding cost for partition k is given by

[5] $$E\{TYC_k\} = m_k\,E\{YC_k\}$$

and the total yarding cost for the setting

[6] $$TYC = \sum_k E\{TYC_k\}$$

The cost of spur-road construction and maintenance is combined with the cost of hauling all of the harvested logs over the length of the spur. Spur-road decommissioning costs should also be added in when appropriate. The spur road has a takeoff from the existing road at (x_e, y_e) and runs in a straight line to the landing at (x_0, y_0). Total cost associated with the decision to build and use a spur road is estimated by

[7] $$TRC = \beta_3 s$$

where

[8] $$s = [(x_e - x_0)^2 + (y_e - y_0)^2]^{\frac{1}{2}}$$

and β_3 is the cost of construction, maintenance, and hauling per unit length of the spur road.

Based on the preceding model development, the total cost minimization problem can now be restated in mathematical notation:

[9] $$\underset{(x_0,\,y_0)}{\text{MIN}}\ TC = TYC + TRC$$

Model limitations

Before proceeding with the formal analysis, several clarifying observations should be made with regard to limitations of the model. Many implicit assumptions and limitations of the model are apparent from the foregoing description; some, however, warrant explicit recognition and comment.

Certain lump-sum costs have been ignored and will not be directly included in the model. Some of these are fixed costs and thus inconsequential to the decision being addressed in this analysis; e.g., for landing construction cost it is assumed in all cases that the decision to use a single central landing of standard design and cost has already been made. Other lump-sum costs are potentially relevant to a final landing-location decision, but must be addressed outside of the model; e.g., the nonvariable (with road length) cost of road construction, which might include equipment move-in costs as an example. Given the existence of a significant road construction cost of this nature, the possibility of obtaining lower total costs, by constructing no road, rather than the model-indicated amount must be considered. The examination of such discrete alternatives, when they exist, and the inclusion of additional costs, when appropriate, are left as some of the practical niceties of any specific application.

One final observation on lump-sum costs: all the nonvariable costs of yarding partition k with a specific yarding system of given productivity can be included in the constant value $m_k\beta_{0,k}$. The coefficient $\beta_{0,k}$ has been inserted primarily for descriptive purposes, since its magnitude is irrelevant to the immediate results of the model. It has been assumed from the beginning that yarding will take place. Hence this particular part of the cost is fixed and inescapable; it will not influence the final location decision.

There are no model-imposed restrictions on the location or size of the landing; the landing may be located at any point on the horizontal plane. Constrained optimization techniques might be employed when locational restrictions exist, but they are beyond the scope of this paper. Within the model the landing position is represented by a point. Physical size of the landing is not treated as being relevant to the location decision.

The model assumes that the entire region of interest is adequately described by a horizontal plane. There is no consideration of ground slope or topographic relief and their potential impact on yarding or roading activity. Likewise the ground conditions are assumed to be uniform or spatially homogeneous at a relatively small scale of resolution; e.g., large contiguous areas of swamp cannot be intermingled with large contiguous areas of firm ground.

The selection of an alternative mix of yarding systems for the various partitions may result in different costs and a different landing location. Use of the model to select the optimal combination of yarding systems will not be examined. Likewise, the comparison of alternative spur-road design standards or takeoff locations must be handled outside of this model.

The current analysis will only examine a yarding cost function [1] that is strictly convex in x_0 and y_0. This restriction insures that there is one, and only one, landing location that minimizes total cost. This restriction could be removed, but then there would be no guarantee that the algorithmically optimized location was the overall best possible solution; i.e., the global optimum. The restriction on non-negativity of the β coefficients does not seem unreasonable, at least when the coefficients appear singly, since it is generally true that as yarding distance increases yarding costs do likewise. The potential difficulty arises when both $\beta_{1,k}$ and $\beta_{2,k}$ appear in the same cost function. A review of the time study literature for both cable and tractive yarder systems suggests that under these conditions it would not be unusual to find a negative sign attached to $\beta_{2,k}$. Under these circumstances the cost function should be re-estimated at a lower precision, with one of the two β coefficients removed from the equation.

Model analysis

The analysis of the mathematical model defined by eqs. 1-9 is covered in this section. This analysis eventually leads to an easily implemented general solution of the model. This solution procedure always yields the least-cost location for the landing. Data entry is a parochial process; it requires no training and is quickly done. There are no restrictions on the shape of the harvest area. The solution procedure uses a standard algorithm with an extremely short computer execution time.

Optimization, discussed in the next section, is done by computer using a well-established numerical procedure. This procedure is based on the first and second partial derivatives of the total cost equation with respect to x_0 and y_0. While these partial derivative formulas are found to be of quite simple form, their development is exceedingly long, and for this reason only a summary presentation is given. Formulas developed in a previous paper (Greulich 1987) are used extensively. Wherever possible notational continuity has been maintained with this previous publication so that reference to it might be facilitated.

Consider the planar region, a portion of which is shown in Fig. 1. This region on the horizontal plane is defined by N line segments that establish its boundary. These line segments and their starting points are sequentially numbered such that the enclosed area of interest is always to the left of the line of traverse. The arbitrarily selected starting point for the traverse is also its ending point. The landing is located at (x_0, y_0). Also shown is traverse line segment i, running from point i to point $i + 1$. Triangle i is formed by connecting the end points of line segment i with the landing location point. The side lengths of triangle i are $L_{i,1}$, $L_{i,2}$, and $L_{i,3}$. The shown positional relationship between the sides of the triangle and the length variables is important. Note that $L_{i,2} \equiv L_{i+1,1}$ and that $L_{i,3}$ is always the length of the traverse segment for triangle i. Because the region is located in the horizontal plane these side lengths are horizontal distances.

The following formula is obtained from eq. 9 upon substitution from eqs. 5-7:

$$[10] \qquad \underset{(x_0,\, y_0)}{\text{MIN}}\ \text{TC} = \beta_3 s + \sum_k m_k\, \text{E}\{\text{YC}_k\}$$

To expedite presentation, the analytical development that follows will focus on the case where there is only one partition. This partition constitutes the entire harvest area. Additional partitions are easily accommodated and will be discussed later. Rewriting eq. 10 after substitution from eq. 4 and deletion of the subscript k yields

$$[11] \qquad \underset{(x_0,\, y_0)}{\text{MIN}}\ \text{TC} = m\beta_0 + m\beta_1 w\, E\{\rho\} + m\beta_2 w^2\, E\{\rho^2\} + \beta_3 s$$

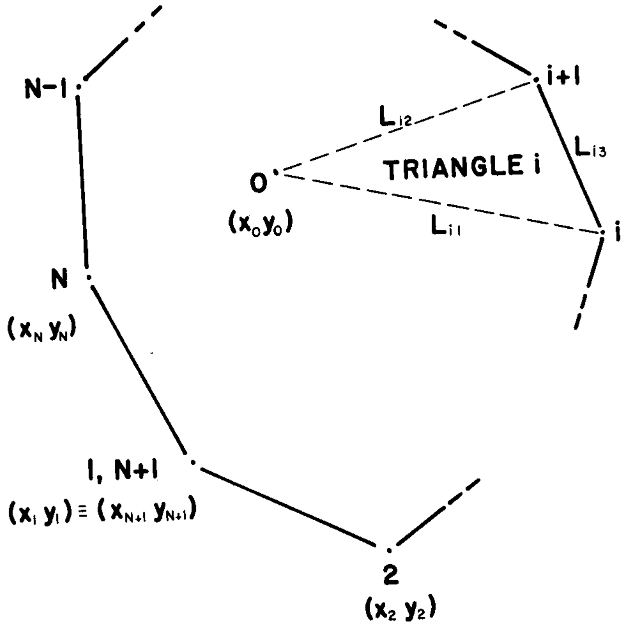

FIG. 1. Portion of a planar region (horizontal) enclosed by N sequentially numbered line segments. The enclosed region of interest lies to the left of the line of traverse.

Consolidate the constant terms

$$[12] \quad \underset{(x_0,\, y_0)}{\text{MIN}}\ \text{TC} = C_0 + C_1\,\text{E}\{\rho\} + C_2\,\text{E}\{\rho^2\} + C_3 s$$

If C_1 and C_2 are non-negative and at least one of them is greater than zero, then eq. 12 can be shown to be strictly convex (see Appendix). Convexity is independent of the signs on C_0 and C_3. Given strict convexity, there exists one, and only one, location that minimizes the total cost. Locating that minimum point is facilitated by the strict convexity of the total cost function. The selected procedure for minimization of the function TC makes use of the first and second partial derivatives of that function:

$$[13a] \quad \frac{\partial \text{TC}}{\partial x_0} = C_1 \frac{\partial \text{E}\{\rho\}}{\partial x_0} + C_2 \frac{\partial \text{E}\{\rho^2\}}{\partial x_0} + C_3 \frac{\partial s}{\partial x_0}$$

$$[13b] \quad \frac{\partial \text{TC}}{\partial y_0} = C_1 \frac{\partial \text{E}\{\rho\}}{\partial y_0} + C_2 \frac{\partial \text{E}\{\rho^2\}}{\partial y_0} + C_3 \frac{\partial s}{\partial y_0}$$

and

$$[14a] \quad \frac{\partial^2 \text{TC}}{\partial x_0^2} = C_1 \frac{\partial^2 \text{E}\{\rho\}}{\partial x_0^2} + C_2 \frac{\partial^2 \text{E}\{\rho^2\}}{\partial x_0^2} + C_3 \frac{\partial^2 s}{\partial x_0^2}$$

$$[14b] \quad \frac{\partial^2 \text{TC}}{\partial y_0^2} = C_1 \frac{\partial^2 \text{E}\{\rho\}}{\partial y_0^2} + C_2 \frac{\partial^2 \text{E}\{\rho^2\}}{\partial y_0^2} + C_3 \frac{\partial^2 s}{\partial y_0^2}$$

$$[14c] \quad \frac{\partial^2 \text{TC}}{\partial y_0 \partial x_0} = C_1 \frac{\partial^2 \text{E}\{\rho\}}{\partial y_0 \partial x_0} + C_2 \frac{\partial^2 \text{E}\{\rho^2\}}{\partial y_0 \partial x_0} + C_3 \frac{\partial^2 s}{\partial y_0 \partial x_0}$$

Each of the terms on the right-hand side of eqs. 13*a*, 13*b*, and 14*a*–14*c* will be examined in turn starting with the first partial derivatives.

From previous work (Greulich 1987) it is known that for a harvest area of triangular shape in the horizontal plane with uniform distribution of turns, the straight-line average yarding distance to an apical landing is given by

$$[15] \quad \text{AYD}_i = \frac{2}{3} \int_0^1 R_i(\lambda)\, d\lambda$$

The variable of integration, λ, is a parameter that within the interval [0, 1] defines all points along the triangle's base (the side opposite the apical landing). The distance from the landing to any point on the base is given by

$$[16] \quad R_i(\lambda) \equiv [H_i^{\text{T}} H_i + 2\lambda H_i^{\text{T}}(H_{i+1} - H_i) + \lambda^2 (H_{i+1} - H_i)^{\text{T}}(H_{i+1} - H_i)]^{\frac{1}{2}}$$

where the transposed H vectors for triangle i are given by

$$[17a] \quad H_i^{\text{T}}\!: (x_i - x_0,\, y_i - y_0)$$

$$[17b] \quad H_{i+1}^{\text{T}}\!: (x_{i+1} - x_0,\, y_{i+1} - y_0)$$

The squared side lengths of triangle i can also be given in terms of the H vectors:

$$[18a] \quad L_{i,1}^2 = H_i^{\text{T}} H_i$$

$$[18b] \quad L_{i,2}^2 = H_i^{\text{T}} H_i + 2H_i^{\text{T}}(H_{i+1} - H_i) + (H_{i+1} - H_i)^{\text{T}}(H_{i+1} - H_i)$$

$$[18c] \quad L_{i,3}^2 = (H_{i+1} - H_i)^{\text{T}}(H_{i+1} - H_i)$$

The area of triangle i is provided by the engineer's well-known coordinate area formula

$$[19] \quad A_i = [-1/2]\,[(x_0)(y_{i+1} - y_i) + (y_0)(x_i - x_{i+1}) + (x_{i+1} y_i - x_i y_{i+1})]$$

and the total area of the partition is then

$$[20] \quad A = \sum_{i=1}^{N} A_i$$

The usual mathematical convention of a counterclockwise traverse of the partition area will yield a positive value for eq. 20 when eq. 19 is used to calculate the individual triangular areas.

Following Donnelly (1978) the average yarding distance for the partition can be written as

$$[21] \quad \text{AYD} = \frac{1}{A} \sum_{i=1}^{N} A_i \text{AYD}_i$$

Donnelly's area weighting procedure with its use of the coordinate area formula is extremely general (see also Blair and Biss 1967). No assumptions with regard to the location of the landing or the shape of the harvest area are required in the application of formula [21]. Equation 19 can be derived as the linear boundary case of a general area formula developed from Green's theorem. Green's theorem broadly defines how eq. 21 may be applied. Justification for the use of this particular area formula in Donnelly's weighting procedure, although not given here, can be shown. The easiest proof involves an alternative geometrical description of the Green's theorem area formula (Davis and Snider 1988) in conjunction with the composite area rule (Suddarth and Herrick 1964).

Observing the definition of AYD and substituting from eq. 15 into eq. 21

$$[22] \quad \text{AYD} \equiv \text{E}\{\rho\} = \frac{1}{A} \sum_{i=1}^{N} \left[\frac{2A_i}{3} \int_0^1 R_i(\lambda)\, d\lambda \right]$$

The first partial derivative of eq. [22] with respect to x_0 is taken:

$$[23] \quad \frac{\partial \mathrm{E}\{\rho\}}{\partial x_0} = \frac{2}{3A} \sum_{i=1}^{N} \left[\frac{\partial A_i}{\partial x_0} \int_0^1 R_i(\lambda)\, \mathrm{d}\lambda + A_i \int_0^1 \frac{\partial R_i(\lambda)}{\partial x_0} \mathrm{d}\lambda \right]$$

Individual terms are evaluated using the calculus and algebraic manipulation

$$[24] \quad \frac{\partial A_i}{\partial x_0} = -\left[\frac{y_{i+1} - y_i}{2}\right]$$

$$[25] \quad \int_0^1 R_i(\lambda)\, \mathrm{d}\lambda = \left[\frac{L_{i,1} + L_{i,2}}{4}\right]\left[1 + \left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}}\right)^2\right] + \left[\frac{2A_i^2}{L_{i,3}^3}\right]\left[\ln\left(\frac{1 + r_i}{1 - r_i}\right)\right]$$

$$[26] \quad \int_0^1 \frac{\partial R_i(\lambda)}{\partial x_0} \mathrm{d}\lambda = \left[(x_{i+1} - x_i)\left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right)\right] - \left[\left(\frac{2A_i}{L_{i,3}^3}\right)(y_{i+1} - y_i)\right]\left[\ln\left(\frac{1 + r_i}{1 - r_i}\right)\right]$$

where

$$[27] \quad r_i \equiv \frac{L_{i,3}}{L_{i,1} + L_{i,2}}$$

Digressing momentarily it is convenient now to write the expected value formulas. From eqs. 22 and 25 is found

$$[28] \quad \mathrm{E}\{\rho\} = \frac{2}{3A} \sum_{i=1}^{N} \left\{\left[(A_i)\left(\frac{L_{i,1} + L_{i,2}}{4}\right)\right]\left[1 + \left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}}\right)^2\right] + \left[\frac{2A_i^3}{L_{i,3}^3}\right]\left[\ln\left(\frac{1 + r_i}{1 - r_i}\right)\right]\right\}$$

in a similar fashion and once again drawing on the results of previous work (Greulich 1987)

$$[29] \quad \mathrm{E}\{\rho^2\} = \frac{1}{12A} \sum_{i=1}^{N} [(A_i)(3L_{i,1}^2 + 3L_{i,2}^2 - L_{i,3}^2)]$$

Continuing with the development of the partial derivations it is noted that for a closed traverse the following relationship holds for arbitrarily selected values of a and b:

$$[30] \quad \sum_{i=1}^{N} x_i^a y_i^b = \sum_{i=1}^{N} x_{i+1}^a y_{i+1}^b$$

Substituting eqs. 19 and 24–26 into eq. 23 and using eq. 30 as required, it is found after tedious algebra that

$$[31a] \quad \frac{\partial \mathrm{E}\{\rho\}}{\partial x_0} = \frac{1}{A}\left\{-\sum_{i=1}^{N}\left[\left(\frac{2A_i^2}{L_{i,3}^3}\right)(y_{i+1} - y_i)\right]\left[\ln\left(\frac{1 + r_i}{1 - r_i}\right)\right] + \sum_{i=1}^{N}\left[\left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right)(A_i)\,(x_{i+1} - x_i)\right]\right\}$$

In a parallel fashion it is found that

$$[31b] \quad \frac{\partial \mathrm{E}\{\rho\}}{\partial y_0} = \frac{1}{A}\left\{\sum_{i=1}^{N}\left[\left(\frac{2A_i^2}{L_{i,3}^3}\right)(x_{i+1} - x_i)\right]\left[\ln\left(\frac{1 + r_i}{1 - r_i}\right)\right] + \sum_{i=1}^{N}\left[\left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right)(A_i)(y_{i+1} - y_i)\right]\right\}$$

Similar, but much less tedious derivations yield

$$[32a] \quad \frac{\partial \mathrm{E}\{\rho^2\}}{\partial x_0} = 2x_0 - \frac{1}{3A}\sum_{i=1}^{N} (x_{i+1} + x_i)\,(x_i y_{i+1} - x_{i+1} y_i)$$

and

$$[32b] \quad \frac{\partial \mathrm{E}\{\rho^2\}}{\partial y_0} = 2y_0 - \frac{1}{3A}\sum_{i=1}^{N} (y_{i+1} + y_i)\,(x_i y_{i+1} - x_{i+1} y_i)$$

It is easily found that

$$[33a] \quad \frac{\partial s}{\partial x_0} = \frac{x_0 - x_e}{s}$$

and

$$[33b] \quad \frac{\partial s}{\partial y_0} = \frac{y_0 - y_e}{s}$$

The six eqs. 31–33 when substituted into eqs. 13*a* and 13*b* give the required first derivatives of the total cost function.

The second partial derivatives are now developed. The basic approach to the derivations is the same as previously applied. Once again tedious algebra and some insight into the structure of the equations are required.

Upon taking the second partial derivative of eq. 31*a* with respect to x_0 and after performing some algebraic manipulation the following simplified equation is obtained

$$[34a] \quad \frac{\partial^2 \mathrm{E}\{\rho\}}{\partial x_0^2} = \frac{1}{A}\left\{\sum_{i=1}^{N}\left[\left(\frac{2A_i}{L_{i,3}^3}\right)(y_{i+1} - y_i)^2\right]\left[\ln\left(\frac{1+r_i}{1-r_i}\right)\right] - \sum_{i=1}^{N}\left[\left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right)(y_{i+1} - y_i)(x_{i+1} - x_i)\right]\right\}$$

and from [31*b*]

$$[34b] \quad \frac{\partial^2 \mathrm{E}\{\rho\}}{\partial y_0^2} = \frac{1}{A}\left\{\sum_{i=1}^{N}\left[\left(\frac{2A_i}{L_{i,3}^3}\right)(x_{i+1} - x_i)^2\right]\left[\ln\left(\frac{1+r_i}{1-r_i}\right)\right] + \sum_{i=1}^{N}\left[\left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right)(y_{i+1} - y_i)(x_{i+1} - x_i)\right]\right\}$$

and from either [31*a*] or [31*b*]

$$[34c] \quad \frac{\partial^2 \mathrm{E}\{\rho\}}{\partial y_0 \partial x_0} = \frac{1}{A}\left\{-\sum_{i=1}^{N}\left[\left(\frac{2A_i}{L_{i,3}^3}\right)(x_{i+1} - x_i)(y_{i+1} - y_i)\right]\left[\ln\left(\frac{1+r_i}{1-r_i}\right)\right] + \sum_{i=1}^{N}\left[\left(\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right)(x_{i+1} - x_i)^2\right]\right\}$$

For the careful reader who may have discerned a lack of anticipated symmetry in eq. 34*c*, it should be noted that it is not too difficult to show the following relationship:

$$[35] \quad \sum_{i=1}^{N}\left[\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right](y_{i+1} - y_i)^2 + \sum_{i=1}^{N}\left[\frac{L_{i,1} - L_{i,2}}{L_{i,3}^2}\right](x_{i+1} - x_i)^2 = 0$$

thus if there were some advantage in doing so the inherent symmetry of eq. 34*c* could be explicitly shown.

From eqs. 32*a* and 32*b* the following are obtained:

$$[36a] \quad \frac{\partial^2 \mathrm{E}\{\rho^2\}}{\partial x_0^2} = \frac{\partial^2 \mathrm{E}\{\rho^2\}}{\partial y_0^2} = 2$$

and

$$[36b] \quad \frac{\partial^2 \mathrm{E}\{\rho^2\}}{\partial x_0 \partial y_0} = 0$$

And from eqs. 33*a* and 33*b* the following are obtained:

$$[37a] \quad \frac{\partial^2 s}{\partial x_0^2} = \frac{(y_e - y_0)^2}{s^3}$$

$$[37b] \quad \frac{\partial^2 s}{\partial y_0^2} = \frac{(x_e - x_0)^2}{s^3}$$

and

$$[37c] \quad \frac{\partial^2 s}{\partial x_0 \partial y_0} = \frac{-(y_e - y_0)(x_e - x_0)}{s^3}$$

It is observed with regard to eqs. 33*a*, 33*b*, and 37 that a discontinuity of the partial derivatives exists at $x_0 = x_e$, $y_0 = y_e$. The significance of this discontinuity and how it is treated will be discussed in the section on numerical optimization.

The substitution of eqs. 34*a*–34*c*, 36*a*, 36*b*, and 37*a*–37*c* into eqs. 14*a*–14*c* yields the required second partial derivatives of the total cost function.

The inclusion of additional partitions entails only minor modification of the total cost objective function, eq. 12:

$$[38] \quad \underset{(x_0,\, y_0)}{\mathrm{MIN}}\ \mathrm{TC} = \sum_k [C_{0,k} + C_{1,k}\mathrm{E}_k\{\rho\} + C_{2,k}\,\mathrm{E}_k\{\rho^2\}] + C_3 s$$

Take partial derivatives of the total cost function, eq. 38; eqs. 34*a*–34*c* and 36*a*, 36*b* are then applied to each separate partition and summed as required.

The formula for the total cost function and its first and second partial derivatives have now been developed. With these formulas in hand the process of numerical optimization is next to be examined.

Numerical optimization

A FORTRAN program was written to provide numerical solutions to the model discussed in the preceding sections. A damped Newton method based on an algorithm recommended by Polak (1971) was selected. The gradient vector and the inverse of the Hessian matrix are calculated at each iteration, a process that is quite fast because of the simplicity of the partial derivatives. Termination criteria are those suggested by Gill et al. (1981) for smooth unconstrained problems. In order to use these termination criteria rather than those for a discontinuous function, a hyperbolic approximation was used for the road-length function:

$$[39] \quad s = [(x_e - x_0)^2 + (y_e - y_0)^2 + \varepsilon^2]^{\frac{1}{2}}$$

This approach to a similar discontinuity problem was suggested by Love and Morris (1975). The use of this approximating function is not required for the numerical solution of the problem. But with the problem definition remaining essentially unchanged, the unfettered use of the termination criteria for smooth unconstrained problems has definite programming advantages.

The first partial derivatives for this function are

$$[40a] \quad \frac{\partial s}{\partial x_0} = \frac{x_0 - x_e}{s}$$

and

TABLE 1. Computer execution time for multiple partitions

Reference	Reported time (s)	Machine	Estimated time (s)[a]
Love (1972)	21	Burroughs B-5500	25.2
Drezner and Weslowsky (1980)	0.1	CDC 6400	0.7
F.E. Greulich[b]	0.9	IBM AT	0.9

[a]If run on a 12-MHz Intel 80286/287.
[b]Based on current work.

$$[40b] \quad \frac{\partial s}{\partial y_0} = \frac{y_0 - y_e}{s}$$

where s is defined by eq. 39. The second partial derivatives are recalculated to be

$$[41a] \quad \frac{\partial^2 s}{\partial x_0^2} = \frac{(y_e - y_0)^2 + \varepsilon^2}{s^3}$$

$$[41b] \quad \frac{\partial^2 s}{\partial y_0^2} = \frac{(x_e - x_0)^2 + \varepsilon^2}{s^3}$$

and

$$[41c] \quad \frac{\partial^2 s}{\partial x_0 \partial y_0} = \frac{-(y_e - y_0)(x_e - x_0)}{s^3}$$

As can be readily seen in comparing these functions with those of eqs. 33*a*, 33*b* and 37*a*–37*c*, the selection of a sufficiently small ε^2 can force an arbitrarily close approximation to the original functional relationship while avoiding discontinuity in the partial derivatives. It should also be recognized that this modification means that the condition $C_3 \geq 0$ must now be imposed for convexity. This sign requirement is not unduly restrictive since in general haul road related costs should increase with road length.

Calculation of the gradient vector and the Hessian matrix can be expedited by exploiting certain structural relationships. Examination of eq. 34*a* with respect to eqs. 24 and 26 reveals that the second partial derivative with respect to x_0 can be written

$$[42a] \quad \frac{\partial^2 \mathrm{E}\{\rho\}}{\partial x_0^2} = \frac{2}{A} \sum_{i=1}^{N} \left[\frac{\partial A_i}{\partial x_0} \int_0^1 \frac{\partial R_i(\lambda)}{\partial x_0} \mathrm{d}\lambda \right]$$

likewise

$$[42b] \quad \frac{\partial^2 \mathrm{E}\{\rho\}}{\partial y_0^2} = \frac{2}{A} \sum_{i=1}^{N} \left[\frac{\partial A_i}{\partial y_0} \int_0^1 \frac{\partial R_i(\lambda)}{\partial y_0} \mathrm{d}\lambda \right]$$

and

$$[42c] \quad \frac{\partial^2 \mathrm{E}\{\rho\}}{\partial y_0 \partial x_0} = \frac{2}{A} \sum_{i=1}^{N} \left[\frac{\partial A_i}{\partial x_0} \int_0^1 \frac{\partial R_i(\lambda)}{\partial y_0} \mathrm{d}\lambda \right]$$

These forms are more computationally efficient, especially since an inspection of eqs. 31*a* and 26 reveals that the first partial derivative with respect to x_0 may be calculated by

$$[43a] \quad \frac{\partial \mathrm{E}\{\rho\}}{\partial x_0} = \frac{1}{A} \sum_{i=1}^{N} \left[A_i \int_0^1 \frac{\partial R_i(\lambda)}{\partial x_0} \mathrm{d}\lambda \right]$$

and likewise for y_0

$$[43b] \quad \frac{\partial \mathrm{E}\{\rho\}}{\partial y_0} = \frac{1}{A} \sum_{i=1}^{N} \left[A_i \int_0^1 \frac{\partial R_i(\lambda)}{\partial y_0} \mathrm{d}\lambda \right]$$

TABLE 2. Computer execution time for single convex polygon

Reference	Reported time (s)	Machine	Estimated time (s)[a]
Cavalier and Sherali (1986)	<0.01	IBM 3081 D24	<0.9
F.E. Greulich[b]	0.1–0.5	IBM AT	0.1–0.5

[a]If run on a 12-MHz Intel 80286/287.
[b]Based on current work.

For computational purposes the integral and partial derivative terms in the above five equations are replaced using the appropriate formulas; e.g., for eq. 42*a* above use eqs. 24 and 26.

The iterative optimization procedure must be supplied with an initial point. Inspection of eqs. 32*a* and 32*b* discloses that when set equal to zero, they easily provide an explicitly solved centroidal value (Blair and Biss 1967). Francis et al. (1983) note that the centroidal solution to the discrete point problem is sometimes used as a starting point for the Weiszfeld algorithm. A centroidal starting point has obvious intuitive appeal and, at the very least, would appear to be a not unreasonable alternative to a user-supplied starting point. Accordingly, a centroidal value calculated on the basis of eqs. 32*a* and 32*b* yields the recommended default initial point.

To obtain some estimate of the relative efficiency of this algorithm, several test runs were made. A program was written in ANSI 77 FORTRAN, compiled using the Microsoft FORTRAN compiler (version 5.0), and run on a 12-MHz 80286/287 personal computer. A test problem given by Love (1972) was used as one basis of comparison. The right-hand column of Table 1 gives estimated execution times for other optimization models previously discussed. Time estimates are based on information provided in the referenced articles as well as on computer execution rates given in Dongarra (1985). Table 2 presents a second comparison based on a convex 15-sided polygon. A range of executive times is given for the Newton method of this paper. The required solution time is a function of the polygon's central symmetry; the greater the departure from central symmetry the longer the execution time. This characteristic of the algorithm is due to use of the centroid as the initial point. Interpretation of these execution time comparisons should be done cautiously. As noted by Dongarra (1985), tabulated and observed times may differ because of subsequent changes in systems' software and hardware. Compilers differ as do programmer skills. In general, however, it is safe to conclude from these figures that this procedure, in spite of its generality, has an execution time that is competitive with alternative methods of optimization for the type of unconstrained optimization considered in this paper.

Concluding observations

Forest engineers now have access to an easily implemented procedure for finding the optimal location of a single landing. Within its other limitations, this optimal location model should find wide application in forest areas of relatively flat terrain where a primary road system is already in place. Data requirements for the model are quite modest, and in many cases data should be either immediately available or quickly acquired with minimal effort. Forest lands that have been put into a geographical information system are particularly amenable to model implementation.

The underlying simplicity of this model suggests that great strides can be made in further model development in the near term. Among current model limitations that merit closer examination are the following.

The model should be extended to sloping planar regions. The productivity of cable systems, and especially tractive yarding systems, is sensitive to ground slope. The inclusion of average yarding slope in the yarding cost function of the model would significantly extend its range of application.

The capability of considering more than one landing location is an obvious extension of the current model. Several major issues will have to be addressed. Among these issues are the loss of model convexity, bifurcation of the road system, and road standard transition points.

Constrained optimization is a third area of particular interest. For spatial location constraints the convexity (or lack of convexity) of the feasible region will be a consideration. The straight-line yarding assumption will also have to be examined if it is desirable to prohibit yarding through designated zones.

Appendix

The mathematical model presented in this paper leads to the unconstrained minimization of a strictly convex function. Because of strict convexity there exists one, and only one, landing location that minimizes total cost. The strict convexity of the model also simplifies the algorithm by which that unique minimum point can be identified. Because of its importance and because it is not immediately obvious that the objective function is strictly convex, it is appropriate to examine that property in more detail. For the sake of brevity standard analytical results that can be found elsewhere will not be given; references for them will be listed.

Love (1972) has supplied a proof that functions of the form

$$\text{[A1]} \qquad \text{TC} = \sum_k C_{1,k}\text{E}_k\{\rho\}$$

are convex. In fact, as asserted by Cavalier and Sherali (1986), this function is strictly convex.

The argument offered here for the strict convexity of eq. A1 follows a quite different route from that taken by Love. In the argument to follow, the second partial derivatives of the average yarding distance (AYD) function for the setting are examined and found to fulfill the conditions for strict convexity. One incidental advantage of this approach is that some additional structural characteristics of this particular function are revealed during the development.

From eq. 21 the second partial derivative with respect to x_0 is obtained:

$$\text{[A2]} \qquad \frac{\partial^2\text{AYD}}{\partial x_0^2} = \frac{1}{A}\sum_{i=1}^{N}\left[2\frac{\partial A_i}{\partial x_0}\frac{\partial \text{AYD}_i}{\partial x_0} + A_i\frac{\partial^2\text{AYD}_i}{\partial x_0^2}\right]$$

Another way of writing eq. 42*a* using equivalent notation is

$$\text{[A3]} \qquad \frac{\partial^2\text{AYD}}{\partial x_0^2} = \frac{3}{A}\sum_{i=1}^{N}\left[\frac{\partial A_i}{\partial x_0}\frac{\partial \text{AYD}_i}{\partial x_0}\right]$$

and upon equating right-hand sides of eqs. A2 and A3

$$\text{[A4]} \qquad \sum_{i=1}^{N}\left[\frac{\partial A_i}{\partial x_0}\frac{\partial \text{AYD}_i}{\partial x_0}\right] = \sum_{i=1}^{N}\left[A_i\frac{\partial^2\text{AYD}_i}{\partial x_0^2}\right]$$

Back substitute into [A3] to obtain

$$\text{[A5]} \qquad \frac{\partial^2\text{AYD}}{\partial x_0^2} = \frac{3}{A}\sum_{i=1}^{N}\left[A_i\frac{\partial^2\text{AYD}_i}{\partial x_0^2}\right]$$

In similar fashion the following are found:

$$\text{[A6]} \qquad \frac{\partial^2\text{AYD}}{\partial y_0^2} = \frac{3}{A}\sum_{i=1}^{N}\left[A_i\frac{\partial^2\text{AYD}_i}{\partial y_0^2}\right]$$

and

$$[A7]\quad \frac{\partial^2 \text{AYD}}{\partial x_0 \partial y_0} = \frac{3}{A}\sum_{i=1}^{N}\left[A_i \frac{\partial^2 \text{AYD}_i}{\partial x_0 \partial y_0}\right]$$

It is interesting to note at this point that the second partial derivatives of the overall AYD for any setting partition may be calculated as the corresponding area-weighted second partial derivatives of the AYDs for the triangles associated with the individual line segments.

For $E\{\rho\}$, equivalently AYD, to be strictly convex, it is sufficient that the principal minors of its Hessian matrix be greater than zero (Peressini et al. 1988). The Hessian matrix is written as

$$[A8]\quad \text{HE}\{\rho\} = \begin{pmatrix} \dfrac{\partial^2 \text{AYD}}{\partial x_0^2} & \dfrac{\partial^2 \text{AYD}}{\partial x_0 \partial y_0} \\ \dfrac{\partial^2 \text{AYD}}{\partial y_0 \partial x_0} & \dfrac{\partial^2 \text{AYD}}{\partial y_0^2} \end{pmatrix}$$

with the requirements for strict convexity then being

$$[A9]\quad \frac{3}{A}\sum_{i=1}^{N}\left[A_i \frac{\partial^2 \text{AYD}_i}{\partial x_0^2}\right] > 0$$

and

$$[A10]\quad \left\{\frac{9}{A^2}\right\}\left\{\left[\sum_{i=1}^{N}\left(A_i \frac{\partial^2 \text{AYD}_i}{\partial x_0^2}\right)\right]\left[\sum_{i=1}^{N}\left(A_i \frac{\partial^2 \text{AYD}_i}{\partial y_0^2}\right)\right] - \left[\sum_{i=1}^{N}\left(A_i \frac{\partial^2 \text{AYD}_i}{\partial x_0 \partial y_0}\right)\right]^2\right\} > 0$$

To show that relationships [A9] and [A10] hold true, consider the planar region shown in Fig. A1. The planar region is bounded by a smooth curve. This boundary curve is divided into two segments defined by the single-valued functions $R_1(\theta)$ and $R_2(\theta)$. Beginning and ending points of the two boundary segments are defined by radial tangents to the region. These radial lines emanate from (x_0, y_0) and form angles θ_1 and θ_2 with the arbitrarily established reference axis.

Parenthetically it is noted that these boundary functions need only be piecewise continuously differentiable; e.g., a sequence of connected line segments. Also, for any segment of the boundary radial to the landing location, the function $R_i(\theta)$ for that segment is defined as a jump discontinuity. Assume that at angle θ_a there is a jump discontinuity, then for the immediately preceding segment interpret

$$[A11a]\quad R_{i-1}(\theta_a) = \lim_{\theta \to \theta_a^+} R_{i-1}(\theta)$$

or

$$[A11b]\quad R_{i-1}(\theta_a) = \lim_{\theta \to \theta_a^-} R_{i-1}(\theta)$$

as appropriate. Do the same for the segment $R_{i+1}(\theta)$ that follows the discontinuity.

From previous work (Greulich 1987) it is known that the AYD from the landing location to a smooth curve that delimits the setting boundary (defined by $R(\theta)$, a single-valued function) is given by

$$[A12]\quad \text{AYD} = E\{\rho\} = \frac{2}{3}\,\frac{\int_{\theta_1}^{\theta_2} R^3(\theta)\,\mathrm{d}\theta}{\int_{\theta_1}^{\theta_2} R^2(\theta)\,\mathrm{d}\theta}$$

The limit is taken and found to be indeterminate:

$$[A13]\quad \lim_{\theta_2 \to \theta_1} E\{\rho\} = \frac{0}{0}$$

By applying L'Hôpital's rule and using the general formula for differentiation under the integral sign (Apostol 1957), the limit is found:

$$[A14]\quad \lim_{\theta_2 \to \theta_1}\left[\frac{2}{3}R(\theta_2)\right] = \frac{2}{3}R(\theta_1)$$

This result was previously known to be true for linear and circular boundary segments, but as seen here, not unexpectedly perhaps, it is true for any smoothly curved boundary segment.

This value for the instantaneous AYD, AYD_I, may be written using either polar or Cartesian coordinates:

$$[A15a]\quad \text{AYD}_\text{I}(\theta_1) \equiv \text{AYD}_\text{I}(x_1, y_1)$$

or

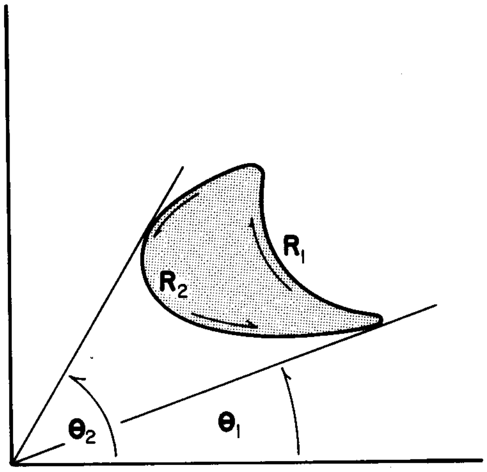

FIG. A1. Simple planar region bounded by a smooth curve.

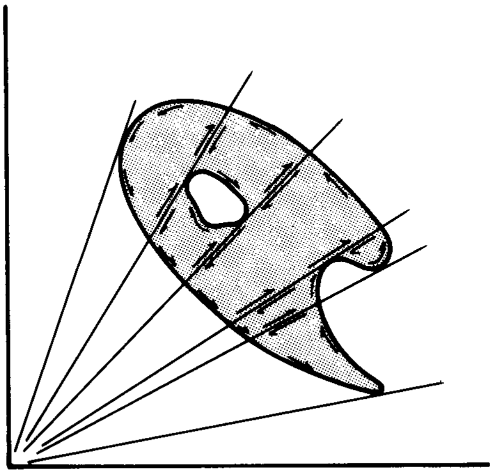

FIG. A2. Complex planar region.

[A15*b*] $\frac{2}{3} R(\theta_1) \equiv \frac{2}{3} L(x_1, y_1)$

where

[A16] $L(x_1, y_1) \equiv [(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\frac{1}{2}}$

Taking partial derivatives of the Cartesian coordinate form of $\mathrm{AYD_I}$, the following results are obtained:

[A17] $$\frac{\partial^2 \mathrm{AYD_I}}{\partial x_0^2} = \frac{2}{3}\left[\frac{(y_1 - y_0)^2}{L^3}\right]$$

[A18] $$\frac{\partial^2 \mathrm{AYD_I}}{\partial y_0^2} = \frac{2}{3}\left[\frac{(x_1 - x_0)^2}{L^3}\right]$$

[A19] $$\frac{\partial^2 \mathrm{AYD_I}}{\partial x_0 \partial y_0} = \frac{2}{3}\left[\frac{-(x_1 - x_0)(y_1 - y_0)}{L^3}\right]$$

These equations are now changed to polar coordinates:

[A20] $$\frac{\partial^2 \mathrm{AYD_I}}{\partial x_0^2} = \frac{2}{3}\left[\frac{\sin^2(\theta)}{R(\theta)}\right]$$

[A21] $$\frac{\partial^2 \mathrm{AYD_I}}{\partial y_0^2} = \frac{2}{3}\left[\frac{\cos^2(\theta)}{R(\theta)}\right]$$

and

[A22] $$\frac{\partial^2 \mathrm{AYD_I}}{\partial x_0 \partial y_0} = \frac{2}{3}\left[\frac{-\sin(\theta)\cos(\theta)}{R(\theta)}\right]$$

As is well known from the calculus

[A23] $dA = (½) R^2(\theta)\, d\theta$

for the differential area between the origin and the smooth curve.

With these results it is seen that as $N \rightarrow \infty$ (and also $A_i \rightarrow 0\ \forall i$) over segment $[\theta_1, \theta_2]$ of a smooth boundary curve described by the single-valued function $R(\theta)$, the following limit is approached:

[A24] $$\sum_{i=1}^{N}\left[A_i \frac{\partial^2 \mathrm{AYD}_i}{\partial x_0^2}\right] \doteq \frac{1}{3}\int_{\theta_1}^{\theta_2} R(\theta)\sin^2(\theta)\, d\theta$$

Hence, referring back to Fig. A1, the left-hand side of eq. A9 may be written in its limit form as

[A25] $$\left[\frac{1}{\mathrm{A}}\right]\left[\int_{\theta_1}^{\theta_2} R_1(\theta)\sin^2(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} R_2(\theta)\sin^2(\theta)\, d\theta\right]$$

Upon rewriting [A25] and collecting terms, it is clearly seen that the first principal minor must be strictly greater than zero:

$$[A26]\quad \frac{1}{A}\left\{\int_{\theta_1}^{\theta_2} [R_1(\theta) - R_2(\theta)][\sin^2(\theta)]\, d\theta\right\} > 0$$

Both bracketed terms of the integrand are everywhere non-negative in $[\theta_1, \theta_2]$, and in fact, the first term must be positive over some portion of the interval for the AYD to exist as it must for any case of interest.

The left-hand side of eq. A10 may also be written in its limit form:

$$[A27]\quad \frac{1}{A^2}\left\{\left[\int_{\theta_1}^{\theta_2} \sin^2(\theta)\, d\mathscr{R}\right]\left[\int_{\theta_1}^{\theta_2} \cos^2(\theta)\, d\mathscr{R}\right] - \left[\int_{\theta_1}^{\theta_2} \sin(\theta)\cos(\theta)\, d\mathscr{R}\right]^2\right\}$$

where

$$[A28]\quad \mathscr{R} \equiv \int [R_1(\theta) - R_2(\theta)]\, d\theta$$

which by the integral form of the Cauchy-Schwarz inequality (Apostol 1957) is also found to be strictly greater than zero since

$$[A29]\quad \sin(\theta) \neq K\cos(\theta)$$

for any real K (Gradshteyn and Ryshik 1980).

Even the most general planar regions can be broken up into subregions of the form just examined; see, for example, Fig. A2. The total AYD is a positive sum of the strictly convex AYDs of these subregions and is itself therefore strictly convex (Peressini et al. 1988).

The Hessian matrix for the expected square of the yarding distance, ED2 or $E\{\rho^2\}$, is easily found from eqs. 36*a* and 36*b*

$$[A30]\quad \text{HE}\{\rho^2\} = \begin{pmatrix} \dfrac{\partial^2\text{ED2}}{\partial x_0^2} & \dfrac{\partial^2\text{ED2}}{\partial x_0 \partial y_0} \\ \dfrac{\partial^2\text{ED2}}{\partial y_0 \partial x_0} & \dfrac{\partial^2\text{ED2}}{\partial y_0^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

It is seen immediately that the principal minors of this matrix are also positive, implying strict convexity of the function. In this situation, as in the case of the AYD function, the appearance of this component in the cost function of additional partitions presents no difficulty. As long as the cost coefficients, $C_{2,k}$, are non-negative for all k, the total cost function will also be strictly convex.

In its original form the road cost function defined by eqs. 7 and 8 is convex, but it is not strictly convex because it exhibits linear change in directions radial to the road takeoff point. This characteristic manifests itself in the zero value of the second principal minor of the Hessian matrix formed from eqs. 37*a*–37*c*. The fact that this function is not strictly convex is inconsequential to whether or not the total cost function is strictly convex. Both of the previously discussed functions associated with yarding cost are strictly convex. The addition of even a single strictly convex function to a sum of convex functions makes the summing function (the total cost function) strictly convex (Peressini et al. 1988). The alternative use of a hyperbolic approximation for computational purposes makes the road cost function strictly convex. This fact can be easily shown by examining the Hessian matrix formed for eqs. 41*a*–41*c*.

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