ECON 401

Advanced Macroeconomics

Fabio Ghironi
University of Washington
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Topic 1

The Stochastic Growth Model

Fabio Ghironi
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Introduction

- This course explores modern theories of macroeconomic fluctuations.

- The goal will be to take you as close as possible to understanding how many macroeconomists at academic and policy institutions think about business cycles and policy questions, including the crisis created by COVID-19, by studying a sequence of models.

- We will start from the stochastic growth model (also known as real business cycle—RBC—model), in which fluctuations are the result of random shocks to technology and economic outcomes are efficient.

Introduction, Continued

- In this course, the word *efficiency* has a very precise meaning:
  - The market economy is efficient when the outcome it generates is the same as the outcome that would be chosen by a benevolent social planner in charge of allocating resources.

- The market economy is efficient in the RBC model: A benevolent planner who acts to maximize social welfare would not do better than the market.

- This implies that this is a model in which there is no role for policy to improve on market outcomes.

Introduction, Continued

- We will study the RBC model not because we believe that it is an accurate, realistic theory of how the macroeconomy works, but because it is a useful starting point to become familiar with concepts, tools, and techniques that we will use many times throughout the quarter.

- We will then introduce a number of more realistic features into our framework: monopoly power, nominal rigidity, financial market frictions, labor market imperfections, producer entry dynamics, heterogeneity across agents, and more.

- These features will imply that the economy we model is no longer efficient: Policy can improve outcomes relative to the market.

- We will conclude the course with an example of how the tools we study can be used to analyze the ongoing COVID-19 crisis.

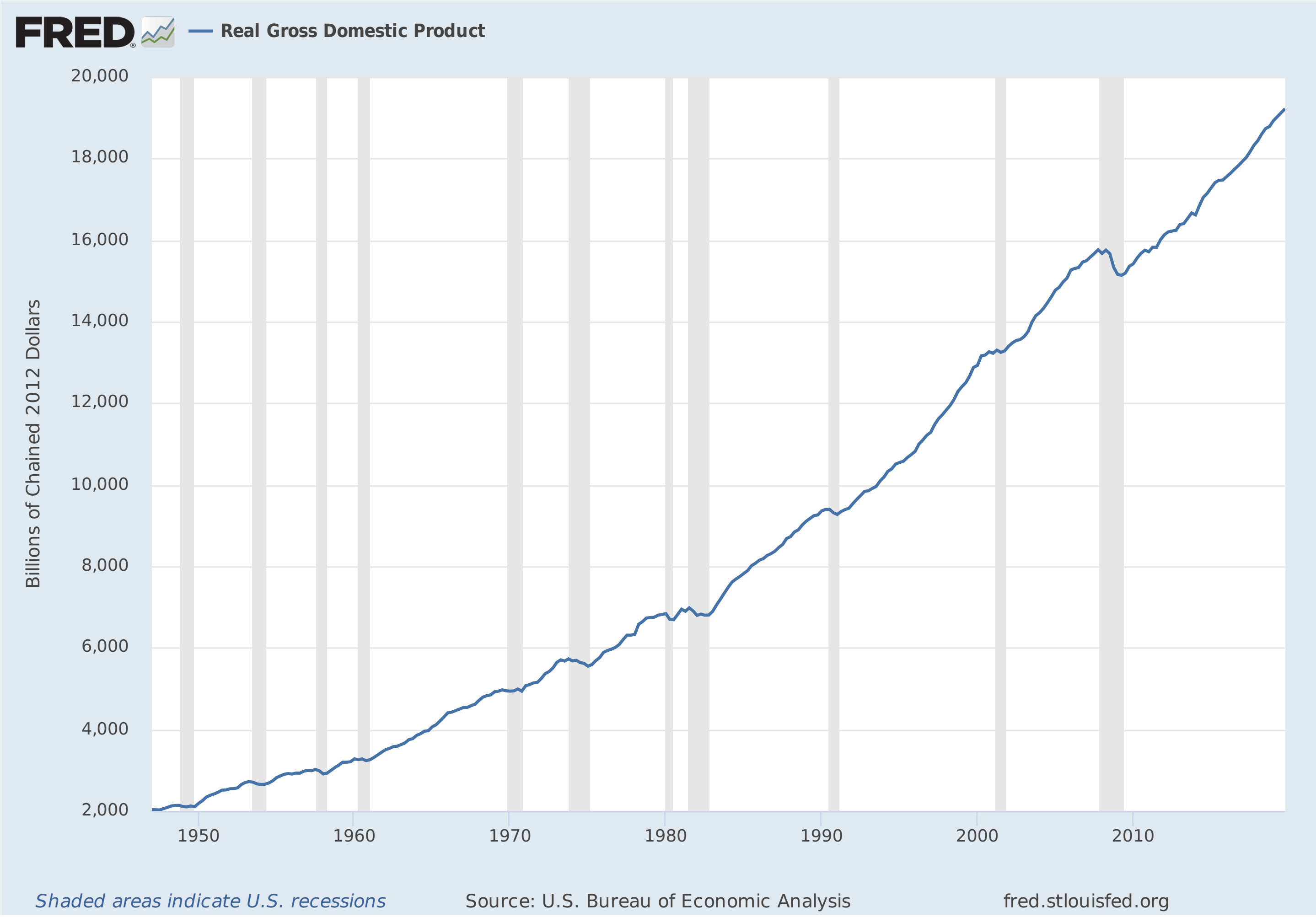
Introduction, Continued

- The tools used in the RBC model became the foundation of the mainstream framework for studying macroeconomic fluctuations in the 1980s, starting with seminal work by Finn Kydland of U.C. Santa Barbara and Edward Prescott of Arizona State University published in *Econometrica* in 1982.

- The model studies fluctuations of the economy around its growth trend (business cycles) triggered by unexpected, random shocks, assuming that agents in the economy act to optimize intertemporal objective functions under rational expectations about the future.

- In its standard versions, the analysis assumes that shocks generate departures from trend that disappear over time: For instance, an unexpected improvement in technology causes the economy's GDP to rise above trend for some time, but eventually the effect of the shock disappears, and the economy is back chugging along its trend growth path.

- The figure in the next slide shows the behavior of U.S. GDP since 1947. It gives you an illustration of situations in which the standard approach can work (much of the time) but also situations in which it will do very poorly (the aftermath of the Great Recession that followed the Global Financial Crisis of 2007-08).
  - We do not have the data yet, but the standard approach to the RBC model that we are going to study will most likely be a very bad tool to think also about the current COVID-19 crisis.



FRED
Real Gross Domestic Product
20,000
18,000
16,000
14,000
12,000
10,000
8,000
6,000
4,000
2,000
Billions of Chained 2012 Dollars
1950
1960
1970
1980
1990
2000
2010
Shaded areas indicate U.S. recessions
Source: U.S. Bureau of Economic Analysis
fred.stlouisfed.org

Introduction, Continued

- As we were saying, market outcomes are efficient in the basic RBC model.

- This happens because there are no *distortions* (or *frictions*) that would cause a benevolent social planner to choose outcomes that differ from the market ones.

- But the usefulness of the model *does not* depend the assumption that business cycles are triggered by technology shocks or that of an efficient model-economy.

- Many scholars have studied the consequences of other types of shocks and of departures from an efficient environment by modifying the model and the approach to it in the directions we will talk about.

Introduction, Continued

- We will introduce distortions, obstacles to the smooth functioning of markets, realistic features that will allow us to tackle questions that the basic RBC setup cannot address—including issues that have taken center stage in discussions on macroeconomics since the Great Recession.

- Studying the RBC model will prepare us to study those more realistic models, and it will help us understand exactly when and why market outcomes in those models are not efficient, and therefore when and why there is a role for policy in addressing those suboptimal outcomes.
  - Put differently, understanding the outcomes in an efficient model-economy helps us understand the mechanisms through which distortions lead to inefficient outcomes when we introduce realistic features in our models, and when and how policy action can be optimal.

Solving Models

- In studying the RBC model, we will pay special attention to the procedure for solving it.
- The difficulty in solving the model is a fundamental non-linearity that arises from the interaction between multiplicative elements, such as Cobb-Douglas production, and additive elements, such as capital accumulation and depreciation.
- This non-linearity makes it impossible, in general, to solve the model without resorting to some kind of approximation.
  - The only case in which this problem does not arise is when capital depreciates fully in one period and agents utility from consumption takes the logarithmic form.
  - This is a very special, unrealistic combination of assumptions.

Solving Models, Continued

- The solution method that we will study for the more general scenario is called *log-linearization*.

- It starts from the model's optimality conditions and budget constraints and transforms them into a system of log-linear expectational difference equations in which all endogenous variables are function of the capital stock and of the exogenous shocks that cause fluctuations.

- Variables in the log-linearized model measure *percentage deviations of original variables from their trend (or steady-state) levels*.
  - We will use the words trend and steady state interchangeably, with the understanding that underlying variables are constant in steady state only if long-run growth is zero, otherwise they are moving at their trend-growth rate.

- The approximated model can then be solved using a method known as the *method of undetermined coefficients*.

- An advantage of this solution method over alternatives is that it can be applied also to models in which the market outcome is not efficient.

Solving Models, Continued

- There are plenty of situations in which you would not want to log-linearize your model (and therefore assume that your variables always display a tendency to return to the steady state around which you approximated the original, non-linear model).
  - There are cases in which log-linearization is used also in non-stationary environments (scenarios in which the economy does not eventually return to its steady state, or trend, after shocks), but this is appropriate only for specific types of exercises.

- Log-linearization limits the range of questions you can study, or it can yield very misleading conclusions.

- For example, log-linearization cannot handle phenomena like the Great Recession and the years that followed and situations in which accounting for nonlinearity (like the zero—or effective—lower bound on central bank policy interest rates) is necessary to understand what is happening.

- But log-linearization is still used to work on many other interesting, important questions and understanding how it works also helps us understand its limitations and why alternative, more complicated techniques become necessary in other scenarios.

Households in the Basic RBC Model

- Consider an economy populated by a large number of identical, infinitely lived households, all subject to the same uncertainty.

- At time t, the representative household maximizes its expected intertemporal utility from t to the infinite future, discounting utility in future periods according to a discount factor β:

$$E_t\left[\sum_{s=t}^{\infty}\beta^{s-t}u(C_s)\right] = E_t\left(\sum_{s=t}^{\infty}\beta^{s-t}\frac{C_s^{1-\gamma}}{1-\gamma}\right), \quad 0<\beta<1, \tag{1}$$

  where E_t denotes the expectation based on the information available at time t, $\sum_{s=t}^{\infty}$ is the summation operator for time that goes from the current period (t) all the way to infinite, and C_s is consumption in period s ($s = t, ..., \infty$).

- We assume that this expectation is rational, *i.e.*, the household uses optimally all the information that is available to it.

  - Much macroeconomic literature studies the consequences of departures from rationality. It is one of the many topics that, unfortunately, we do not have time to study. Michael Woodford of Columbia University has been doing very interesting work in this area recently.

The Intertemporal Utility Function

- The expression

$$E_t\left(\sum_{s=t}^{\infty}\beta^{s-t}\frac{C_s^{1-\gamma}}{1-\gamma}\right)$$

  is a compact way of writing

$$E_t\left(\frac{C_t^{1-\gamma}}{1-\gamma}+\beta\frac{C_{t+1}^{1-\gamma}}{1-\gamma}+\beta^2\frac{C_{t+2}^{1-\gamma}}{1-\gamma}+...+\beta^{\infty}\frac{C_{t+\infty}^{1-\gamma}}{1-\gamma}\right).$$

- Discounting by β captures the idea that households care about utility from current consumption more than they care about utility from future consumption.
  - Very interesting literature has explored the consequences of different forms of discounting. See, for instance, work by David Laibson of Harvard University.

The Intertemporal Utility Function, Continued

- $\gamma > 0$ is the coefficient of relative risk aversion:
  - It measures the attitude of our representative household toward risk (uncertainty).
  - If γ were equal to zero—linear utility—the household would not care about risk. It would be perfectly indifferent between a certain level of consumption and an uncertain one.
  - If you are not familiar with the concept of risk aversion, you find more information in Appendix A.

The Intertemporal Utility Function, Continued

- Let us define the parameter $\sigma \equiv \frac{1}{\gamma}$.

- This is known as the elasticity of intertemporal substitution.
  - As we shall see, it measures the responsiveness of consumption to interest rate changes: the willingness of agents to postpone or anticipate consumption across periods in response to movement in interest rates.

- Tight connection between attitude toward uncertainty (γ) and toward intertemporal substitution (σ) is an undesirable feature of the model when studying important questions (for instance, related to asset pricing).

- Larry Epstein of Boston University and Stanley Zin of NYU developed a framework that unties risk aversion from intertemporal substitution. Their work, published in the *Journal of Political Economy* in 1989, became widely used to address important questions.

- We will stick to the basic framework, keeping in mind that it has significant limitations (see Appendix A for an example).

Capital Accumulation and Labor Supply

- Households can accumulate a single asset, homogeneous physical capital, K_t and

$$K_{t+1} = (1-\delta)K_t + I_t, \quad 0 < \delta \leq 1 \tag{2}$$

  where K_t is the capital stock with which the household begins period t, I_t is investment in period t, δ is the rate of capital depreciation, and K_{t+1} is the stock of capital with which the household will begin period $t+1$.

- Each household supplies a fixed amount of labor ($N_t = 1$) in each period in a perfectly competitive labor market.

The Budget Constraint

- The representative household's consumption is constrained by:

$$C_t + I_t = \widetilde{r}_t K_t + w_t, \tag{3}$$

where $\widetilde{r}_t$ is the rental rate the household receives from the firms that rent its capital in a perfectly competitive rental market, and w_t is the wage ($N_t = 1$ implies $w_t N_t = w_t =$ labor income).

  – This is the household's period budget constraint: A constraint like this applies to every period (to every s, for $s = 0, ..., \infty$).

- (2) and (3) imply that the household's budget constraint can be rewritten as:

$$C_t + K_{t+1} = (1 + \widetilde{r}_t - \delta)K_t + w_t. \tag{4}$$

- The problem of the household is to maximize (1) subject to (4).

- How do we solve such intertemporal optimization problem?

Solving the Household's Problem: Intuitive Approach

- Let us start with in intuitive approach.

- Suppose we have a household that must decide what to do with 1 dollar in the current period (t): it can use it to buy consumption in t (let us assume that 1 unit of consumption costs 1 dollar) or it can invest it in an asset that will generate the uncertain gross return R_{t+1} at time $t+1$.

- Consider what the two possible choices do for the household:

- If it uses the dollar to buy consumption, it obtains the benefit given by the increment in utility from consuming an extra unit of consumption today—the marginal utility of consumption: $u'(C_t)$.

- If it invests the dollar, it will receive the return R_{t+1} at time $t+1$. In terms of the utility increment generated by the extra consumption this allows the household to do at $t+1$, this translates into $u'(C_{t+1})R_{t+1}$.

Notation Digression

- When we are dealing with functions of only one variable, we will denote the first derivative by using a superscript "$'$" and the second derivative by using a superscript "$''$."
- When we are dealing with functions of more than one variable, we will denote the first partial derivative with respect to a variable by having that variable indicated once as subscript and the second derivative by having the variable indicated twice as subscript.
  - Example: The first derivatives of the function $f(x, y)$ with respect to x and to y are denoted $f_x(x, y)$ and $f_y(x, y)$, respectively, and the second derivatives are $f_{xx}(x, y)$ and $f_{yy}(x, y)$ (I am omitting cross derivatives, assuming things are clear).
- An alternative way of indicating partial first derivatives that may appear in these course slides will be to use numerical subscripts referring to the variable with respect to which we are taking the derivative.
  - Example: The first derivatives of the function $f(x, y)$ with respect to x and to y are denoted $f_1(x, y)$ and $f_2(x, y)$, respectively, and the second derivatives are $f_{11}(x, y)$ and $f_{22}(x, y)$ (again omitting cross derivatives).
- Do *not* use "$'$" or "$''$" superscripts when denoting derivatives of functions of more than one variable in this course.
- Why? Because $f'(x, y)$ does not tell anyone with respect to what variable the derivative is taken!

Solving the Household's Problem: Intuitive Approach, Continued

- But the household does not know $C_{t+1}$and R_{t+1} with certainty at time t, when it is taking its decision. Hence, it will compute its expectation of $R_{t+1}u'(C_{t+1})$ based on the information it has at time t: $E_t\left[u'(C_{t+1})R_{t+1}\right]$.

- Moreover, in comparing the benefit of consuming today to that of investing (and thus postponing consumption to the next period), the household will discount the future benefit with the discount factor β.

- Hence, the household will compare $u'(C_t)$ and $\beta E_t\left[u'(C_{t+1})R_{t+1}\right]$.

- When is the household happy with the allocation of its resources across periods?

Solving the Household's Problem: Intuitive Approach, Continued

- When it is indifferent between the two alternatives!

- In other words, for the household's behavior to be optimized, it must be the case that:

$$u'(C_t) = \beta E_t \left[u'(C_{t+1}) R_{t+1} \right].$$

- This optimality condition is known as Euler equation.

- In our model, the asset the household can invest in is capital, and the return that an extra unit of capital today generates at $t+1$ is $1 + \widetilde{r}_{t+1} - \delta$: the undepreciated portion of that unit of capital plus the rental rate that it generates.

- Hence, the Euler equation for optimal capital accumulation in our model is:

$$u'(C_t) = \beta E_t \left[u'(C_{t+1})(1 + \widetilde{r}_{t+1} - \delta) \right],$$

or, given the assumed form of the period utility function,

$$C_t^{-\gamma} = \beta E_t \left[C_{t+1}^{-\gamma}(1 + \widetilde{r}_{t+1} - \delta) \right]. \tag{5}$$

Solving the Household's Problem: Doing the Math

- Now let us show how we can obtain this equation by doing math.
- The budget constraint (4) can be rearranged as:

$$C_t = -K_{t+1} + (1 + \widetilde{r}_t - \delta)K_t + w_t. \tag{6}$$

- Recall that the household faces a constraint like (4) in every period—put differently, it faces a sequence of constraints like (4) for time that goes from t to ∞.
- In the generic period s, it has to be:

$$C_s = -K_{s+1} + (1 + \widetilde{r}_s - \delta)K_s + w_s. \tag{7}$$

- We can substitute this constraint for C_s in the objective of the household, which will therefore be maximizing:

$$E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \frac{[-K_{s+1} + (1 + \widetilde{r}_s - \delta)K_s + w_s]^{1-\gamma}}{1-\gamma} \right\}. \tag{8}$$

Solving the Household's Problem: Doing the Math

- What does the household choose?
- The household takes the rental rate and the wage as given—as we mentioned above, they are determined in perfectly competitive markets in which all agents are price takers.
- Moreover, at any time s, $\acute{K}_s$ is *predetermined*: It is the capital stock with which the household *begins* the period. It was determined in the previous period.
- Having substituted investment and consumption out of the problem through our manipulation of constraints and substitutions (the substitution of (2) into (3), and the substitution of (7) into (1)) leaves K_{s+1} as the only variable that the household actually chooses at any time s.

Solving the Household's Problem: Doing the Math

- Without loss of generality, focus on $s = t$. The first-order condition for the household's optimal behavior follows from setting the derivative of (8) with respect to $K_{t+1:}$ equal to 0.

- To find this derivative most transparently, note what happens if we write the summation in (8) explicitly. The household maximizes:

$$\frac{\left[-K_{t+1} + (1 + \widetilde{r}_t - \delta)K_t + w_t\right]^{1-\gamma}}{1-\gamma} + \beta E_t \left\{ \frac{\left[-K_{t+2} + (1 + \widetilde{r}_{t+1} - \delta)K_{t+1} + w_{t+1}\right]^{1-\gamma}}{1-\gamma} \right\}$$
$$+\beta^2 E_t \left\{ \frac{\left[-K_{t+3} + (1 + \widetilde{r}_{t+2} - \delta)K_{t+2} + w_{t+2}\right]^{1-\gamma}}{1-\gamma} \right\} + ...$$

- Note also that everything in the first term is known at time t (K_{t+1} is chosen at t). Therefore, we can drop the expectation operator from that term.

- As you see, $K_{t+1:}$ appears in two consecutive terms of this expression. Hence, taking the derivative yields:

$$-(1-\gamma)\frac{\left[-K_{t+1} + (1 + \widetilde{r}_t - \delta)K_t + w_t\right]^{-\gamma}}{1-\gamma}$$
$$+\beta E_t \left\{ \frac{(1-\gamma)\left[-K_{t+2} + (1 + \widetilde{r}_{t+1} - \delta)K_{t+1} + w_{t+1}\right]^{-\gamma}}{1-\gamma} (1 + \widetilde{r}_{t+1} - \delta) \right\}$$

Solving the Household's Problem: Doing the Math

- If you simplify the $1-\gamma$ terms, substitute (7) for $s=t$ and $s=t+1$, respectively, in the first and in the second term of this expression, and set it equal to 0, you immediately find:

  $$-C_t^{-\gamma}+\beta E_t\left[C_{t+1}^{-\gamma}(1+\widetilde{r}_{t+1}-\delta)\right]=0,$$

  or

  $$C_t^{-\gamma}=\beta E_t\left[C_{t+1}^{-\gamma}(1+\widetilde{r}_{t+1}-\delta)\right],$$

  i.e., the Euler equation (5).

- A sequence of such equations (one for every $s=t,...,\infty$) must be satisfied for the household to be optimizing its consumption and investment behavior over time.

The Transversality Condition

- It turns out that the Euler equation is actually not the only optimality condition the household must satisfy:

- The Euler equation describes optimal behavior between any two consecutive periods (s and $s+1$, for $s = t, ..., \infty$), but the household is solving an infinite horizon problem that requires it to look beyond any pair of consecutive periods.

- The additional condition that must be satisfied is known as *transversality conditions*, and it has this form:

$$\lim_{T \longrightarrow \infty} E_t \left[\beta^T u'(c_{t+T})(1 + \widetilde{r}_{t+T} - \delta) K_{t+T} \right] = 0. \tag{9}$$

- We are not going to do the math to show why this condition must hold.

The Transversality Condition, Continued

- Intuitively, if the expression on the left-hand side were strictly positive, the household would be overaccumulating capital, so that a higher expected lifetime utility could be achieved by increasing consumption today.

- The counterpart to such non-optimality in a *finite* horizon model would be that the household dies with positively valued capital holdings: There is no bequest motive in our model for which anyone would want to die with positively valued assets!

- Condition (9) cannot be violated on the negative side because the marginal utility of consumption is never negative, $0 < \delta \leq 1$, and capital (a factor of production) must be positive.

Euler Equations and Transversality Conditions

- One way to look at Euler equations and transversality conditions is to observe that Euler equations rule out arbitrage opportunities between consecutive periods (when the Euler equation holds, the household cannot increase its utility by changing consumption and capital holdings between two consecutive periods).

- Transversality conditions rule out permanent/infinite-horizon arbitrages (the household cannot increase its utility by increasing consumption permanently).

- Euler equations represent short-run optimality conditions, which all candidate paths for optimality of consumption and investment must satisfy, while the transversality condition gives an additional long-run optimality condition, which (under the assumptions we are making on the shape of the period utility function) only one of the short-run optimal paths satisfies.

  – Concavity of the utility function ensures that we do not need to compute second-order conditions for the household's maximization problem.

The Rental Rate and Production

- Households rent capital to firms and, with competitive markets,

$$\widetilde{r}_t = \frac{\partial Y_t}{\partial K_t} \quad \text{(marginal product of capital)},$$

  where Y_t is output.

- We assume that output in the economy is given by a Cobb-Douglas production function. In aggregate per capita terms,

$$Y_t = (A_t N_t)^{\alpha} K_t^{1-\alpha} = A_t{}^{\alpha} K_t^{1-\alpha} \tag{10}$$

  where $0 < \alpha < 1$ and A_t denotes exogenous technology (which is subject to random shocks).

- Therefore,

$$\widetilde{r}_t = (1-\alpha)\left(\frac{A_t}{K_t}\right)^{\alpha}$$

  and the Euler equation (5) becomes:

$$C_t^{-\gamma} = \beta E_t \left\{ C_{t+1}^{-\gamma} \left[(1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta \right] \right\} \tag{11}$$

Efficiency and the Planner's Outcome

- There are no distortions in the model-economy we are considering (markets are perfectly competitive).
- Then, the decentralized, competitive equilibrium generated by market behavior coincides with the solution of the problem that a benevolent social planner would solve.
- Specifically, the planner would recognize that the following aggregate per capita resource constraint must be satisfied in each period:

$$Y_t = C_t + I_t.$$

- Thus, from (3) and (**??**),

$$Y_t = \widetilde{r}_t K_t + w_t,$$

or

$$w_t = Y_t - \widetilde{r}_t K_t$$

(as implied by Euler's output exhaustion theorem).

Efficiency and the Planner's Outcome, Continued

- So, (4) becomes:

$$C_t + K_{t+1} = (1-\delta)K_t + Y_t,$$

or, taking (10) into account,

$$C_t + K_{t+1} = (1-\delta)K_t + A_t{}^{\alpha} K_t^{1-\alpha}. \tag{12}$$

- A planner would recognize that the gross return at $t+1$ from investing one unit of the consumption good at t in capital would be:

$$R_{t+1} \equiv (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta,$$

i.e., the marginal product of capital at $t+1$ plus undepreciated capital.

Efficiency and the Planner's Outcome, Continued

- Now, maximizing (1) subject to (12) yields:

$$C_t^{-\gamma} = \beta E_t \left\{ C_{t+1}^{-\gamma} \left[(1-\alpha) \left(\frac{A_{t+1}}{K_{t+1}} \right)^{\alpha} + 1 - \delta \right] \right\},$$

  or

$$C_t^{-\gamma} = \beta E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right), \tag{13}$$

  i.e., at an optimum, the cost of investing one unit of consumption today in capital accumulation (the marginal utility of one unit of consumption today) must be equal to the expected discounted marginal utility value of the gross return from investing one unit of consumption good in capital accumulation.

- As expected, (13) (the Euler equation from the solution of the planner's problem) and (11) (the Euler equation for the decentralized, market solution) are identical once the definition of R_{t+1} is taken into account.

Steady-State Growth

- Let us look for a steady-state, or balanced growth path of the model, in which technology, capital, and consumption all grow at a constant common growth rate.

- We denote this growth rate as:

$$G \equiv \frac{\overline{A}_{t+1}}{\overline{A}_t}$$

  (overbars denote steady-state levels).

- In steady-state, the gross rate of return on capital, R_{t+1}, becomes a constant R, while the first-order condition (13) becomes:

$$G^\gamma = \beta R, \tag{14}$$

  or, in logs (letting $r \equiv \log R$ and $g \equiv \log G$):

$$g = \frac{\log \beta + r}{\gamma} = \sigma \log \beta + \sigma r.$$

- This condition, relating the equilibrium growth rate of consumption to the intertemporal elasticity of substitution times the real interest rate, is a standard result of models with power utility.

Steady-State Growth, Continued

- The definition of R and equation (14) imply that, in steady state, the constant technology-capital ratio is:

$$\frac{\overline{A}_t}{\overline{K}_t} = \left[\frac{G^\gamma/\beta - (1-\delta)}{1-\alpha}\right]^{1/\alpha}.$$

- A higher rate of technology growth leads to a lower capital stock for a given level of technology.
  - The reason is that, in steady state, faster technology growth must be accompanied by faster consumption growth.
  - Agents will accept a steeper consumption path only if the rate of return on capital is higher, which requires a lower capital stock.
- Setting $G^\gamma/\beta = R \approx 1 + r$ yields:

$$\frac{\overline{A}_t}{\overline{K}_t} \approx \left(\frac{r+\delta}{1-\alpha}\right)^{1/\alpha}. \tag{15}$$

Steady-State Growth, Continued

- It is possible to solve for various ratios of variables that are constant along a steady-state growth path.
- These ratios can be expressed in terms of four underlying parameters:
  - g, the log technology growth rate;
  - r, the log real return on capital;
    - strictly speaking, r is an endogenous variable of our model, but we treat is as a parameter as we recognize that it must satisfy $r = -\log\beta + \frac{g}{\sigma} = -\log\beta + \gamma g$.
  - α, the exponent on labor and technology in the production function, or equivalently, labor's share of output;
  - and δ, the rate of capital depreciation.

Steady-State Growth, Continued

- For purposes of "calibration," interpreting periods as quarters, benchmark values of these parameters may be:

$$\begin{aligned} g &= .005 \quad \text{(}2\%\text{ at annual rate),} \\ r &= .015 \quad \text{(}6\%\text{ at annual rate),} \\ \alpha &= .667, \\ \delta &= .025 \quad \text{(}10\%\text{ at annual rate).} \end{aligned}$$

- These are all plausible numbers for the U.S. economy.

- Given $r = .015$ and $g = .005$, $r = -\log\beta + \gamma g$ defines the pairs of values for γ and β such that $r = .015$ and $g = .005$.

Steady-State Growth, Continued

- Using the production function and (15), we find the constant steady-state output capital ratio:

$$\frac{\overline{Y}_t}{\overline{K}_t} = \left(\frac{\overline{A}_t}{\overline{K}_t}\right)^{\alpha} \approx \frac{r+\delta}{1-\alpha}. \tag{16}$$

- Similarly, in steady state, the consumption-output ratio is constant at (see below for $\overline{C}_t/\overline{K}_t$):

$$\frac{\overline{C}_t}{\overline{Y}_t} = \frac{\overline{C}_t/\overline{K}_t}{\overline{Y}_t/\overline{K}_t} \approx 1 - \frac{(1-\alpha)(g+\delta)}{r+\delta}. \tag{17}$$

- At the benchmark parameter values above, it must be:

$$\frac{\overline{Y}_t}{\overline{K}_t} = .118 \;\; (.472 \text{ at annual rate}) \;\; \text{and} \;\; \frac{\overline{C}_t}{\overline{Y}_t} = .745,$$

  fairly reasonable values.

A Non-Linear Model of Fluctuations

- Outside the steady state, the model we laid out is a system of non-linear equations for consumption, capital, output, and technology.

- Nonlinearities are caused by incomplete capital depreciation ($\delta < 1$ in (12) and in $R_{t+1} = (1-\alpha)\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha} + 1 - \delta$) and by time variation in the consumption-output ratio (or the savings rate).

- *Exact* analytical solution of the model is possible only in the unrealistic special case in which capital depreciates fully in one period ($\delta = 1$) and agents have log utility ($\gamma = 1$), so that the consumption-output ratio (and therefore the savings rate) is always constant.

- You find the details on how this special case works in Appendix B.

- The problem is that $\delta = 1$ and $\gamma = 1$ are extremely restrictive hypotheses. In all other cases, the model features a mixture of multiplicative and additive terms that make an exact solution impossible.

- How do we proceed?

Log-Linearization

- Our solution approach will be to seek an *approximate* analytical solution by transforming the model into a system of approximate log-linear difference equations.

- In doing so, we are going to rely on the following result: For small deviations of the variable X_t from the steady state $\overline{X}_t$, it is:

$$\frac{dX_t}{\overline{X}_t} = \frac{X_t - \overline{X}_t}{\overline{X}_t} \approx d\log X_t = \log X_t - \log \overline{X}_t,$$

  and we are going to define:

$$x_t \equiv \frac{dX_t}{\overline{X}_t}.$$

- Now, interpret all lower-case variables below as zero-mean percentage deviations from the steady state of the model that we obtained above.

Log-Linearization, Continued

- From the production function,

$$y_t = \alpha a_t + (1-\alpha)k_t. \tag{18}$$

This one is easy: Just take logs of the production function and remember that $N_t = 1$; (18) holds exactly, it is not an approximation.

Log-Linearization, Continued

- Things are harder for equations that are not log-linear.
- For example:

$$C_t + K_{t+1} = (1 - \delta)K_t + Y_t. \tag{19}$$

- John Campbell of Harvard University used Taylor expansions to approximate the model in a 1994 article in the *Journal of Monetary Economics* that is a standard reference on how to find the approximation below.
- I find it more transparent and efficient to proceed as follows.

Log-Linearization, Continued

- The differential of (19) is:

$$dC_t + dK_{t+1} = (1-\delta)dK_t + dY_t.$$

- Thus,

$$\overline{C}_t \frac{dC_t}{\overline{C}_t} + \overline{K}_{t+1} \frac{dK_{t+1}}{\overline{K}_{t+1}} = (1-\delta)\overline{K}_t \frac{dK_t}{\overline{K}_t} + \overline{Y}_t \frac{dY_t}{\overline{Y}_t},$$

or

$$\frac{\overline{C}_t}{\overline{K}_t} c_t + \frac{\overline{K}_{t+1}}{\overline{K}_t} k_{t+1} = (1-\delta)k_t + \frac{\overline{Y}_t}{\overline{K}_t} y_t. \quad (20)$$

Log-Linearization, Continued

- Now, we know that

$$\frac{\overline{K}_{t+1}}{\overline{K}_t} = G \approx 1 + g.$$

Also,

$$\frac{\overline{Y}_t}{\overline{K}_t} \approx \frac{r+\delta}{1-\alpha}$$

- Then, a steady-state version of (19) implies:

$$\frac{\overline{K}_{t+1}}{\overline{K}_t} = (1-\delta) + \frac{\overline{Y}_t}{\overline{K}_t} - \frac{\overline{C}_t}{\overline{K}_t}.$$

- Using $\frac{\overline{K}_{t+1}}{\overline{K}_t} \approx 1 + g$ and $\frac{\overline{Y}_t}{\overline{K}_t} \approx \frac{r+\delta}{1-\alpha}$ and solving for $\frac{\overline{C}_t}{\overline{K}_t}$ yields:

$$\frac{\overline{C}_t}{\overline{K}_t} \approx \frac{r+\delta}{1-\alpha} - (g+\delta)$$

Log-Linearization, Continued

- Therefore, substituting these results into (20), we can rewrite it as:

$$\left(\frac{r+\delta}{1-\alpha}-(g+\delta)\right)c_t+(1+g)k_{t+1}=(1-\delta)k_t+\frac{r+\delta}{1-\alpha}y_t,$$

  or:

$$(1+g)k_{t+1}=(1-\delta)k_t+\frac{r+\delta}{1-\alpha}y_t+\left(g+\delta-\frac{r+\delta}{1-\alpha}\right)c_t.$$

  This is a linear equation in the variables k_{t+1}, k_t, y_t, and c_t—the percentage deviations of the variables K_{t+1}, K_t, Y_t, and C_t from their steady-state levels!

- Moreover, substituting $y_t=\alpha a_t+(1-\alpha)k_t$, we have:

$$k_{t+1}=\frac{1+r}{1+g}k_t+\frac{\alpha\left(r+\delta\right)}{\left(1-\alpha\right)\left(1+g\right)}a_t+\left[\frac{g+\delta}{1+g}-\frac{r+\delta}{\left(1+g\right)\left(1-\alpha\right)}\right]c_t.$$

Log-Linearization, Continued

- Let

$$\lambda_1 \equiv \frac{1+r}{1+g} \quad \text{and} \quad \lambda_2 \equiv \frac{\alpha\left(r+\delta\right)}{\left(1-\alpha\right)\left(1+g\right)}.$$

Then:

$$k_{t+1} = \lambda_1 k_t + \lambda_2 a_t + \left(1-\lambda_1-\lambda_2\right) c_t. \tag{21}$$

At the benchmark parameter values,

$$\lambda_1 = 1.01, \quad \lambda_2 = .08, \quad \text{and} \quad 1-\lambda_1-\lambda_2 = -.09.$$

- To understand these coefficients, note that

$$1-\lambda_1-\lambda_2 = -\frac{\overline{C}_t}{\overline{Y}_t}\frac{\overline{Y}_t}{\overline{K}_t}\left(1+g\right)^{-1} = -(.118)(.745)(1.005)^{-1}.$$

  – This is the negative of the steady-state ratio of this period's consumption to next period's capital stock:

    · A \$$1$ increase in consumption today lowers tomorrow's capital stock by \$$1$, but a 1 percent increase in consumption this period lowers next period's capital stock by only $.09$ percent because in steady state one period's consumption is only $.09$ times as big as the next period's capital stock.

Log-Linearization, Continued

- Now focus on the Euler equation:

$$C_t^{-\gamma} = \beta E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right).$$

- Assume that the variables on the right-hand side are *jointly log-normal and homoskedastic.*
  - The first assumption means that the log-variables are normally distributed and the second means that they have constant second moments (variances and covariances).
  - The assumptions are consistent with a log-normal productivity shock being the source of fluctuations and with the approximations we use.

Log-Linearization, Continued

- Taking logs of both sides of the Euler equation:

$$-\gamma \log C_t = \log \beta + \log \left[E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right]. \tag{22}$$

- But log-normality implies the following property:

$$\log \left[E_t \left(X_{t+1} \right) \right] = E_t \left[\log \left(X_{t+1} \right) \right] + \frac{1}{2} var_t \left[\log \left(X_{t+1} \right) \right].$$

- Therefore:

$$\begin{aligned} \log \left[E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right] &= E_t \left[\log \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right] + \frac{1}{2} var_t \left[\log \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right] \\ &= -\gamma E_t \left(\log C_{t+1} \right) + E_t \left(\log R_{t+1} \right) \\ &\quad + \frac{\gamma^2}{2} \sigma^2_{t, \log C_{t+1}} + \frac{1}{2} \sigma^2_{t, \log R_{t+1}} - \gamma \sigma_{t, \log C_{t+1}, \log R_{t+1}}, \end{aligned}$$

where for any variable X_{t+1}, $\sigma^2_{t, \log X_{t+1}}$ denotes the conditional variance at time t of $\log X_{t+1}$, and $\sigma_{t, \log C_{t+1}, \log R_{t+1}}$ denotes the conditional covariance at time t of $\log C_{t+1}$ and $\log R_{t+1}$.

Log-Linearization, Continued

- Hence, (22) becomes:

$$-\gamma \log C_t = \log \beta + E_t \left(-\gamma \log C_{t+1} + \log R_{t+1}\right) + \frac{\gamma^2}{2}\sigma^2_{t,\log C_{t+1}} + \frac{1}{2}\sigma^2_{t,\log R_{t+1}} - \gamma\sigma_{t,\log C_{t+1},\log R_{t+1}}.$$

- Now differentiate this equation to obtain:

$$-\gamma d \log C_t \approx -\gamma E_t \left(d \log C_{t+1}\right) + E_t \left(d \log R_{t+1}\right).$$

- Why did second moments disappear?

- Remember: We are assuming that variables are homoskedastic. Hence, conditional second moments are constant, and they drop out when we differentiate!

Log-Linearization, Continued

- Given

$$x_t \equiv \frac{dX_t}{\overline{X}_t} = \frac{X_t - \overline{X}_t}{\overline{X}_t} \approx d\log X_t = \log X_t - \log \overline{X}_t,$$

it finally follows that we can write the Euler equation in log-linear form as:

$$\gamma E_t \left(c_{t+1} - c_t\right) \approx E_t r_{t+1}$$

where $r_{t+1} = d\log R_{t+1}$.

- Or, recalling $\sigma = \frac{1}{\gamma}$,

$$E_t \left(c_{t+1} - c_t\right) \approx \frac{1}{\gamma} E_t r_{t+1} = \sigma E_t r_{t+1}. \tag{23}$$

  – The intertemporal elasticity of substitution σ measures the responsiveness of consumption to a change in the return to asset accumulation.

Log-Linearization, Continued

- Now,

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta.$$

- Recall $d\log R_{t+1} \approx \frac{dR_{t+1}}{\overline{R}_{t+1}}$. Then:

$$dR_{t+1} = (1-\alpha)\,\alpha\frac{(dA_{t+1})\,\overline{K}_{t+1} - (dK_{t+1})\,\overline{A}_{t+1}}{\overline{K}_{t+1}^2}\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha-1},$$

or:

$$\overline{R}_{t+1}r_{t+1} \approx \alpha\,(1-\alpha)\,\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\,(a_{t+1}-k_{t+1})\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha-1} = \alpha\,(1-\alpha)\,(a_{t+1}-k_{t+1})\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha}.$$

Log-Linearization, Continued

- Recall also:

$$\frac{\overline{A}_{t+1}}{\overline{\overline{K}}_{t+1}} \approx \left(\frac{r+\delta}{1-\alpha}\right)^{1/\alpha}.$$

- Then:

$$\overline{R}_{t+1} \approx (1-\alpha)\,\frac{r+\delta}{1-\alpha} + 1 - \delta = 1 + r.$$

Log-Linearization, Continued

- So,

$$(1+r)\,r_{t+1} \approx \alpha\,(1-\alpha)\,(a_{t+1}-k_{t+1})\,\frac{r+\delta}{1-\alpha},$$

and

$$r_{t+1} \approx \lambda_3\,(a_{t+1}-k_{t+1})\,, \tag{24}$$

with:

$$\lambda_3 \equiv \frac{\alpha\,(r+\delta)}{1+r}.$$

  – The same result can be obtained by taking the differential of

$$\log R_{t+1} = \log\left[1-\delta+(1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha}\right].$$

    You should try doing it as an exercise.

Log-Linearization, Continued

- At the benchmark parameter values, $\lambda_3 = .03$.
  - This coefficient is very small. One way to understand this is to note that changes in technology have only small proportional effects on the one-period return on capital because capital depreciates only slowly, so most of the return R is undepreciated capital rather than marginal output from the Cobb-Douglas production function.
  - Alternatively, we can note that $r_{t+1} \approx R_{t+1} - 1 \approx (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha}$ when δ is negligible. In this case, a 1 percent increase in the technology-capital ratio raises r_{t+1} by about α percent. But α percent of r_{t+1} is only αr_{t+1} percentage points.
- Equations (23) and (24) together imply:

$$E_t\left(c_{t+1} - c_t\right) \approx \sigma\lambda_3 E_t\left(a_{t+1} - k_{t+1}\right). \tag{25}$$

- To close the model, we only need to specify a process for the technology shock a_t, the percentage deviation of A_t from its steady-state level $\overline{A}_t$: $a_t = \frac{A_t - \overline{A}_t}{\overline{A}_t}$.

- We assume an $AR(1)$ process:

$$a_t = \phi a_{t-1} + \varepsilon_t, \quad -1 \leq \phi \leq 1. \tag{26}$$

- We assume that the innovations to technology, ε_t, are normally distributed and such that $E_{t-1}(\varepsilon_t) = 0$.

- The $AR(1)$ coefficient ϕ measures the *persistence of technology shocks*, with the extreme case $\phi = 1$ being a random walk for technology.

Log-Linearization, Continued

- We did a ton of math and it looked awfully complicated, but look what we have now: Equations (21), (25), and(26) form a system of linear expectational difference equations in (the percentage deviations from the steady state of) technology, capital, and consumption!

- In other words, we boiled down the non-linear model we started from to the linear system:

$$\begin{aligned} k_{t+1} &\approx \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2) c_t, \\ E_t (c_{t+1} - c_t) &\approx \sigma \lambda_3 E_t (a_{t+1} - k_{t+1}), \\ a_t &= \phi a_{t-1} + \varepsilon_t. \end{aligned} \tag{27}$$

- The parameter of these equations include λ_1, λ_2 and λ_3 (where $\lambda_1 \equiv \frac{1+r}{1+g}$, $\lambda_2 \equiv \frac{\alpha(r+\delta)}{(1-\alpha)(1+g)}$, and $\lambda_3 \equiv \frac{\alpha(r+\delta)}{1+r}$), $\sigma = \frac{1}{\gamma}$, the $AR(1)$ coefficient ϕ that measures the persistence of technology shocks, and the variance of the technology innovations ε_t.

The Calibration Approach to RBC Analysis

- In Campbell's interpretation, the "calibration" approach to real business cycle analysis takes λ_1, λ_2, and λ_3 as known, and searches for values of σ and ϕ (and a variance for the technology innovation, ε) to match the moments of observed macroeconomic the series.
  - If λ_1, λ_2, and λ_3 are not taken as given, one can search for values of all the structural parameters of the model–including those of which λ_1, λ_2, and λ_3 are functions–to match moments of observed data.
- One can then verify if the values of parameters such that the model matches moments of the data are reasonable.
- An alternative interpretation of the calibration approach runs in the opposite direction and asks the following questions:
  - Given reasonable values of the structural parameters of the model and a process for a_t that is roughly consistent with the data, how far are the moments of endogenous variables implied by the model from the moments of actual data?
  - How far are the *impulse responses* (the responses of endogenous variables to exogenous shocks) from those implied by the data?

Determinacy of the Solution

- To compute impulse responses (*i.e.*, the responses of consumption and capital to technology innovations) or the second-moment properties (*i.e.*, the variances or covariances of consumption and capital implied by assumptions on the variance of technology innovations) to compare them to properties of the data, we must solve the system (27).

- But whenever we solve a system of linear, expectational, difference equations such as (27), in principle we need to check that there is a *unique* solution, *i.e.*, that the solution (if it exists) is *determinate*.

- If the solution is indeterminate, the economy is subject to fluctuations that are not caused by changes in the fundamentals–*sunspot fluctuations*.

- We will not study how to prove that the system (27) has a unique solution. You find the information in Appendix C. For our purposes, trust that the system does have a unique solution.

Determinacy of the Solution, Continued

- Important: A very interesting branch of macroeconomics that time limitations do not allow us to study focuses precisely on what happens in model-economies that do not have a unique solution (for dynamics around the steady state or even for the steady state itself).

- These models are best suited to capture John Maynard Keynes' idea of *animal spirits*, fluctuations in sentiment that trigger economic fluctuations and that are not captured by so-called fundamental-driven fluctuations we are focusing on. (Fundamental in the sense that technology is among the "fundamentals" of our model as opposed to, say, the color of sunspots.)

- This is a fascinating, very important branch of macroeconomics. Roger Farmer of the University of Warwick, UK, and Karl Shell of Cornell University are among the major contributors.

The Method of Undetermined Coefficients

- Once we trust that the system (27) has a unique solution, it can be solved with a method known as the method of undetermined coefficients.
- Let η_{zx} denote the partial elasticity of variable z with respect to variable x.
- Guess that the solution for consumption takes the form:

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t, \tag{28}$$

  where η_{ck} and η_{ca} are unknown but assumed constant.

- We are going to verify the guess by finding values of η_{ck} and η_{ca} that satisfy the restrictions of the approximate model.

The Method of Undetermined Coefficients, Continued

- **Note 1:** The guess (28) is consistent with the logic of a technique for solving dynamic models known as dynamic programming: Optimal behavior maps the *state of the economy at time* t (described in our model by capital—k_t—and technology—a_t) into the variables that are endogenous during that period (in this case, consumption during that period—c_t).
  - Think about it: From the perspective of households and firms in the economy, what can summarize the state (the initial condition) of the economy at the time when they take a decision?
  - Well, that's the capital stock they entered the period with and the current realization of the available technology.
  - So, given a set of linear equations that we are trying to solve, we are going to guess that the solution is a linear function of the endogenous state (capital) and the exogenous one (technology).

The Method of Undetermined Coefficients, Continued

- **Note 2:** Having verified determinacy allows us to be confident that the solution we are guessing (referred to also as the *minimum state vector solution*, since it depends on the smallest number of state variables) is the unique solution of the system (27).
  - If the solution were indeterminate, (28) would be only one of the possible solutions of (27).
  - Alternative solutions would exist—among them, solutions that map so-called non-fundamental states (such as the color of sunspots...) into the endogenous variables.

The Method of Undetermined Coefficients, Continued

- Given our guessed solution for consumption and equation (21), we can write the guessed solution for capital as:

$$k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t, \tag{29}$$

  with:

$$\eta_{kk} \equiv \lambda_1 + (1 - \lambda_1 - \lambda_2)\, \eta_{ck}, \quad \eta_{ka} \equiv \lambda_2 + (1 - \lambda_1 - \lambda_2)\, \eta_{ca}.$$

- **Note 3:** As expected (again, consistent with the logic of dynamic programming), the solution maps the state at time t into the choice of assets entering $t+1$ (k_{t+1}).

The Method of Undetermined Coefficients, Continued

- **Note 4:** As noted above, the solution in (28) and (29) is also called the *minimum state variable solution*, in the sense that it is the solution that expresses the endogenous variables as functions of the minimum state vector—the vector consisting of the endogenous, predetermined state variable k_t and of the exogenous state variable a_t.
  - The concept of minimum state variable (MSV) solution has been proposed by Bennett McCallum of Carnegie-Mellon University as the only solution that is relevant in practice even in situations in which there is indeterminacy, and, therefore, the MSV solution is not the unique rational expectation equilibrium of the model.
  - Many scholars see determinacy of the equilibrium (which ensures that the MSV solution is the unique equilibrium) as an important, desirable property of macroeconomic models.
  - Others, like Roger Farmer and Karl Shell, view allowing for multiple possible solutions (and therefore multiple possible equilibria) as central to understanding fluctuations.

The Method of Undetermined Coefficients, Continued

- Substituting the conjectured solution into $E_t(c_{t+1} - c_t) = \sigma\lambda_3 E_t(a_{t+1} - k_{t+1})$ yields:

$$\eta_{ck}(k_{t+1} - k_t) + \eta_{ca}E_t(a_{t+1} - a_t) = \sigma\lambda_3 E_t a_{t+1} - \sigma\lambda_3 k_{t+1} \tag{30}$$

(k_{t+1} is known at time t, when it is determined).

- Substitute (29) into (30) and use $E_t a_{t+1} = \phi a_t$. The result (taking the definitions of η_{kk} and η_{ka} into account) is an equation in only the two state variables, k_t and a_t:

$$\begin{aligned} & \eta_{ck}[\lambda_1 - 1 + (1 - \lambda_1 - \lambda_2)\eta_{ck}]k_t + \eta_{ck}[\lambda_2 + (1 - \lambda_1 - \lambda_2)\eta_{ca}]a_t + \eta_{ca}(\phi - 1)a_t \\ = \ & \sigma\lambda_3\phi a_t - \sigma\lambda_3[\lambda_1 + (1 - \lambda_1 - \lambda_2)\eta_{ck}]k_t - \sigma\lambda_3[\lambda_2 + (1 - \lambda_1 - \lambda_2)\eta_{ca}]a_t. \end{aligned} \tag{31}$$

The Method of Undetermined Coefficients, Continued

- To solve this equation, we first equate coefficients on k_t to find η_{ck} and then equate coefficients on a_t to find η_{ca}, given η_{ck}.
- Equating coefficients on k_t gives the quadratic equation:

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0, \tag{32}$$

with:

$$\begin{aligned} Q_2 &\equiv 1 - \lambda_1 - \lambda_2, \\ Q_1 &\equiv \lambda_1 - 1 + \sigma\lambda_3 (1 - \lambda_1 - \lambda_2), \\ Q_0 &\equiv \sigma\lambda_3\lambda_1. \end{aligned}$$

The Method of Undetermined Coefficients, Continued

- The quadratic formula gives two solutions to (32).
- With the benchmark set of parameter values, one of these is positive.
  - Equation (21), with $\lambda_1 > 1$, shows that η_{ck} must be positive for the steady state to be locally stable.
    - If $\eta_{ck} < 0$, then $\lambda_1 + (1 - \lambda_1 - \lambda_2)\,\eta_{ck} > 1$, which implies $\eta_{kk} > 1$ in (29), or an unstable steady state to which the economy never returns after shocks.
  - Hence, the positive solution is the appropriate one:

$$\eta_{ck} = \frac{1}{2Q_2}\left(-Q_1 - \sqrt{Q_1^2 - 4Q_0Q_2}\right).$$

  - Intuitively: It makes economic sense that, if the economy invested more during period $t-1$, and therefore it enters period t with more capital, consumption during period t is higher.
- Note that η_{ck} depends only on σ and the λ parameters and is invariant to the persistence of the technology shock, ϕ.

The Method of Undetermined Coefficients, Continued

- The solution of the model is then completed by finding η_{ca} as:

$$\eta_{ca} = \frac{-\eta_{ck}\lambda_2 + \sigma\lambda_3\left(\phi - \lambda_2\right)}{\phi - 1 + \left(1 - \lambda_1 - \lambda_2\right)\left(\eta_{ck} + \sigma\lambda_3\right)}.$$

- To obtain this, equate coefficients on a_t at the left and right side of equation (31), substitute the solution for η_{ck}, and solve the resulting equation for η_{ca}.

The Method of Undetermined Coefficients, Continued

- To summarize, we have:

$$\begin{aligned} a_t &= \phi a_{t-1} + \varepsilon_t, \\ k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t, \\ c_t &= \eta_{ck} k_t + \eta_{ca} a_t, \end{aligned}$$

  as solution of the model, with the parameters obtained above.

- These equations make it possible to study the dynamics of consumption, capital, and technology following an innovation to the latter.

- In other words, the equations can be used to analyze the response of the economy to technology shocks, *i.e.*, to perform *impulse response analysis*.

The Method of Undetermined Coefficients, Continued

- Given numerical values for parameters, we can use the equations to compute the paths of technology, capital, and consumption over time in response to an initial innovation to technology $\varepsilon_0 = 1$ at time $t = 0$ (assuming that the economy was in steady state until—and including—period $t = -1$).
  - We will want to use the fact that $k_0 = 0$ (since it was determined at $t = -1$, before the innovation to technology).

- We will explore how to do this using a simple Excel spreadsheet that will illustrate how the consumption elasticities, η_{ck} and η_{ca}, and the capital elasticities, η_{kk} and η_{ka}, derived from them determine the dynamic behavior of our model economy.

- For those of you who have taken time series econometrics, Appendix D goes over some time series implications of the model.

A Summary of the Dynamic Properties of the Model

- Three characteristics of the fixed-labor model deserve note.

- **First**, analysis of impulse responses shows that *capital accumulation has an important effect on the dynamics of the economy only when the underlying technology shock is persistent*, lasting long enough for significant changes in capital to occur.

- *The stochastic growth model–or at least this version–is unable to generate persistent effects from transitory shocks*.

A Summary of the Dynamic Properties of the Model, Continued

- To understand this point, recall the solution equations:

$$
\begin{aligned}
c_t &= \eta_{ck} k_t + \eta_{ca} a_t, \\
y_t &= \eta_{yk} k_t + \eta_{ya} a_t, \\
k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t.
\end{aligned}
$$

- Persistence in the dynamics of consumption and other endogenous variables follows from their dependence on the exogenous state a_t and on the endogenous, predetermined state k_t.

- The persistence of technology (ϕ) is an exogenous parameter.

- Therefore, the persistence of dynamics that comes from dependence on a_t is exogenous.

- Instead, we refer to the persistence that arises as a consequence of dependence on the endogenous state k_t as *endogenous persistence*.

- However, if ϕ is small, the technology shock does not last long enough to generate significant changes in capital, and the effect of capital dynamics on the economy is consequently small, so that the deviation of consumption and output from the steady state becomes very small once a_t has returned to the steady state.

A Summary of the Dynamic Properties of the Model, Continued

- **Second**, *technology shocks do not have strong effects on realized or expected returns on capital*.
- The reason is that the gross rate of return on capital largely consist of undepreciated capital rather than the net output that is affected by technology shocks.
- The realized return on capital equals λ_3, and $\lambda_3 = .03$ at benchmark parameter values.
- Thus, a 1 percent technology shock changes the realized return on capital on impact by only 3 basis points (12 at annual rate).
- The expected return on capital is even more stable (constant if the representative agent is risk neutral) because capital accumulation lowers the marginal product of capital one period after a positive technology shock occurs, partially offsetting any persistent effects of the shock.

A Summary of the Dynamic Properties of the Model, Continued

- **Third**, *capital accumulation does not generate a short or long-run "multiplier"* in the sense of an output response to a technology shock that is larger (in percentage terms) than the underlying shock itself.
- This means that slower-than-normal technology growth can generate only slower-than-normal output growth and not actual declines in output.
- The model with fixed labor supply can explain output declines only by appealing to implausible declines in the *level* of technology.

Variable Labor Supply

- We now move to a model in which labor supply is allowed to vary over time and is determined endogenously.
- The production function is unchanged:

$$Y_t = (A_t N_t)^{\alpha} K_t^{1-\alpha}. \tag{33}$$

- Also the law of motion for capital remains:

$$K_{t+1} = (1-\delta) K_t + Y_t - C_t. \tag{34}$$

- However, we now assume that the period utility function is:

$$u(C_t, 1-N_t) = \log C_t + \theta \frac{(1-N_t)^{1-\gamma_n}}{1-\gamma_n}. \tag{35}$$

Variable Labor Supply, Continued

$$u\left(C_t, 1-N_t\right) = \log C_t + \theta \frac{\left(1-N_t\right)^{1-\gamma_n}}{1-\gamma_n}.$$

- The total amount of time available to agents in each period is normalized to 1. Thus, $1-N_t$ is leisure in period t.

- Utility is additively separable in consumption and leisure. Robert King of Boston University, Charles Plosser of the Hoover Institution, and Sergio Rebelo of Northwestern University showed in a 1988 *Journal of Monetary Economics* article that log utility from consumption is required to obtain constant steady-state labor supply (i.e., balanced growth) when utility is additively separable over consumption and leisure.

  - The balanced growth requirement does not restrict the form of the utility function for leisure.

- Power utility nests several cases in the literature (for example, log when $\gamma_n = 1$, linear when $\gamma_n = 0$). Let $\sigma_n \equiv \frac{1}{\gamma_n}$ denote the elasticity of intertemporal substitution for leisure.

Variable Labor Supply, Continued

- The Euler equation for consumption is still:

$$C_t^{-1} = \beta E_t \left(C_{t+1}^{-1} R_{t+1} \right). \tag{36}$$

- But now:

$$R_{t+1} = (1-\alpha) \left(\frac{A_{t+1} N_{t+1}}{K_{t+1}} \right)^{\alpha} + 1 - \delta. \tag{37}$$

- And the key new feature of the model is that there is now a *static* first-order condition for the optimal choice of leisure relative to consumption at each point in time.

Variable Labor Supply, Continued

- We refer to this first-order condition as the *labor-leisure tradeoff*: the agent needs wage income to consume, but labor has a utility cost.
- Intuitively, it must be that, for the household to be optimizing, the marginal utility of leisure equals the real wage evaluated in terms of the marginal utility of consumption.
  - *i.e.*, the marginal utility of leisure must equal how much marginal utility of consumption the real wage earned by supplying an extra unit of labor generates:

$$\theta (1 - N_t)^{-\gamma_n} = C_t^{-1} w_t. \tag{38}$$

  - *Ceteris paribus*, if consumption increases, its marginal utility (and thus the marginal utility of wage income to buy consumption) decreases, and so does labor supply.
- Condition (38) also states that the marginal rate of substitution between leisure and consumption has to be equal to the real wage.
- You can obtain this condition by solving a properly modified version of the household's maximization problem with period utility (35) and $N_t \neq 1$. Maximizing with respect to N_t gives (38).

Variable Labor Supply, Continued

- With competitive markets, the real wage equals the marginal product of labor:

$$w_t = \alpha A_t^{\alpha} \left(\frac{K_t}{N_t} \right)^{1-\alpha}. \tag{39}$$

- Thus, in an efficient economy, the marginal rate of substitution between leisure (or labor) and consumption in household utility has to be equal to the marginal rate at which labor is transformed into output in firm production.

- Combining (38) and (39) yields the labor market clearing condition that determines equilibrium employment:

$$\theta \left(1 - N_t\right)^{-\gamma_n} = \alpha \frac{A_t^{\alpha}}{C_t} \left(\frac{K_t}{N_t} \right)^{1-\alpha}. \tag{40}$$

The Steady State with Variable Labor Supply

- It turns out that the analysis of the steady state from the model with fixed labor carries over directly to the variable-labor model.
- It is still the case that, in a steady state with $\frac{\overline{A}_{t+1}}{\overline{A}_t} = G$, $G^\gamma = \beta R$, or $G = \beta R$, as $\gamma = 1$ in this model. Thus, $g = \log \beta + r$.
- The steady-state values of the ratios $\frac{A_t}{K_t}$, $\frac{Y_t}{K_t}$, and $\frac{C_t}{Y_t}$ can be obtained following similar steps to those above.
  - See Appendix E slides for some details.

The Log-Linear Model with Variable Labor Supply

- We can linearize the model's equations around the steady sate as with did for the fixed-labor model, using $d \log X_t \approx \frac{dX_t}{\overline{X}_t} = x_t$.

- The log-linear version of the capital accumulation equation, using

$$y_t = \alpha \left(a_t + n_t\right) + \left(1 - \alpha\right) k_t,$$

is:

$$k_{t+1} \approx \lambda_1 k_t + \lambda_2 \left(a_t + n_t\right) + \left(1 - \lambda_1 - \lambda_2\right) c_t, \tag{41}$$

with λ_1 and λ_2 the same as in the fixed-labor model. (See the Appendix F slides.)

- The interest rate is:

$$r_{t+1} \approx \lambda_3 \left(a_{t+1} + n_{t+1} - k_{t+1}\right), \tag{42}$$

with λ_3 the same as before.

- Linearizing the Euler equation, using the log-normality and homoskedasticity assumptions, and using (42) yields:

$$E_t \left(c_{t+1} - c_t\right) \approx \lambda_3 E_t \left(a_{t+1} + n_{t+1} - k_{t+1}\right). \tag{43}$$

The Log-Linear Model with Variable Labor Supply, Continued

- Now focus on (40). Taking logs:

$$\log\theta - \gamma_n \log(1-N_t) = \log\alpha + \alpha\log A_t - \log C_t + (1-\alpha)\log K_t - (1-\alpha)\log N_t.$$

- Then:

$$-\gamma_n d\log(1-N_t) = \alpha d\log A_t - d\log C_t + (1-\alpha)\, d\log K_t - (1-\alpha)\, d\log N_t.$$

- Observe that:

$$d\log(1-N_t) = -\frac{dN_t}{1-\overline{N}} = -\frac{\overline{N}}{1-\overline{N}}\frac{dN_t}{\overline{N}} = -\frac{\overline{N}}{1-\overline{N}} n_t.$$

- Thus,

$$\gamma_n \frac{\overline{N}}{1-\overline{N}} n_t \approx \alpha a_t + (1-\alpha)(k_t - n_t) - c_t,$$

or:

$$n_t \approx \frac{1-\overline{N}}{\overline{N}} \sigma_n \left[\alpha a_t + (1-\alpha)(k_t - n_t) - c_t\right]. \tag{44}$$

The Log-Linear Model with Variable Labor Supply, Continued

- $\overline{N}$ solves a non-linear equation shown in Appendix E.
- If we assume that households allocate on average one-third of their time to market activities, then $\overline{N} = \frac{1}{3}$ and $\frac{1-\overline{N}}{\overline{N}} = 2$.
- We take this as benchmark (and the equation in Appendix E can be used to solve for the combinations (γ_n, θ) such that $\overline{N} = \frac{1}{3}$ with the other parameters at their benchmark values).
- Equation (44) can be rewritten as:

$$n_t \left[1 + \frac{(1-\alpha)\left(1-\overline{N}\right)}{\overline{N}}\sigma_n\right] \approx \frac{1-\overline{N}}{\overline{N}}\sigma_n \left[\alpha a_t + (1-\alpha)\,k_t - c_t\right],$$

or:

$$n_t \left[\frac{\overline{N} + (1-\alpha)\left(1-\overline{N}\right)\sigma_n}{\overline{N}}\right] \approx \frac{1-\overline{N}}{\overline{N}}\sigma_n \left[\alpha a_t + (1-\alpha)\,k_t - c_t\right],$$

which implies:

$$n_t \approx \mu \left[(1-\alpha)\,k_t + \alpha a_t - c_t\right], \tag{45}$$

with

$$\mu = \mu(\sigma_n) \equiv \frac{\left(1-\overline{N}\right)\sigma_n}{\overline{N} + (1-\alpha)\left(1-\overline{N}\right)\sigma_n}.$$

The Log-Linear Model with Variable Labor Supply, Continued

- μ measures the responsiveness of labor supply to shocks that change the real wage or consumption, taking into account the fact that, if labor supply increases, the real wage is driven down.

- To see this, observe that

$$
\begin{aligned}
w_t &= \alpha A_t^{\alpha} \left(\frac{K_t}{N_t} \right)^{1-\alpha} \Rightarrow \\
\omega_t &= \alpha a_t + (1-\alpha)(k_t - n_t) = (1-\alpha) k_t + \alpha a_t - (1-\alpha) n_t,
\end{aligned}
$$

  where $\omega_t \equiv \frac{dw_t}{\overline{w}}$.

- Because of this effect, even when utility from leisure is linear ($\sigma_n \to \infty$), μ is finite and equal to $\frac{1}{1-\alpha}$ (as implied by $\omega_t = (1-\alpha) k_t + \alpha a_t - (1-\alpha) n_t$).

- As the curvature of the utility function for leisure increases, μ falls and becomes zero when $\gamma_n \to \infty$ ($\sigma_n \to 0$). This corresponds to the fixed-labor-supply model we studied before.

- Note that the value of $\overline{N}$ affects only the relation between σ_n and μ and not any other aspect of the model.

The Log-Linear Model with Variable Labor Supply, Continued

- Substituting equation (45) into equations (41) and (43) and maintaining the assumption $a_t = \phi a_{t-1} + \varepsilon_t$, $E_{t-1}(\varepsilon_t) = 0$, returns a system of equations in capital, consumption, and technology similar to the one we studied when labor was fixed at 1.

- The system can be solved using the same method of undetermined coefficients.

- Assuming that there is a unique solution, we can conjecture the MSV solution:

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t. \tag{46}$$

  - A good exercise for you would be to verify determinacy of the equilibrium as done in the case of fixed labor supply.

- η_{ck} solves a quadratic equation of the type:

$$Q_2 \eta_{ck}^2 + Q_1 \eta_{ck} + Q_0 = 0,$$

  where the coefficients Q_2, Q_1, and Q_0 are now more complicated than before. (See Appendix G.)

- As before, we pick the positive solution. The solution for η_{ca} can be obtained straightforwardly from η_{ck} and the other parameters. These solutions are the same as in the fixed-labor-supply model when labor supply is completely inelastic, so that $\mu = 0$.

Dynamics with Variable Labor Supply

- The dynamics of the economy take the same form as in the fixed-labor model. Once again:

$$k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t. \tag{47}$$

(The definitions/expressions for η_{kk} and η_{ka} are in Appendix G.)

- Substituting (46) into (45) yields:

$$n_t = \mu \left[(1-\alpha) k_t + \alpha a_t - \eta_{ck} k_t - \eta_{ca} a_t \right] = \eta_{nk} k_t + \eta_{na} a_t, \tag{48}$$

with $\eta_{nk} \equiv \mu (1 - \alpha - \eta_{ck})$ and $\eta_{na} \equiv \mu (\alpha - \eta_{ca})$.

- Increases in capital raise the real wage by a factor $1-\alpha$ (recall $\omega_t = (1-\alpha) k_t + \alpha a_t - (1-\alpha) n_t$).
- This stimulates labor supply.
- But capital also increases consumption by a factor η_{ck}, and this can have an offsetting effect.
- Similarly, increases in technology raise the real wage by a factor α, but the stimulating effect on labor supply is offset by the effect η_{ca} of technology on consumption.

Dynamics with Variable Labor Supply, Continued

- Finally, using $y_t = \alpha\left(a_t + n_t\right) + \left(1 - \alpha\right) k_t$ and substituting for n_t from (48) gives:

$$y_t = \eta_{yk} k_t + \eta_{ya} a_t, \tag{49}$$

  with $\eta_{yk} \equiv 1 - \alpha + \alpha\mu\left(1 - \alpha - \eta_{ck}\right)$, $\eta_{ya} \equiv \alpha + \alpha\mu\left(\alpha - \eta_{ca}\right)$.

- As before, if we use the lag operator notation, it turns out that output is an $ARMA(2, 1)$ process.

- However, capital and technology now affect output both directly (with coefficients α and $1 - \alpha$, respectively) and indirectly through labor supply.

- The initial response to a technology shock is now $\alpha + \alpha\mu\left(\alpha - \eta_{ca}\right)$ rather than α.

- Thus, the variable-labor model can produce an amplified output response to technology shocks, even in the very short run.

Dynamics with Variable Labor Supply, Continued

- *At the benchmark parameter values*, if $\sigma_n = 0$, the model reduces to the fixed-labor case: $\eta_{nk} = \eta_{na} = 0$.
- As σ_n increases, η_{nk} becomes increasingly negative, while η_{na} becomes increasingly positive.
- Thus, an increase in capital lowers the work effort because it increases consumption more than the real wage.
- A positive technology shocks increases the work effort.
- η_{nk} is independent of the persistence parameter ϕ, but η_{na} declines with ϕ.
- The reason is that, if an innovation has persistent effects on technology, it increases consumption more than a transitory shock (η_{ca} increases with ϕ),
- The increase in consumption lowers the marginal utility of income and reduces the work effort.
- Put another way, *transitory shocks produce a stronger intertemporal substitution effect in labor supply* (if ϕ is small, η_{na} is large).

Dynamics with Variable Labor Supply, Continued

- Campbell analyzes a number of special cases and properties of the model, along with the responses of the return on capital and the real wage to shocks.
- It turns out that the responses of the return on capital to capital and technology are $\lambda_3 (\eta_{nk} - 1)$ and $\lambda_3 (1 + \eta_{na})$, respectively.
- These responses remain very small also in the variable-labor model.
- It is possible to check that:

$$\omega_t = y_t - n_t, \tag{50}$$

so that $\eta_{\omega k} = \eta_{yk} - \eta_{nk}$ and $\eta_{\omega a} = \eta_{ya} - \eta_{na}$. (Verify this as an exercise.)

- $\eta_{\omega a}$ is smallest when utility is linear in leisure ($\gamma_n = 0$). In this case, $\eta_{\omega a} = \eta_{ca}$, because linear utility from leisure generates a constant wage-consumption ratio.
- $\eta_{\omega a}$ rises as labor supply becomes less elastic (*i.e.*, as γ_n increases, or σ_n decreases).

Dynamics with Variable Labor Supply, Continued

- Variable labor supply has important implications for the short-run elasticity of output with respect to technology, η_{ya}.

- When labor supply is fixed, $\eta_{ya} = \alpha = .667$.

- With variable labor supply, $\eta_{ya} = \alpha + \alpha\mu\left(\alpha - \eta_{ca}\right)$, which can exceed 1 (it falls with ϕ, however, and it cannot exceed $.99$ when $\phi = 1$).

- This is important, because an elasticity greater than 1 allows absolute declines in output to be generated by positive but slower-than-normal growth in technology, which is surely more plausible than the notion of absolute declines in technology.

RBC Wrapping Up

- This concludes our analysis of the real business cycle model.
- As we noted when we began, key assumptions and results of the framework are not supported by evidence, but the model gives us a methodological starting point and conceptual foundation.
- Chapter 14 of Sanjay Chugh's textbook gives you a diagrammatic analysis of the model and its properties.
- We will begin departing from this basic framework by introducing the consequences of monopoly power next.

Appendix A: Risk Aversion and the Equity Premium Puzzle

- Consider the utility function $u = \frac{c^{1-\gamma}}{1-\gamma}$, which we are using in the RBC model and we will use again in many other models. $\gamma = -\frac{cu''(c)}{u'(c)}$ is called *coefficient of relative risk aversion*.

- **Interpretation** (Pratt, 1964, *Econometrica*)**:** Suppose we offer two alternatives to a consumer who starts off with risk-free consumption level c: (s)he can receive $c - \pi$ with certainty or a lottery paying $c - y$ with probability $.5$ and $c + y$ with probability $.5$.

- For given values of c and y, we want to find the value of $\pi = \pi(y, c)$ that leaves the consumer indifferent between the two choices (the maximum amount the consumer is willing to pay in order to avoid the bet).

- That is, we want to find $\pi(y, c)$ such that:

$$u\left[c - \pi\left(y, c\right)\right] = .5u\left(c + y\right) + .5u\left(c - y\right)$$

- For given c and y, this non-linear equation can be solved for π.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Alternatively, for small y, use Taylor expansions and a local argument:
  - Expansion of $u(c-\pi)$:
$$u(c-\pi) = u(c) - \pi u'(c) + O\left(\pi^2\right), \tag{51}$$
    where we let $c-\pi$ be the variable x in our expansion of $f(x) = u(c-\pi)$ around $x_0 = c$ and $O(\cdot)$ means terms of order at most $(\cdot)$.
  - Expansion of $u(c+\widetilde{y})$:
$$u(c+\widetilde{y}) = u(c) + \widetilde{y}u'(c) + \frac{1}{2}\widetilde{y}^2 u''(c) + O\left(\widetilde{y}^3\right),$$
    where $\widetilde{y}$ is the random variable that takes value y with probability $.5$ and $-y$ with probability $.5$, and we let $c+\widetilde{y}$ be the variable x in our expansion of $f(x) = u(c+\widetilde{y})$ around $x_0 = c$.
  - We consider a second-order expansion here due to the randomness of $\widetilde{y}$, which requires us to include second moments in the expansion.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Taking expectations of both sides of this equation yields:

$$Eu\left(c+\widetilde{y}\right)=u\left(c\right)+\frac{1}{2}y^{2}u''\left(c\right)+o\left(y^{2}\right), \tag{52}$$

  where $o\left(\cdot\right)$ means terms of smaller order than $\left(\cdot\right)$.

- Equating (51) and (52) and ignoring higher-order terms gives:

$$\pi\left(y,c\right)\approx\frac{1}{2}y^{2}\left[\frac{-U''\left(c\right)}{U'\left(c\right)}\right],$$

  or, if $u\left(c\right)=\frac{c^{1-\gamma}}{1-\gamma}$,

$$\pi\left(y,c\right)\approx\frac{1}{2}y^{2}\frac{\gamma}{c},$$

  which can be rearranged to

$$\frac{\pi\left(y,c\right)}{y}\approx\frac{1}{2}\gamma\frac{y}{c}.$$

- This tells us that the *premium* that the consumer is willing to pay to avoid a fair bet of size y is (approximately) equal to $\frac{1}{2}\gamma$ times the ratio between the size of the bet and the consumer's initial level of consumption. γ characterizes the consumer's attitude toward uncertainty and is key to determine the premium (s)he is willing to pay to avoid it.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Now, think of confronting someone with initial consumption of $\$50,000$ per year with a 50-50 chance of winning or losing y dollars.

- Consider $y = 10, 100, 1000, 5000$. How much would the person be willing to pay to avoid that risk?

- Based on $\pi = \frac{1}{2}\gamma\frac{y^2}{c}$:

	y	**10**	**100**	**1000**	**5000**
γ	**2**	.002	.2	20	500
	5	.005	.5	50	1250
	10	.01	1	100	2500

- A common reaction to these premia is that for γ as high as 5, they are too big. This motivates most macroeconomists' view that γ should not be much higher than 2 or 3.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Mehra and Prescott (1985, *Journal of Monetary Economics*) consider data on average yields on relatively riskless bonds and risky equity in the U.S. for the period 1889 -1978.
- The average real yield on the S&P 500 index was 7 percent. The average yield on short-term debt was only 1 percent, *i.e.*, there was an equity premium of 6 percent.
- Let $1+r^i_{t+1}$ denote the real rate of return on asset i between t and $t+1$, $i=b$ for bonds, $i=s$ for stocks and look at the summary statistics below:

	Mean	**Var-Cov**		
		$1+r^s_{t+1}$	$1+r^b_{t+1}$	$\frac{c_{t+1}}{c_t}$
$1+r^s_{t+1}$	1.070	.0274	.00104	.00219
$1+r^b_{t+1}$	1.010		.00308	-.000193
$\frac{c_{t+1}}{c_t}$	1.018			.00127

- The presence of an equity premium is consistent with the theory: Stocks are riskier than bonds and therefore agents require a premium in order to hold them.
- But is a 6 percent spread justifiable within basic models given actual riskiness of stocks and bonds?
- No—and addressing this puzzle resulted in its own literature in macro-finance.

Appendix B: A Log-Linear Model of Fluctuations

- To see what happens when $\gamma = 1$ and $\delta = 1$, observe that, with $\gamma = 1$, the Euler equation becomes:

$$C_t^{-1} = \beta E_t \left(C_{t+1}^{-1} R_{t+1} \right) \tag{53}$$

- Recall that the economy we are modeling is such that:

$$Y_t = C_t + I_t, \quad \text{or} \quad 1 = \frac{C_t}{Y_t} + \frac{I_t}{Y_t}.$$

- Let

$$\frac{I_t}{Y_t} \equiv \tilde{s}_t = \text{saving rate.}$$

- Then,

$$\frac{C_t}{Y_t} = 1 - \tilde{s}_t, \quad \text{or} \quad C_t = (1 - \tilde{s}_t) Y_t.$$

Appendix B: A Log-Linear Model of Fluctuations, Continued

- Thus, (53) implies:

$$-\log(1-\tilde{s}_t) - \log Y_t = \log\beta + \log\left\{E_t\left[\frac{R_{t+1}}{(1-\tilde{s}_{t+1})Y_{t+1}}\right]\right\} \tag{54}$$

- $Y_t = A_t{}^\alpha K_t^{1-\alpha}$ and $\delta = 1$ imply:

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^\alpha = (1-\alpha)\frac{Y_{t+1}}{K_{t+1}}.$$

- Also, $\delta = 1$ implies $K_{t+1} = Y_t - C_t = \tilde{s}_t Y_t$. Hence,

$$R_{t+1} = \frac{1-\alpha}{s_t}\left(\frac{Y_{t+1}}{Y_t}\right).$$

Appendix B: A Log-Linear Model of Fluctuations, Continued

- Thus, (54) reduces to:

$$
\begin{aligned}
-\log(1-\tilde{s}_t) - \log Y_t &= \log\beta + \log\left\{E_t\left[\frac{(1-\alpha)}{\tilde{s}_t(1-\tilde{s}_{t+1})Y_t}\right]\right\} \\
&= \log\beta + \log(1-\alpha) - \log\tilde{s}_t - \log Y_t + \log\left[E_t\left(\frac{1}{1-\tilde{s}_{t+1}}\right)\right],
\end{aligned}
$$

  which implies:

$$
\log\tilde{s}_t - \log(1-\tilde{s}_t) = \log\beta + \log(1-\alpha) + \log\left[E_t\left(\frac{1}{1-\tilde{s}_{t+1}}\right)\right]. \tag{55}
$$

- Because technology and capital do not enter (55), there is a constant value of $\tilde{s}_t$, $\widehat{s}$, that satisfies (55).

Appendix B: A Log-Linear Model of Fluctuations, Continued

- To verify this, note that, if $\tilde{s}_t = \widehat{s}\,\forall t$, then

$$E_t\left(\frac{1}{1-\tilde{s}_{t+1}}\right) = \frac{1}{1-\widehat{s}},$$

and (55) becomes:

$$\log \widehat{s} = \log \beta + \log(1-\alpha),$$

or

$$\widehat{s} = \beta(1-\alpha).$$

- Now, if $s_t = \widehat{s} = \beta(1-\alpha)$, it follows that:

$$\frac{C_t}{Y_t} = 1 - \beta(1-\alpha),$$

and:

$$K_{t+1} = \beta(1-\alpha)Y_t.$$

Appendix B: A Log-Linear Model of Fluctuations, Continued

- Given the production function $Y_t = A_t{}^\alpha K_t^{1-\alpha}$, $K_t = \beta(1-\alpha)Y_{t-1}$ yields:

$$Y_t = A_t{}^\alpha \left[\beta(1-\alpha)\right]^{1-\alpha} Y_{t-1}^{1-\alpha},$$

  or

$$\log Y_t = (1-\alpha)\log \widehat{s} + (1-\alpha)\log Y_{t-1} + \alpha \log A_t. \tag{56}$$

- Equation (56) implies that, *given assumptions on the process for* $\log A_t$*, it is the possible to obtain exact solutions for the paths of all endogenous variables*:
  - Given assumptions on $\log A_t$, we can use (56) to reconstruct the exact path of $\log Y_t$.
  - We can then use $\log K_{t+1} = \log\left[\beta(1-\alpha)\right] + \log Y_t$ and $\log C_t = \log\left[1-\beta(1-\alpha)\right] + \log Y_t$ to reconstruct the exact paths of capital and consumption.
    - Since capital is predetermined (capital at $t+1$ is chosen at t), K_{t+1} is a function of Y_t.)
    - Recall that the assumptions on utility and constraints that ensure a unique solution for the competitive equilibrium/planner's problem are satisfied here.
    - Hence, $\widehat{s}$ is the unique optimal saving rate when $\delta = 1$ and $\gamma = 1$, so that we are assured that the model can be solved exactly under these assumptions.

Appendix C: Determinacy of the Solution

- To verify determinacy, we proceed as follows.
- Focus on the endogenous variables (consumption and capital) and on a perfect foresight version of (27).
- We do not need to worry about the exogenous shock variable a_t and about the expectation operator when verifying determinacy. Use the symbol $=$ instead of $\approx$ to simplify notation.

Appendix C: Determinacy of the Solution, Continued

- We can write:

$$
\begin{aligned}
k_{t+1} &= \lambda_1 k_t + (1-\lambda_1-\lambda_2)\, c_t, \\
c_{t+1} &= c_t - \sigma\lambda_3 k_{t+1} = c_t - \sigma\lambda_3 \left[\lambda_1 k_t + (1-\lambda_1-\lambda_2)\, c_t\right] \\
&= -\sigma\lambda_1\lambda_3 k_t + \left[1-(1-\lambda_1-\lambda_2)\,\sigma\lambda_3\right] c_t.
\end{aligned}
$$

- Or, in matrix form:

$$
\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = M \begin{bmatrix} k_t \\ c_t \end{bmatrix}, \quad M \equiv \begin{bmatrix} \lambda_1 & 1-\lambda_1-\lambda_2 \\ -\sigma\lambda_1\lambda_3 & 1-(1-\lambda_1-\lambda_2)\,\sigma\lambda_3 \end{bmatrix}. \tag{57}
$$

Appendix C: Determinacy of the Solution, Continued

- Blanchard and Kahn (1980, *Econometrica*) showed that, for a system of linear, expectational difference equations such as (57) to have a unique solution, the number of eigenvalues of the matrix M that lie (strictly) outside the unit circle must be equal to the number of non-predetermined variables in the vector $\left[\begin{array}{cc} k_t & c_t \end{array}\right]'$.

- Capital at time t was chosen at time $t-1$. Hence, k_t is a predetermined variable. Consumption–c_t–is not predetermined. Therefore, we need an eigenvalue of M outside the unit circle and one inside for the system (57) (and (27)) to have a determinate solution.

- To calculate the eigenvalues of M, we must solve:

$$\det\left[\begin{array}{cc} \lambda_1 - q & 1-\lambda_1-\lambda_2 \\ -\sigma\lambda_3\lambda_1 & 1-(1-\lambda_1-\lambda_2)\,\sigma\lambda_3 - q \end{array}\right] = q^2 - [1+\lambda_1-(1-\lambda_1-\lambda_2)\,\sigma\lambda_3]\,q + \lambda_1 = 0. \quad (58)$$

Appendix C: Determinacy of the Solution, Continued

- We can try to solve equation (58) by brute force or we can be smart. The latter method is best. :-)
- Consider:

$$J(q) = q^2 - [1 + \lambda_1 - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3]\,q + \lambda_1.$$

- $J(q)$ is a parabola. It is strictly convex, since $J''(q) = 2 > 0$.
- Graph $J(q)$. If the parabola intersects the q-axis once inside the unit circle and once outside, we are done: The matrix M has an eigenvalue outside and an eigenvalue inside the unit circle, and our system of expectational difference equations has a unique solution.
  - The solution is stable (the model displays the desired property that variables return to the steady state after temporary shocks) if the eigenvalue inside the unit circle is strictly inside.

Appendix C: Determinacy of the Solution, Continued

- To graph $J(q)$, compute:

$$
\begin{aligned}
J(0) &= \lambda_1 > 0, \\
J(1) &= (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3, \\
J(-1) &= 2(1 + \lambda_1) - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3, \\
\lim_{q \to -\infty} J(q) &= \lim_{q \to +\infty} J(q) = +\infty.
\end{aligned}
$$

- Using the expressions for λ_1, λ_2, and λ_3, you can verify that:

$$
J(1) = -\frac{\sigma\alpha (r + \delta)[r + \alpha\delta - g(1 - \alpha)]}{(1 + r)(1 - \alpha)(1 + g)} < 0 \Leftrightarrow r + \alpha\delta > g(1 - \alpha).
$$

- To simplify our analysis, assume that structural parameter values are such that $r + \alpha\delta > g(1 - \alpha)$. Note that:

$$
J(1) < 0 \Rightarrow J(-1) > 0.
$$

Appendix C: Determinacy of the Solution, Continued

- Compute also:

$$J'(q) = 2q - \left[1 + \lambda_1 - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3\right].$$

- Hence:

$$\begin{aligned} J'(0) &= -\left[1 + \lambda_1 - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3\right], \\ J'(1) &= 1 - \lambda_1 + (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3, \\ J'(-1) &= -3 - \lambda_1 + (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3. \end{aligned}$$

- Note that $J(1) = (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3 < 0$ implies

$$J'(0) < 0 \quad \text{and} \quad J'(-1) < 0.$$

Appendix C: Determinacy of the Solution, Continued

- Therefore, the graph of $J(q)$ is strictly positive and decreasing at $q = -1$ and $q = 0$.
- It crosses the q-axis *once* between 0 and 1 (this is a consequence of $J(0) > 0$, $J'(0) < 0$, and $J(1) < 0$).
- At $q = 1$, $J(q)$ may be increasing or decreasing, but, regardless of the sign of $J'(1)$, the second intersection of $J(q)$ with the q-axis must happen to the right of 1.
  - The fact that $J(q)$ is a parabola, *i.e.*, it switches from decreasing to increasing only once, and $J(1) < 0$ rule out a second intersection to the left of 1.
- It follows that the roots of $J(q) = q^2 - [1 + \lambda_1 - (1 - \lambda_1 - \lambda_2)\sigma\lambda_3] q + \lambda_1 = 0$ lie one inside and one outside the unit circle.
- Therefore, the eigenvalues of M are one inside and one outside the unit circle, and the system of expectational difference equations (27) has a determinate solution.
  - If $J(q)$ never intersects the q-axis, the eigenvalues of M are complex. In this case, verifying determinacy would involve checking the norm of the eigenvalues. We focus on the real case for simplicity.

Appendix D: Some Time Series Implications of the Basic Model

- Now define the lag operator L by:

$$Lx_t = x_{t-1}.$$

- Using the lag operator, the log-linearized solution for capital, $k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t$, can be written as:

$$k_{t+1} = \frac{\eta_{ka}}{1 - \eta_{kk} L} a_t. \tag{59}$$

- Using the same notation, the $AR(1)$ technology process that we have assumed can be written as:

$$a_t = \frac{1}{1 - \phi L} \varepsilon_t. \tag{60}$$

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Equations (59) and (60) imply that capital follows an $AR(2)$ process:

$$k_{t+1} = \frac{\eta_{ka}}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t, \tag{61}$$

or:

$$\begin{aligned}
(1-\eta_{kk}L)(1-\phi L)\,k_{t+1} &= \eta_{ka}\varepsilon_t \Rightarrow \\
\left[1-(\phi+\eta_{kk})\,L+\phi\eta_{kk}L^2\right]k_{t+1} &= \eta_{ka}\varepsilon_t \Rightarrow \\
k_{t+1}-(\phi+\eta_{kk})\,k_t+\phi\eta_{kk}k_{t-1} &= \eta_{ka}\varepsilon_t \Rightarrow
\end{aligned}$$

$$k_{t+1} = (\phi+\eta_{kk})\,k_t-\phi\eta_{kk}k_{t-1}+\eta_{ka}\varepsilon_t.$$

Appendix D: Some Time Series Implications of the Basic Model, Continued

$$k_{t+1} = (\phi + \eta_{kk})\, k_t - \phi\eta_{kk}k_{t-1} + \eta_{ka}\varepsilon_t.$$

- Two points on this:
  - (*a*) The roots of the capital stock process are η_{kk} and ϕ, which are both *real* numbers.
    - Thus, the model does not produce oscillating responses to shocks (which would happen with complex roots).
  - (*b*) The shock to capital at $t+1$ is the technology innovation realized at time t.
    - The capital stock is known one period in advance as it is an endogenous *state* variable, determined by lagged investment and a non-stochastic depreciation rate.

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Recall $y_t = (1-\alpha)k_t + \alpha a_t$.
- With fixed labor supply, $\eta_{yk} = 1-\alpha$ and $\eta_{ya} = \alpha$. Substitute (60) and (61) into

$$y_t = (1-\alpha)Lk_{t+1} + \alpha a_t.$$

- It follows that:

$$\begin{aligned} y_t &= \frac{(1-\alpha)\eta_{ka}L}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t + \frac{\alpha}{1-\phi L}\varepsilon_t \\ &= \frac{\alpha + [(1-\alpha)\eta_{ka} - \alpha\eta_{kk}]L}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t. \end{aligned} \tag{62}$$

- Technology innovations affect output both directly ($\frac{\alpha}{1-\phi L}\varepsilon_t$) and indirectly, through their impact on capital accumulation ($\frac{(1-\alpha)\eta_{ka}L}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t$).

- The sum of the two effects is what we call an $ARMA(2,1)$ process for output:

$$\left(1-\eta_{kk}L\right)\left(1-\phi L\right)y_t = \alpha\varepsilon_t + \left[(1-\alpha)\eta_{ka} - \alpha\eta_{kk}\right]L\varepsilon_t,$$

  or

$$y_t = \left(\phi+\eta_{kk}\right)y_{t-1} - \phi\eta_{kk}y_{t-2} + \alpha\varepsilon_t + \left[(1-\alpha)\eta_{ka} - \alpha\eta_{kk}\right]\varepsilon_{t-1},$$

  where $\left(\phi+\eta_{kk}\right)y_{t-1} - \phi\eta_{kk}y_{t-2}$ is the $AR(2)$ component of the process and $\alpha\varepsilon_t + \left[(1-\alpha)\eta_{ka} - \alpha\eta_{kk}\right]\varepsilon_{t-1}$ is the $MA(1)$ part.

- The process for consumption comes from substituting (59) and (60) into $c_t = \eta_{ck}k_t + \eta_{ca}a_t$:

$$\begin{aligned} c_t &= \frac{\eta_{ck}\eta_{ka}L}{\left(1-\eta_{kk}L\right)\left(1-\phi L\right)}\varepsilon_t + \frac{\eta_{ca}}{1-\phi L}\varepsilon_t \\ &= \frac{\eta_{ca} + \left(\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk}\right)L}{\left(1-\eta_{kk}\right)\left(1-\phi L\right)}\varepsilon_t. \end{aligned} \tag{63}$$

- This too is an $ARMA(2,1)$ process:

$$c_t = \left(\phi+\eta_{kk}\right)c_{t-1} - \phi\eta_{kk}c_{t-2} + \eta_{ca}\varepsilon_t + \left(\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk}\right)\varepsilon_{t-1}.$$

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Note that capital, output, and consumption processes all have the same autoregressive roots η_{kk} and ϕ.
- Thus, we can reconstruct the entire path of the dynamic responses of k, y, and c to a technology innovation at an initial point in time (impulse responses).
- Generally, we will let the computer do this job for us and plot the responses.
  - A set of Matlab codes written by Harald Uhlig of the University of Chicago in 1999 essentially implements the method of undetermined coefficients.
- However, there are cases—like the basic RBC model—in which models are sufficiently simple that we can solve for the elasticities η with pencil and paper, and, as noted above, we can calculate impulse responses using Excel.

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Of curse, the nature of the response (size of initial movement, shape, speed of return to the steady state–if this happens) depends on parameter values.
  - If $\phi = 1$, technology innovations have permanent effects, and the economy does not return to the original steady state.
  - Output converges to a new, permanently higher (or lower) steady-state path after a one-time positive (or negative) technology shock with $\phi = 1$.
- Campbell's paper analyzes the consequences of different parameter values for the elasticities η and the characteristics of impulse responses.
  - Note that Campbell allows for $\beta > 1$, which is not the usual assumption.
  - Read this part of Campbell's paper for more information.

Appendix D: Using the Time Series Process Equations to Obtain Impulse Responses

- Suppose the economy was in steady state until time 0, and suppose $\varepsilon_0 > 0$, $\varepsilon_t = 0 \ \forall t > 0$.
- We can use the equations we obtained above for the processes followed by capital, output, and consumption to compute the responses to the innovation ε_0.
- At time $t = 0$:

$$
\begin{aligned}
k_0 &= 0 \text{ (because capital is predetermined)}, \\
y_0 &= \alpha \varepsilon_0, \\
c_0 &= \eta_{ca} \varepsilon_0.
\end{aligned}
$$

- At time $t = 1$:

$$
\begin{aligned}
k_1 &= \eta_{ka} \varepsilon_0, \\
y_1 &= (\phi + \eta_{kk})\, y_0 + [(1-\alpha)\eta_{ka} - \alpha \eta_{kk}]\, \varepsilon_0 \\
&= (\phi + \eta_{kk})\, \alpha \varepsilon_0 + [(1-\alpha)\eta_{ka} - \alpha \eta_{kk}]\, \varepsilon_0 \\
&= [\alpha \phi + (1-\alpha)\eta_{ka}]\, \varepsilon_0, \\
c_1 &= (\phi + \eta_{kk})\, c_0 + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})\, \varepsilon_0 \\
&= (\phi + \eta_{kk})\, \eta_{ca} \varepsilon_0 + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})\, \varepsilon_0 \\
&= (\eta_{ck}\eta_{ka} + \eta_{ca}\phi)\, \varepsilon_0,
\end{aligned}
$$

Appendix D: Using the Time Series Process Equations to Obtain Impulse Responses, Continued

- At time $t = 2$:

$$
\begin{aligned}
k_2 &= (\phi + \eta_{kk})\, k_1 - \phi \eta_{kk} k_0 \\
&= (\phi + \eta_{kk})\, \eta_{ka} \varepsilon_0, \\
y_2 &= (\phi + \eta_{kk})\, y_1 - \phi \eta_{kk} y_0, \text{ where the solutions for } y_1 \text{ and } y_0 \text{ are above,} \\
c_2 &= (\phi + \eta_{kk})\, c_1 + \phi \eta_{kk} c_0, \text{ where the solutions for } c_1 \text{ and } c_0 \text{ are above.}
\end{aligned}
$$

- And so on...

- As I suggested above, you can calculate exactly the same impulse responses directly using the equations:

$$
\begin{aligned}
k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t, \\
y_t &= (1 - \alpha)\, k_t + \alpha a_t, \\
c_t &= \eta_{ck} k_t + \eta_{ca} a_t, \\
a_t &= \phi a_{t-1} + \varepsilon_t.
\end{aligned}
$$

- Try it as an exercise: Set $\varepsilon_0 > 0$, $\varepsilon_t = 0\ \forall t > 0$ in the equations above, figure out the paths of k, y, and c, and verify that they coincide with those in the Excel example that used the process equations for these variables.

Appendix E: The Steady State with Variable Labor Supply

•

$$\overline{R}_{t+1} = R = (1-\alpha)\left(\frac{\overline{A}_{t+1}\overline{N}_{t+1}}{\overline{\overline{K}}_{t+1}}\right)^{\alpha} + 1 - \delta = (1-\alpha)\left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{\overline{K}}_{t+1}}\right)^{\alpha} + 1 - \delta \Rightarrow$$

$$\begin{aligned}
\left[\frac{\frac{G}{\beta}-(1-\delta)}{1-\alpha}\right]^{\frac{1}{\alpha}} &= \frac{\overline{A}_{t+1}\overline{N}}{\overline{\overline{K}}_{t+1}} \Rightarrow \\
\frac{\overline{A}_{t+1}\overline{N}}{\overline{\overline{K}}_{t+1}} &\approx \left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}} \Rightarrow \\
\frac{\overline{A}_{t+1}}{\overline{\overline{K}}_{t+1}} &\approx \frac{1}{\overline{\overline{N}}}\left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

• Now, (40) implies:

$$\theta\left(1-\overline{N}\right)^{-\gamma_n} = \alpha\left(\frac{\overline{A}_t}{\overline{\overline{K}}_t}\right)^{\alpha}\left(\frac{\overline{K}_t}{\overline{\overline{C}}_t}\right)\overline{N}^{-(1-\alpha)}. \tag{64}$$

Appendix E: The Steady State with Variable Labor Supply, Continued

- So, recalling $\frac{\overline{A}_t}{\overline{K}_t} \approx \frac{1}{\overline{N}} \left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}}$,

$$\theta \left(1 - \overline{N}\right)^{-\gamma_n} \approx \alpha \overline{N}^{-\alpha} \frac{r+\delta}{1-\alpha} \left(\frac{\overline{K}_t}{\overline{C}_t}\right) \overline{N}^{-(1-\alpha)}. \tag{65}$$

- Also, (34) implies:

$$\begin{aligned} \frac{\overline{K}_{t+1}}{\overline{K}_t} &= 1 - \delta + \frac{\overline{Y}_t}{\overline{K}_t} - \frac{\overline{C}_t}{\overline{K}_t}, \\ 1 + g &= 1 - \delta + \frac{\left(\overline{A}_t \overline{N}\right)^{\alpha} \overline{K}_t^{1-\alpha}}{\overline{K}_t} - \frac{\overline{C}_t}{\overline{K}_t}, \end{aligned} \tag{66}$$

or:

$$\frac{\overline{C}_t}{\overline{K}_t} = \left(\frac{\overline{A}_t}{\overline{K}_t}\right)^{\alpha} \overline{N}^{\alpha} - (g + \delta).$$

Appendix E: The Steady State with Variable Labor Supply, Continued

- But, using $\frac{\overline{A}_t}{\overline{K}_t} \approx \frac{1}{\overline{N}} \left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}}$,

$$\begin{aligned} \frac{\overline{C}_t}{\overline{K}_t} &\approx \left(\frac{1}{\overline{N}}\right)^{\alpha} \frac{r+\delta}{1-\alpha} \overline{N}^{\alpha} - (g+\delta) = \frac{r+\delta-(1-\alpha)(g+\delta)}{1-\alpha} \\ &= \frac{r+\delta-g-\delta+\alpha g+\alpha\delta}{1-\alpha} = \frac{r+\alpha\delta-g(1-\alpha)}{1-\alpha}, \end{aligned} \tag{67}$$

  which shows that $\frac{\overline{C}_t}{\overline{K}_t}$ is the same as in the fixed-labor-supply model, and using (66) shows that $\frac{\overline{Y}_t}{\overline{K}_t}$ is also the same as before.

- Equations (65) and (67) imply:

$$\theta\left(1-\overline{N}\right)^{-\gamma_n} \approx \alpha \overline{N}^{-1} \frac{r+\delta}{r+\alpha\delta-g(1-\alpha)}. \tag{68}$$

- Thus, $\overline{N}$ solves the non-linear equation (68).

Appendix F: Log-Linearizing the Variable-Labor Supply Model

- From the production function:

$$y_t = \alpha \left(a_t + n_t\right) + \left(1 - \alpha\right) k_t.$$

- The law of motion of capital implies:

$$dK_{t+1} = \left(1 - \delta\right) dK_t + dY_t - dC_t,$$

or:

$$\frac{\overline{K}_{t+1}}{\overline{K}_t} \frac{dK_{t+1}}{\overline{K}_{t+1}} = \left(1 - \delta\right) \frac{dK_t}{\overline{K}_t} + \frac{\overline{Y}_t}{\overline{K}_t} \frac{dY_t}{\overline{Y}_t} - \frac{\overline{C}_t}{\overline{K}_t} \frac{dC_t}{\overline{C}_t},$$

from which:

$$\left(1 + g\right) k_{t+1} = \left(1 - \delta\right) k_t + \frac{\overline{Y}_t}{\overline{K}_t} y_t - \frac{\overline{C}_t}{\overline{K}_t} c_t,$$

where we showed before that $\frac{\overline{Y}_t}{\overline{K}_t}$ and $\frac{\overline{C}_t}{\overline{K}_t}$ are the same as in the fixed-labor model.

Appendix F: Log-Linearizing the Variable-Labor Supply Model, Continued

- Using the log-linear production function yields:

$$k_{t+1} \approx \lambda_1 k_t + \lambda_2 (a_t + n_t) + (1 - \lambda_1 - \lambda_2) c_t,$$

with λ_1 and λ_2 the same as before.

- Finally,

$$\begin{aligned}
dR_{t+1} &= (1-\alpha)\alpha \frac{\left(dA_{t+1}\overline{N} + dN_{t+1}\overline{A}_{t+1}\right)\overline{K}_{t+1} - d\overline{K}_{t+1}\left(\overline{A}_{t+1}\overline{N}\right)}{\overline{K}_{t+1}^2} \left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}}\right)^{\alpha-1} \\
&= (1-\alpha)\alpha \frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}} (a_{t+1} + n_{t+1} - k_{t+1}) \left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}}\right)^{\alpha-1} \\
&= (1-\alpha)\alpha \left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}}\right)^{\alpha} (a_{t+1} + n_{t+1} - k_{t+1}).
\end{aligned}$$

- Then, using our results above,

$$r_{t+1} \approx \frac{(1-\alpha)\alpha\frac{r+\delta}{1-\alpha}(a_{t+1} + n_{t+1} - k_{t+1})}{(1-\alpha)\frac{r+\delta}{1-\alpha} + 1 - \delta} = \lambda_3 (a_{t+1} + n_{t+1} - k_{t+1}),$$

where $\lambda_3 \equiv \frac{\alpha(r+\delta)}{1+r}$, as in the fixed-labor model.

Appendix G: Solving the Variable Labor-Supply Model

- Recall equations (41), (43), and (45):

$$\begin{aligned} k_{t+1} &\approx \lambda_1 k_t + \lambda_2 (a_t + n_t) + (1 - \lambda_1 - \lambda_2) c_t, \\ E_t (c_{t+1} - c_t) &\approx \lambda_3 E_t (a_{t+1} + n_{t+1} - k_{t+1}), \\ n_t &\approx \mu [(1-\alpha) k_t + \alpha a_t - c_t] \end{aligned}$$

- Substitute (45) into (41):

$$k_{t+1} \approx \lambda_1 k_t + \lambda_2 a_t + \lambda_2 \mu (1-\alpha) k_t + \lambda_2 \mu \alpha a_t - \lambda_2 \mu c_t + (1 - \lambda_1 - \lambda_2) c_t,$$

or:

$$k_{t+1} \approx [\lambda_1 + \lambda_2 \mu (1-\alpha)] k_t + \lambda_2 (1 + \mu\alpha) a_t + [1 - \lambda_1 - \lambda_2 (1+\mu)] c_t. \tag{69}$$

- Substitute (45) into (43):

$$\begin{aligned} E_t (c_{t+1} - c_t) &\approx \lambda_3 E_t [a_{t+1} + \mu (1-\alpha) k_{t+1} + \mu\alpha a_{t+1} - \mu c_{t+1} - k_{t+1}] \\ &= \lambda_3 E_t \{[\mu (1-\alpha) - 1] k_{t+1} + (1 + \mu\alpha) a_{t+1} - \mu c_{t+1}\}. \end{aligned} \tag{70}$$

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Guess $c_t = \eta_{ck} k_t + \eta_{ca} a_t$ and substitute into (69):

$$
\begin{aligned}
k_{t+1} &\approx [\lambda_1 + \lambda_2 \mu (1-\alpha)]\, k_t + \lambda_2 (1 + \mu\alpha)\, a_t \\
&\quad + [1 - \lambda_1 - \lambda_2 (1+\mu)]\, \eta_{ck} k_t + [1 - \lambda_1 - \lambda_2 (1+\mu)]\, \eta_{ca} a_t \\
&= \eta_{kk} k_t + \eta_{ka} a_t,
\end{aligned}
$$

  with

$$
\begin{aligned}
\eta_{kk} &\equiv \lambda_1 + \lambda_2 \mu (1-\alpha) + [1 - \lambda_1 - \lambda_2 (1+\mu)]\, \eta_{ck}, \\
\eta_{ka} &\equiv \lambda_2 (1 + \mu\alpha) + [1 - \lambda_1 - \lambda_2 (1+\mu)]\, \eta_{ca}.
\end{aligned}
$$

- Substitute $k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t$ and $c_t = \eta_{ck} k_t + \eta_{ca} a_t$ into (70).

- Use $a_t = \phi a_{t-1} + \varepsilon_t$, so that $E_t (a_{t+1}) = \phi a_t$, and the fact that k_{t+1} is known at time t.

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Then,

$$\begin{aligned}
&\eta_{ck}\left(k_{t+1}-k_t\right)+\eta_{ca}\left(\phi a_t-a_t\right) \\
\approx\ &\lambda_3\left\{\left[\mu\left(1-\alpha\right)-1\right]k_{t+1}+\left(1+\mu\alpha\right)\phi a_t-\mu\eta_{ck}k_{t+1}-\mu\eta_{ca}\phi a_t\right\},
\end{aligned}$$

  or:

$$\begin{aligned}
&\eta_{ck}\left(\eta_{kk}-1\right)k_t+\eta_{ck}\eta_{ka}a_t+\eta_{ca}\left(\phi-1\right)a_t \\
\approx\ &\lambda_3\left\{\left[\mu\left(1-\alpha\right)-1-\mu\eta_{ck}\right]\left(\eta_{kk}k_t+\eta_{ka}a_t\right)+\left(1+\mu\alpha-\mu\eta_{ca}\right)\phi a_t\right\},
\end{aligned}$$

  and η_{ck} solves:

$$\eta_{ck}\left(\eta_{kk}-1\right)=\lambda_3\left[\mu\left(1-\alpha\right)-1-\mu\eta_{ck}\right]\eta_{kk}.$$

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Recalling $\eta_{kk} \equiv \lambda_1 + \lambda_2\mu\,(1-\alpha) + [1-\lambda_1-\lambda_2\,(1+\mu)]\,\eta_{ck}$, this equation becomes:

$$\begin{aligned} & \eta_{ck}\left\{\lambda_1 + \lambda_2\mu\,(1-\alpha) + [1-\lambda_1-\lambda_2\,(1+\mu)]\,\eta_{ck}\right\} - \eta_{ck} \\ = \; & \lambda_3\left[\mu\,(1-\alpha) - 1 - \mu\eta_{ck}\right]\left\{\lambda_1 + \lambda_2\mu\,(1-\alpha) + [1-\lambda_1-\lambda_2\,(1+\mu)]\,\eta_{ck}\right\}, \end{aligned}$$

which has the form:

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0,$$

with:

$$\begin{aligned} Q_2 &\equiv (1+\lambda_3\mu)\left[1-\lambda_1-\lambda_2\,(1+\mu)\right], \\ Q_1 &\equiv (1+\lambda_3\mu)\left[\lambda_1+\lambda_2\,(1-\alpha)\,\mu\right] - \lambda_3\left[\mu\,(1-\alpha)-1\right]\left[1-\lambda_1-\lambda_2\,(1+\mu)\right] - 1, \\ Q_0 &\equiv -\lambda_3\left[\mu\,(1-\alpha)-1\right]\left[\lambda_1+\lambda_2\,(1-\alpha)\,\mu\right]. \end{aligned}$$

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Finally, η_{ca} solves:

$$\begin{aligned}
& \eta_{ck}\eta_{ka} + \eta_{ca}(\phi - 1) \\
= \;& \lambda_3\left[\mu(1-\alpha) - 1 - \mu\eta_{ck}\right]\eta_{ka} + \lambda_3(1 + \mu\alpha - \mu\eta_{ca})\phi,
\end{aligned}$$

or:

$$\begin{aligned}
& \eta_{ca}(\phi - 1) + \lambda_3\mu\phi\eta_{ca} \\
= \;& \lambda_3\left[\mu(1-\alpha) - 1 - \mu\eta_{ck}\right]\eta_{ka} + \lambda_3(1 + \mu\alpha)\phi - \eta_{ck}\eta_{ka}.
\end{aligned}$$

- But $\eta_{ka} \equiv \lambda_2(1+\mu\alpha) + \left[1 - \lambda_1 - \lambda_2(1+\mu)\right]\eta_{ca}$. Hence, substituting and rearranging,

$$\eta_{ca} = \frac{(1+\alpha\mu)\left\{\lambda_3\phi - \lambda_2\left[\eta_{ck}(1+\lambda_3\mu) - \lambda_3\left[\mu(1-\alpha) - 1\right]\right]\right\}}{\left[1-\lambda_1-\lambda_2(1+\mu)\right]\left\{\eta_{ck}(1+\lambda_3\mu) - \lambda_3\left[\mu(1-\alpha)-1\right]\right\} - \left[1 - \phi(1+\lambda_3\mu)\right]}.$$

ECON 401

Advanced Macroeconomics

Topic 2

New Keynesian Macroeconomics

Fabio Ghironi

University of Washington

New Keynesian Theory: Monopoly Pricing

RELEVANT MARKET STRUCTURE(S)?

- Real business cycle (RBC)/neoclassical theory
  - All (goods) prices are determined in perfect competition
  - In both consumption-leisure and consumption-savings dimensions
  - Critical assumption: no firm is a price ***setter*** → no firm has any market power
- New Keynesian theory
  - Starting point: firms ***do*** wield (at least some) market power
  - Critical assumption: firms ***do*** set their (nominal) prices
  - Purposeful setting/re-setting of (nominal) prices may entail costs of some sort
    - "Menu costs," but soon interpret more broadly
    - Central issue in macro: how do "costs of adjusting prices" ("sticky prices") affect monetary policy insights and recommendations?
- Upcoming analysis
  - Step 1: Develop theory in which firms are purposeful price setters, not price takers
  - Step 2: Superimpose on the theory some "costs" of setting/re-setting nominal prices
  - Step 3: Study optimal monetary policy

MONOPOLISTIC COMPETITION

- Monopolistically-competitive view of goods markets the foundation of NK theory
  - An intermediate market structure between pure perfect competition and pure monopoly

- Framework
  - Allows for purposeful price setting by firms
  - Retains some competitive features of pure supply-and-demand theory
  - Assumes that goods are imperfect substitutes
    - The foundation/essence of market power
    - In contrast to the perfect substitutability of goods in theory of pure perfect competition

- Markup
  - The ratio of a firm's unit sales price to its marginal cost of production
  - A key concept in the theory of monopoly/monopolistic competition
  - A key measurable (sort of…) empirical concept

  - Perfect competition: markup = ______

NK Model Overview

- Monopolistic competition in goods markets the underlying market structure
- Operationalize by dividing goods markets into two "sectors"

- **Retail firms**
  - Each sells a perfectly-substitutable "retail good" in a perfectly-competitive market
  - Purchase differentiated "wholesale goods" in monopolistically-competitive markets

- **Wholesale firms**
  - An "infinite" number of them
  - Each produces a "wholesale good" imperfectly substitutable with any other "wholesale good"
  - → each wholesale firm is a price setter
  - "Wholesale goods" sold to retail firms

- Conceptual separation allows for separate consideration of
  - Price setting at the microeconomic level
  - Determination of market outcomes at the macroconomic level

Retail Firms

- A representative retail firm
- Operates a "production function" that "bundles together" wholesale goods into retail goods
  - Inputs: wholesale goods ONLY, no labor or other inputs required
  - Example: a retail store that produces no goods of its own

- Dixit-Stiglitz aggregator function/"production function"
  - Workhorse building block of NK theory

$$y_t = \left[\int_0^1 y_{it}^{1/\varepsilon} di \right]^{\varepsilon}$$

(Also a basic building block of theory of international trade)

Output of the retail good — An infinity (continuum) or differentiated wholesale goods

  - Parameter ε measures curvature
  - Elasticity of substitution between any pair of differentiated wholesale goods is $\varepsilon/(\varepsilon-1)$
  - ε also the critical determinant of profit-maximizing markup
  - Restriction for NK model to make any sense: $\varepsilon > 1$
  - Setting ε = _____ recovers perfect competition (i.e., RBC, not NK)

See all this soon

RETAIL FIRMS

- A continuous infinity of wholesale goods
  - A metaphor for "many varieties of goods"
  - Easier to deal with mathematically than discrete infinity (tools of calculus can be applied!)
  - And normalize to continuum [0,1] (could also say, i.e., [0,2], etc…)

- Schematic structure of goods markets

- Representative retail firm's profit function

Nominal price of the retail good **Nominal price of wholesale good *i***

$$P_t y_t - \int_0^1 P_{it} y_{it} di$$

Substitute Dixit-Stiglitz production function

$$P_t \left[\int_0^1 y_{it}^{1/\varepsilon} di \right]^{\varepsilon} - \int_0^1 P_{it} y_{it} di$$

Retail Firms

- Representative retail firm's profit-maximization problem

$$\max_{\{y_{it}\}_{i=0..\infty}} P_t \left[\int_0^1 y_{it}^{1/\varepsilon} di \right]^{\varepsilon} - \int_0^1 P_{it} y_{it} di$$

Chooses profit-maximizing quantity of input of *each* wholesale good. Focus analysis on any arbitrary wholesale good – call it y_{jt}.

- FOC with respect to y_{jt} (for any j)

.

.

.

- …after several rearrangements

$$y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{\frac{\varepsilon}{1-\varepsilon}} y_t$$

DEMAND FUNCTION FOR GOOD *j*

WHOLESALE FIRMS

- Focus on the activities of an arbitrary wholesale firm *j*
- Symmetry: assume that every wholesale firm makes decisions analogously
  - Consistent with the representative-agent approach
- So can speak of a "representative" wholesale firm

- Assume zero fixed costs of production

- Operates a constant-returns-to-scale (CRS) production technology in order to produce its unique, differentiated output
  - CRS: if all inputs are scaled up by the factor *x*, total output is scaled up by the factor *x*
  - Implementation of theory requires specifying neither the factors of production (i.e., labor, capital, etc) nor a production function (*f*(.))

Together, these imply a simple description of production

- Marginal cost of production
  - = average cost of production
  - is invariant to the quantity produced
    - i.e., mc is NOT a function mc(quantity)

WHOLESALE FIRMS

- **Representative wholesale firm's profit-maximization problem**

Total revenue depends on firm's production and its *own* product price.

mc is NOT a function of quantity produced – CRS assumption.

FC = 0 → $mc = ac$

$$\max_{P_{jt}} P_{jt} y_{jt} - P_t mc_{jt} y_{jt}$$

Conversion of production costs into nominal terms requires factor P_t, NOT P_{jt}. Because costs are not denominated in the firm's *own* prices.

Substitute in demand function for wholesale good j.

The critical point of analysis of monopoly: the firm *understands* and *internalizes* the effect of *its* price on the quantity that it sells.

$$\max_{P_{jt}} P_{jt} \left(\frac{P_{jt}}{P_t} \right)^{\frac{\varepsilon}{1-\varepsilon}} y_t - P_t mc_{jt} \left(\frac{P_{jt}}{P_t} \right)^{\frac{\varepsilon}{1-\varepsilon}} y_t$$

- **The sole choice object is P_{jt}**
  - **Compute FOC!**

WHOLESALE FIRMS

- Representative wholesale firm's profit-maximization problem

$$\max_{P_{jt}} P_{jt}\left(\frac{P_{jt}}{P_t}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t - P_t mc_{jt}\left(\frac{P_{jt}}{P_t}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t$$

- FOC with respect to ***P***

$$\frac{1}{1-\varepsilon} P_{jt}^{\frac{\varepsilon}{1-\varepsilon}} P_t^{\frac{\varepsilon}{\varepsilon-1}} y_t - \frac{\varepsilon}{1-\varepsilon} P_{jt}^{\frac{2\varepsilon-1}{1-\varepsilon}} P_t^{\frac{2\varepsilon-1}{\varepsilon-1}} mc_t y_t = 0$$

.
.
.

- …after several rearrangements

$$\frac{P_{jt}}{P_t} = \varepsilon mc_{jt}$$

Optimal relative price of wholesale firm *j* is a markup *ε* over marginal cost of production.

KEY PRICING RESULT OF DIXIT-STIGLITZ THEORY.

- Define relative price as $p_{jt} = \frac{P_{jt}}{P_t}$

In which case can express pricing rule as → $p_{jt} = \varepsilon mc_{jt}$

Or as optimal markup rule → $\mu_{jt} = p_{jt} / mc_{jt} = \varepsilon$

The Dixit-Stiglitz Framework

- **Key prediction of basic Dixit-Stiglitz theory**

$$\mu_{jt} = \frac{p_{jt}}{mc_{jt}} = \varepsilon$$

  - **Firms aim to keep their prices at a constant markup over marginal cost**

- **Empirical relevance of DS constant markup prediction?**
  - **Not very in the short run…**
  - **…but maybe in the long run**

- **Markups generally observed to be countercyclical (with respect to GDP)**
  - **During expansions, markups decline; during recessions, markups rise**
  - **(detrended, business cycle frequencies)**

- **DS framework has long been the main starting point for pricing theories; recent incorporation into studying**
  - **Customer switching effects**
  - **Brand loyalty**
  - **Search costs**

NEXT: The Dixit-Stiglitz framework as the foundation of New Keynesian sticky-price models.

NEW KEYNESIAN THEORY: THE MODERN STICKY-PRICE MODEL

RELEVANT MARKET STRUCTURE(S)?

- Real business cycle (RBC) / neoclassical theory
  - All (goods) prices are determined in perfect competition
  - In both consumption-leisure and consumption-savings dimensions
  - Critical assumption: no firm is a price ***setter*** → no firm has any market power

- New Keynesian theory
  - Starting point: firms ***do*** wield (at least some) market power
  - Critical assumption: firms ***do*** set their (nominal) prices
  - Purposeful setting / re-setting of (nominal) prices may entail costs of some sort
    - "Menu costs," but soon interpret more broadly
    - Central issue in macro: how do "costs of adjusting prices" ("sticky prices") affect monetary policy insights and recommendations?

- Upcoming analysis
  - Step 1: Develop theory in which firms are purposeful price setters, not price takers
  - NOW → Step 2: Superimpose on the theory some "costs" of setting / re-setting nominal prices
  - Step 3: Study optimal monetary policy

MENU COSTS

- Do firms incur "costs" in the very act of setting/re-setting nominal prices?
- If so, what is the nature and prevalence of these costs?
- A central issue in price theory

- Menu cost – any and all costs incurred directly due to the price (re-)setting process
  - Independent of any physical production costs – i.e., NOT a cost captured by standard "production functions"

- Two common views of nature of menu costs
  - Fixed menu cost: total menu cost is independent of the magnitude of the price change being considered
  - Example: cost of printing new prices on restaurant menus is probably independent of what the new prices are

  - Convex menu cost: total menu cost is convex and increasing in the magnitude of the price change being considered
  - Example: if "menu cost" includes "cost of angering customers," "managerial time," etc., convexity assumption may be more appropriate

Anderson and Simester; Zbracki et al papers

- Both fixed and convex are likely aspects of menu costs
- Formal theoretical NK model typically focuses only on convex menu costs

MODELING CONVEX MENU COSTS

- Introduce menu costs at level of wholesale firms
  - Because they actually (re-)set prices!
  - What does it mean for a firm that is not a price-setter to incur costs of setting prices?...

- Wholesale firm j incurs real menu cost of nominal price adjustment

$$\frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}}-1\right)^2$$

  - REAL cost of price adjustment – denominated in goods
  - Parameter $\Psi > 0$ governs "importance" of menu costs
    - $\Psi = 0$ means no menu cost, which recovers basic Dixit-Stiglitz framework

- Convex: the larger the percentage deviation of P_{jt} from P_{jt-1}, the larger the menu cost
  - Implication: a disincentive to adjusting prices "too quickly"
- Question: are downward adjustments just as costly as upward adjustments?
  - Intuition: "no"   Anderson and Simester evidence: "maybe?..."

RETAIL FIRMS

- **Representative retail firm's profit-maximization problem**

$$\max_{\{y_{it}\}_{i=0..\infty}} P_t \left[\int_0^1 y_{it}^{1/\varepsilon} di \right]^{\varepsilon} - \int_0^1 P_{it} y_{it} di$$

Chooses profit-maximizing quantity of input of *each* wholesale good. Focus analysis on any arbitrary wholesale good – call it y_{jt}.

- **FOC with respect to y_{jt} (for any j)**

.

.

.

- **…after several rearrangements**

$$y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{\frac{\varepsilon}{1-\varepsilon}} y_t$$

DEMAND FUNCTION FOR GOOD j

- **IDENTICAL TO BASIC (FLEXIBLE-PRICE) DIXIT-STIGLITZ FRAMEWORK!**

WHOLESALE FIRMS

- Focus on the activities of an arbitrary wholesale firm *j*
- Symmetry: assume that every wholesale firm makes decisions analogously
  - Consistent with the representative-agent approach
- So can speak of "the" wholesale firm

- Assume zero fixed costs of production

- Operates a constant-returns-to-scale (CRS) production technology in order to produce its unique, differentiated output
  - CRS: if all inputs are scaled up by the factor *x*, total output is scaled up by the factor *x*
  - Implementation of theory requires specifying neither the factors of production (i.e., labor, capital, etc) nor a production function (*f*(.))

Together, these imply a simple description of production

- Marginal cost of production
  - = average cost of production
  - is invariant to the quantity produced
    - i.e., mc is NOT a function mc(quantity)
- AND ALSO INCUR QUADRATIC MENU COSTS $\frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}}-1\right)^2$

The basis for "sticky" or "sluggish" nominal price adjustment

WHOLESALE FIRMS

- **Representative wholesale firm's period-*t* profit function**

Total revenue depends on firm's production and its *own* product price.

mc is NOT a function of quantity produced – CRS assumption. FC = 0 → *mc* = *ac* OF PRODUCTION!

$$P_{jt}y_{jt} - P_t mc_{jt} y_{jt} - \underbrace{\frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}} - 1\right)^2 P_t}$$

Conversion of production costs into nominal terms requires factor P_t, NOT P_{jt}. Because costs are not denominated in the firm's *own* prices.

Period-t menu costs, conversion of which into nominal terms requires factor P_t, NOT P_{jt}. Because costs are not denominated in the firm's *own* prices.

- **Presence of menu cost makes wholesale firm's profit-maximization problem a DYNAMIC one**
  - **Because any nominal price chosen in a given period has consequences for profits in the subsequent period through menu costs**
  - **Firm pricing problem is forward-looking**

Discount factor requires inflation adjustment.

And background assumption: no agency problem.

- **Dynamic (two-period) profit function**

$$P_{jt}y_{jt} - P_t mc_{jt} y_{jt} - \frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}} - 1\right)^2 P_t + \frac{\beta}{1+\pi_{t+1}}\left[P_{jt+1}y_{jt+1} - P_{t+1}mc_{t+1}y_{jt+1} - \frac{\psi}{2}\left(\frac{P_{jt+1}}{P_{jt}} - 1\right)^2 P_{t+1}\right]$$

WHOLESALE FIRMS

- **Representative wholesale firm's profit-maximization problem**

$$\max_{P_{jt}} P_{jt}y_{jt} - P_t mc_{jt}y_{jt} - \frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}} - 1\right)^2 P_t + \frac{\beta}{1+\pi_{t+1}}\left[P_{jt+1}y_{jt+1} - P_{t+1}mc_{t+1}y_{jt+1} - \frac{\psi}{2}\left(\frac{P_{jt+1}}{P_{jt}} - 1\right)^2 P_{t+1}\right]$$

Substitute in demand function for wholesale good *j* in both period t and t+1 (and t+2, t+3, t+4, …)

The critical point of analysis of monopoly: the firm *understands* and *internalizes* the effect of *its* price on the quantity that it sells.

$$\max_{P_{jt}} P_{jt}\left(\frac{P_{jt}}{P_{jt-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t - P_t mc_{jt}\left(\frac{P_{jt}}{P_{jt-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t - \frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}} - 1\right)^2 P_t$$

$$+\frac{\beta}{1+\pi_{t+1}}\left[P_{jt+1}\left(\frac{P_{jt+1}}{P_{jt}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_{t+1} - P_{t+1}mc_{t+1}\left(\frac{P_{jt+1}}{P_{jt}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_{t+1} - \frac{\psi}{2}\left(\frac{P_{jt+1}}{P_{jt}} - 1\right)^2 P_{t+1}\right]$$

In period t, firm chooses P_{jt}.

So FOC with respect to P_{jt}…

WHOLESALE FIRMS

- **Representative wholesale firm's profit-maximization problem**

$$\max_{P_{jt}} P_{jt}\left(\frac{P_{jt}}{P_{jt-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t - P_t mc_{jt}\left(\frac{P_{jt}}{P_{jt-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t - \frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}}-1\right)^2 P_t$$

$$+\frac{\beta}{1+\pi_{t+1}}\left[P_{jt+1}\left(\frac{P_{jt+1}}{P_{jt}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_{t+1} - P_{t+1} mc_{t+1}\left(\frac{P_{jt+1}}{P_{jt}}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_{t+1} - \frac{\psi}{2}\left(\frac{P_{jt+1}}{P_{jt}}-1\right)^2 P_{t+1}\right]$$

- **FOC with respect to P_{jt}**

$$\frac{1}{1-\varepsilon} P_{jt}^{\frac{\varepsilon}{1-\varepsilon}} P_t^{\frac{\varepsilon}{\varepsilon-1}} y_t - \frac{\varepsilon}{1-\varepsilon} P_{jt}^{\frac{2\varepsilon-1}{1-\varepsilon}} P_t^{\frac{2\varepsilon-1}{\varepsilon-1}} mc_t y_t - \psi\left(\frac{P_{jt}}{P_{jt-1}}-1\right)\frac{P_t}{P_{jt-1}} + \frac{\beta\psi}{1+\pi_{t+1}}\left(\frac{P_{jt+1}}{P_{jt}}-1\right)\frac{P_{t+1}}{P_{jt}}\frac{P_{jt+1}}{P_{jt}} = 0$$

- **If ψ = 0, collapses to**

$$\frac{P_{jt}}{P_t} = \varepsilon mc_t$$

Exactly the flexible-price Dixit-Stiglitz pricing rule

- **Existence of menu costs (ψ > 0) complicates pricing rule**

SYMMETRIC EQUILIBRIUM

- Now drop the distinction between "retail goods" and "wholesale goods"
  - Suppose "goods" are all identical

- A macro perspective
  - The "representative good"…
  - …since macro analysis is most concerned with aggregates

- Impose symmetry by now dropping *j* indexes – i.e., now suppose $P_{jt} = P_t$

$$\frac{1}{1-\varepsilon}\underbrace{P_t^{\frac{\varepsilon}{1-\varepsilon}}P_t^{\frac{\varepsilon}{\varepsilon-1}}}_{=1}y_t-\frac{\varepsilon}{1-\varepsilon}\underbrace{P_t^{\frac{2\varepsilon-1}{1-\varepsilon}}P_t^{\frac{2\varepsilon-1}{\varepsilon-1}}}_{=1}mc_t y_t-\psi\left(\frac{P_t}{P_{t-1}}-1\right)\frac{P_t}{P_{t-1}}+\frac{\beta\psi}{1+\pi_{t+1}}\left(\frac{P_{t+1}}{P_t}-1\right)\frac{P_{t+1}}{P_t}\frac{P_{t+1}}{P_t}=0$$

Several terms combine…

$$\frac{1}{1-\varepsilon}\left[1-\varepsilon mc_t\right]y_t-\psi\left(\frac{P_t}{P_{t-1}}-1\right)\frac{P_t}{P_{t-1}}+\frac{\beta\psi}{1+\pi_{t+1}}\left(\frac{P_{t+1}}{P_t}-1\right)\frac{P_{t+1}}{P_t}\frac{P_{t+1}}{P_t}=0$$

…and use definition of inflation $P_t / P_{t-1} = 1 + \pi_t$

NEW KEYNESIAN PHILLIPS CURVE

- The New Keynesian Phillips Curve (NKPC)

$$\frac{1}{1-\varepsilon}\left[1-\varepsilon mc_t\right]y_t-\psi\pi_t(1+\pi_t)+\beta\psi\pi_{t+1}(1+\pi_{t+1})=0$$

- Links period-t inflation to period-t marginal costs of production and period-(t+1) inflation

- "Classical" Phillips Curve
  - A link between period-t inflation and one component of period-t marginal costs of production (employment)
  - No "forward-looking" elements in it

- Forward-looking pricing/inflation behavior the key idea articulated by NKPC
  - Pricing decisions are inherently dynamic

- NKPC the cornerstone idea in New Keynesian theory

- Here derived from Rotemberg framework…can derive off alternative theories

ECON 401

Advanced Macroeconomics

Alternative New Keynesian Foundation

Fabio Ghironi
University of Washington

Introduction

- When we studied the monopolistic competition foundation of the New Keynesian model that Sanjay Chugh introduces in his textbook and slides, I mentioned that we could have set up the model without introducing the separation between wholesalers and retailers by just assuming that the representative consumer consumers a bundle of differentiated final products produced by firms with monopoly power that operate under monopolistic competition.

- These notes/slides show you how to do that.

A Consumption Bundle

- Suppose that instead of the homogeneous retail good produced by Chugh's perfectly competitive retailers, households in our economy consume a bundle of differentiated products that combines these products according to:

$$C_t = \left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}},$$

  where $\theta > 1$ is the elasticity of substitution between differentiated goods in the bundle, and $c_t(j)$ denoted consumption of product j, which is produced under monopolistic competition by firm j.

- The consumer has period utility $U(C_t)$ (or $U(C_t, 1 - N_t)$ if we want to have endogenous labor supply in the model) and the dynamics of C_t are determined by an intertemporal maximization problem as usual.

- We are interested in the expressions for the CPI of this economy (P_t) and for how the consumer allocates the C_t determined by the appropriate Euler equation to the individual differentiated $c_t(j)$'s.

The Welfare-Consistent CPI

- The welfare-based consumer price index of this economy is obtained as solution to the problem of finding the minimum amount of spending needed to purchase one unit of the bundle C_t.

- Formally,

$$P_t \equiv \min_{c_t(j)} \int_0^1 p_t(j)c_t(j)dj \quad \text{subject to} \quad C_t = 1,$$

where $p_t(j)$ is the price of good j, or, substituting the expression for C_t,

$$P_t = \min_{c_t(j)} \int_0^1 p_t(j)c_t(j)dj \quad \text{subject to} \quad \left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}} = 1.$$

- The Lagrangian for this problem is:

$$L = \int_0^1 p_t(j)c_t(j)dj + \lambda_t \left[1 - \left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}\right],$$

where λ_t is the Lagrange multiplier.

The Welfare-Consistent CPI, Continued

- Taking the derivative of this expression with respect to $c_t(j)$, setting it equal to zero, using $\left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}} = C_t$, and rearranging yields:

$$c_t(j) = \left(\frac{p_t(j)}{\lambda_t}\right)^{-\theta} C_t,$$

or:

$$c_t(j) = \left(\frac{p_t(j)}{\lambda_t}\right)^{-\theta}$$

once we recall the constraint $C_t = 1$.

- Now substitute the expression $c_t(j) = (p_t(j)/\lambda_t)^{-\theta}$ into $\left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}} = 1$.

- Simple algebra then allows you to obtain:

$$\lambda_t = \left(\int_0^1 p_t(j)^{1-\theta} dj\right)^{\frac{1}{1-\theta}}.$$

The Welfare-Consistent CPI, Continued

- Therefore,

$$c_t(j) = \left(\frac{p_t(j)}{\lambda_t}\right)^{-\theta} = p_t\left(j\right)^{-\theta}\left(\int_0^1 p_t(j)^{1-\theta}dj\right)^{\frac{\theta}{1-\theta}}.$$

- Recall that P_t is defined as the value of spending ($\int_0^1 p_t(j)c_t(j)dj$) such that spending needed for $C_t = 1$ is minimized, i.e., such that $c_t(j)$ obeys this expression.
  - As usual in this class, the shapes of objective function and constraint are such that it is not necessary to verify second-order conditions.

- Hence,

$$P_t = \int_0^1 p_t(j)^{1-\theta}\left(\int_0^1 p_t(j)^{1-\theta}dj\right)^{\frac{\theta}{1-\theta}} dj = \left(\int_0^1 p_t(j)^{1-\theta}dj\right)^{\frac{1}{1-\theta}}.$$

- Note that this expression for the price index implies the usual property that $P_t = p_t$ in the symmetric equilibrium of the model.

Differentiated Good Demand

- The optimal demand for each individual differentiated good j is found by solving the problem:

$$\max_{c_t(j)} C_t \quad \text{subject to...} \int_0^1 p_t(j) c_t(j) dj = S_t,$$

where S_t is an exogenously imposed amount of spending.

  - Note that this does not mean finding a different C_t from the one implied by the intertemporal utility maximization problem.
  - C_t remains determined by the relevant Euler equation.
  - The solution to the problem we are studying now will tell us how best to allocate C_t across the individual differentiated goods.

Differentiated Good Demand, Continued

- Substituting the expression for C_t in the maximization problem above yields:

$$\max_{c_t(j)} \left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \quad \text{subject to...} \int_0^1 p_t(j) c_t(j) dj = S_t,$$

and the Lagrangian for this problem is

$$L = \left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} + \phi_t \left(S_t - \int_0^1 p_t(j) c_t(j) dj \right).$$

- Taking the derivative of the Lagrangian with respect to $c_t(j)$, setting it equal to zero, using $\left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} = C_t$, and rearranging yields:

$$c_t(j) = \left(\phi_t p_t(j)\right)^{-\theta} C_t.$$

- Substituting this into $C_t = \left(\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}$ yields an equation that can be solved for ϕ_t to obtain:

$$\phi_t = \left(\int_0^1 p_t(j)^{1-\theta} dj \right)^{-\frac{1}{1-\theta}}.$$

- But comparing this to the expression for P_t obtained above immediately implies that $\phi_t = P_t^{-1}$.

Differentiated Good Demand, Continued

- Hence,

$$c_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\theta} C_t.$$

- Note that this demand function has the same expression as the demand of a differentiated wholesale good by the retailer in Sanjay Chugh's version of the New Keynesian framework.

- Hence, everything else follows identically.

- In particular, producer j's profit maximization problem will yield:

$$\frac{p_t\left(j\right)}{P_t} = \frac{\theta}{\theta - 1} mc_t(j).$$

- Identical production functions and symmetry across producers will imply $mc_t(j) = mc_t$, $p_t\left(j\right) = p_t$, and the price index expression will imply $P_t = p_t$, as we already noted.

- The demand expression for $c_t(j)$ in the symmetric equilibrium will in turn imply $c_t = C_t$, very much like we had Y_t (output of the retail bundle) $= y_t$ (output of each wholesale good) in Chugh's framework.

- All the properties of the framework, under flexible or sticky prices, follow identically.

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Advanced Macroeconomics

Topic 3

Macroeconomic Policy

Fabio Ghironi
University of Washington

OPTIMAL MONETARY POLICY: THE FLEXIBLE PRICE CASE

BASICS OF OPTIMAL POLICY ANALYSIS

- Describe the demand-side environment (i.e., consumers)
  - Arguments of utility function?
  - Which assets trade in private-sector financial markets?
  - Derive consumer optimality conditions

- Describe the supply-side environment (i.e., firms)
  - Which inputs are used in production process?
  - Derive firm profit-maximizing conditions
    - So far: simple factor price = marginal product conditions (i.e., wage = mpn, etc.)
    - Soon: New Keynesian firm analysis more involved (price-setting decisions)

- Describe actions/role of government
  - How is monetary policy conducted? How is fiscal policy conducted?
  - How do government policy choices affect private sector behavior?

- Describe resource constraint

- Describe private-sector equilibrium
  - For given policy choices by government, how does market equilibrium arise?
  - How does price adjustment/setting affect market clearing?

- Optimal policy analysis best thought of as picking a government policy that induces the "best" private-sector equilibrium

CASH-IN-ADVANCE FRAMEWORK

- Cash-in-advance (CIA) an alternative way of modeling role of money
  - Alternative to MIU framework
  - Highlights medium-of-exchange role of money

- Representative consumer
  - Period-*t* utility function *u*(*c*, 1-*n*)
  - Subjective discount factor *β*
  - Period-*t* budget constraint just as in MIU model

Total time is "1 unit."

$$P_t c_t + P_t^b B_t + M_t + S_t a_t + \tau_t = P_t w_t n_t + B_{t-1} + M_{t-1} + (S_t + D_t) a_{t-1}$$

…and so on for period *t*+1, *t*+2, etc.

Lump-sum tax (used to effect changes in money supply – more soon…)

  - In each period, a cash-in-advance constraint

$$P_t c_t = M_t$$ in period *t*

$$P_{t+1} c_{t+1} = M_{t+1}$$ in period *t*+1

$$P_{t+2} c_{t+2} = M_{t+2}$$ and so on…

Captures idea that (nominal) consumption expenditures limited by how much cash (money) an individual has

(Technically want to model as inequality constraint, $P_t c_t \le M_t$, but equality suffices to illustrate main ideas)

CASH-IN-ADVANCE FRAMEWORK: ANALYSIS

- **Lagrangian**

λ is multiplier on budget constraint

μ is multiplier on CIA constraint

$$
\begin{aligned}
& u(c_t, 1-n_t) + \beta u(c_{t+1}, 1-n_{t+1}) + \beta^2 u(c_{t+2}, 1-n_{t+2}) + \ldots \\
& + \lambda_t \left[W_t n_t + M_{t-1} + B_{t-1} + (S_t + D_t) a_{t-1} + \tau_t - P_t c_t - P_t^b B_t - M_t - S_t a_t \right] \\
& + \mu_t \left[M_t - P_t c_t \right] \\
& + \beta \lambda_{t+1} \left[W_{t+1} n_{t+1} + M_t + B_t + (S_{t+1} + D_{t+1}) a_t + \tau_{t+1} - P_{t+1} c_{t+1} - P_{t+1}^b B_{t+1} - M_{t+1} - S_{t+1} a_{t+1} \right] \\
& + \beta \mu_{t+1} \left[M_{t+1} - P_{t+1} c_{t+1} \right] \\
& + \ldots.
\end{aligned}
$$

- **FOCs**

c_t:

n_t:

M_t:

B_t:

a_t:

- **Combine into "MRS = price ratio" type of optimality conditions**

Consumption-leisure optimality condition

$$\frac{u_2(c_t, 1-n_t)}{u_1(c_t, 1-n_t)} = w_t \left[1 + \frac{i_t}{1+i_t} \right]^{-1}$$

$$\frac{u_2(c_t, 1-n_t)}{u_2(c_{t+1}, 1-n_{t+1})} = \beta(1+r_t) \frac{W_t}{W_{t+1}} \frac{P_{t+1}}{P_t}$$

Consumption-savings optimality condition

CASH-IN-ADVANCE FRAMEWORK: ANALYSIS

- Consumption-leisure optimality condition

$$\frac{u_2(c_t, 1-n_t)}{u_1(c_t, 1-n_t)} = w_t \left[1 + \frac{i_t}{1+i_t}\right]^{-1}$$

  - Relevant relative price ratio depends on real wage….
    …but also here on nominal interest rate
  -
    - Nominal interest rate (i.e., fact that transactions are monetary) acts as a tax on consumption-leisure margin

- Optimal policy analysis in CIA framework
  - Reduces to determining the welfare-maximizing (aka utility-maximizing) tax to impose on consumption-leisure margin
  - → What is the optimal nominal interest rate?

- Efficiency concerns will shape the answer to the optimal policy question

CASH-IN-ADVANCE FRAMEWORK

- Firms
  - Very simple model of production
  - Production technology in every period

$$y_t = f(n_t) = n_t$$

  - Can think of Cobb-Douglas with capital share = 0

- Profit maximization
  - In every period, representative firm maximizes profit (in real terms)

$$f(n_t) - w_t n_t$$

  - → Labor demand function

$$w_t = 1 \forall t$$

  - Perfectly elastic labor demand function reflects *lack of* diminishing marginal product

CASH-IN-ADVANCE FRAMEWORK

- Government
  - Assume only monetary policy is operative
  - Ignore fiscal policy considerations

- Central bank's budget constraint – in every period *t*,

$$M_t - M_{t-1} = \tau_t$$

- Lump-sum tax τ used to ***implement changes in*** nominal money supply

- Lump-sum assumption allows for
  - Ignoring fiscal considerations – i.e., monetary policy is independent of fiscal policy
  - Private sector views central bank's policy decisions as independent of any individual market participant's decisions
  - "Independent of" is the crux of the idea of "lump sum"…

- Suppose policy set according to a money-growth-rate rule

$$M_t = (1+g_t)M_{t-1} \Leftrightarrow \tau_t = g_t M_{t-1}$$

  - g_t the growth rate of money supply in period t; isomorphic to interest rate rule

CASH-IN-ADVANCE FRAMEWORK

- **Resource constraint – in every period *t*,**

$$c_t = n_t$$

  - **All output used for (only) consumption**

- **Summarize private-sector equilibrium conditions**

Describes demand side

  - **Consumption-leisure optimality condition** $\frac{u_2(c_t,1-n_t)}{u_1(c_t,1-n_t)} = w_t\left[1+\frac{i_t}{1+i_t}\right]^{-1}$
  - **Consumption-savings optimality condition** $\frac{u_2(c_t,1-n_t)}{u_2(c_{t+1},1-n_{t+1})} = \beta(1+r_t)\frac{W_t}{W_{t+1}}\frac{P_{t+1}}{P_t}$
  - **Cash-in-advance constraint** $c_t = \frac{M_t}{P_t}$

Describes supply side

  - **Labor-demand condition** $w_t = 1$

Describes market clearing

  - **Resource constraint** $c_t = n_t$

CASH-IN-ADVANCE FRAMEWORK

- Condense private-sector equilibrium conditions….
  - …by imposing $w_t = 1$ and $n_t = c_t$ everywhere
  - Consumption-leisure optimality condition $$\frac{u_2(c_t, 1-c_t)}{u_1(c_t, 1-c_t)} = \left[1 + \frac{i_t}{1+i_t}\right]^{-1}$$
  - Consumption-savings optimality condition $$\frac{u_2(c_t, 1-c_t)}{u_2(c_{t+1}, 1-c_{t+1})} = \beta(1+r_t)$$
  - Cash-in-advance constraint $c_t = \frac{M_t}{P_t} \Rightarrow \frac{c_t}{c_{t-1}} = \frac{1+g_t}{1+\pi_t}$ Rewriting in terms of growth rates
- Limit attention to steady-state (i.e., long run) policy questions
- Can express entire steady-state private-sector equilibrium as

$$\frac{u_2\left(\bar{c}, 1-\bar{c}\right)}{u_1\left(\bar{c}, 1-\bar{c}\right)} = \frac{1+g}{1+g+1+g-\beta}$$

Optimal Policy Analysis

- Can express entire steady-state private-sector equilibrium as

$$\frac{u_2\left(\bar{c},1-\bar{c}\right)}{u_1\left(\bar{c},1-\bar{c}\right)}=\frac{1+g}{1+g+1+g-\beta}$$

- Defines implicitly a reaction function $\bar{c}(g)$

  - A (potentially complicated..) summary description of how private-sector equilibrium quantities depend on any given choice of government policy ***g***

- Maintained assumption
  - Central bank knows/understand perfectly the private-sector reaction function
  - Realism? Impossible for a central bank to literally know this…
  - …but provides a starting point for analysis

- → Central bank takes into account the reaction function $\bar{c}(g)$ when setting (optimal) policy

OPTIMAL POLICY ANALYSIS

- Goal of policy makers
  - Maximize welfare (utility) of representative consumer
  - In steady-state

$$\sum_{s=0}^{\infty} \beta^s u\left(\bar{c}, 1-\bar{c}\right) = \frac{u\left(\bar{c}, 1-\bar{c}\right)}{1-\beta}$$

Recall infinite summation formula

$$\downarrow$$

$$\sum_{s=0}^{\infty} \beta^s u\left(\bar{c}(g), 1-\bar{c}(g)\right) = \frac{u\left(\bar{c},(g)1-\bar{c}(g)\right)}{1-\beta}$$

Taking into account private-sector reaction function

- The formal policy problem
  - Choose *g* that maximizes private-sector welfare
  - The *c*(*g*) function summarizes the behavior of private markets

OPTIMAL POLICY ANALYSIS

- **Optimal monetary policy problem**

$$\max_{g} \sum_{s=0}^{\infty} \beta^{s} u\left(\bar{c}(g), 1-\bar{c}(g)\right) = \frac{u\left(\bar{c},(g)1-\bar{c}(g)\right)}{1-\beta}$$

- **FOC with respect to *g***
  - **KEY: NOW need to take into account the dependence of private-market outcomes on the policy in place**

- **Compare with private-sector equilibrium**

$$\frac{u_2\left(\bar{c}, 1-\bar{c}\right)}{u_1\left(\bar{c}, 1-\bar{c}\right)} = \frac{1+g}{1+g+1+g-\beta}$$

- **If policy is set optimally**

$$\frac{u_2\left(\bar{c}, 1-\bar{c}\right)}{u_1\left(\bar{c}, 1-\bar{c}\right)} = 1$$

- **QUESTION: What money growth rate *g* achieves this outcome?**
  - ***g* = *β* – 1 aligns the private-sector outcome with the policymaker's desired outcome**

ALIGNING THESE IS THE GOAL!

FRIEDMAN RULE

- **Optimal long-run money growth rate is $g = \beta - 1$**
- **With $\beta < 1$…**
- **…money supply should decrease in the long run!**

- **Optimal long-run inflation**
  - **Monetarist link**

$$\pi = g = \beta - 1 < 0$$

A normative statement

  - **Long-run deflation!**
  - **Central bank should seek to target negative inflation on average**

- **Optimal long-run nominal interest rate**
  - **$i = 0$ from Fisher relation (aka consumption-savings optimality condition)**

- **The Friedman Rule**
  - **Seminal 1969 analysis**
  - **Whether stated in terms of long-run target for nominal interest rate or long-run target for inflation / money growth**

UNDERSTANDING THE FRIEDMAN RULE

- **What does $g = \beta - 1$ achieve?**

- **Eliminates the wedge in the consumption-leisure dimension**
  - **Economic efficiency achieved if**

$$\frac{u_2\left(\bar{c},1-\bar{c}\right)}{u_1\left(\bar{c},1-\bar{c}\right)}=1$$

Derive based on SOCIAL PLANNER problem.

Distinct from OPTIMAL POLICY problem!

  - **Private-sector outcome**

$$\frac{u_2\left(\bar{c},1-\bar{c}\right)}{u_1\left(\bar{c},1-\bar{c}\right)}=\frac{1+g}{1+g+1+g-\beta}$$

  - **Friedman Rule allows policy makers to achieve economic efficiency in private markets – even though central bank is NOT a Social Planner**
    - **Notice very nuanced/precise statements/logic…**

- **Sets the (opportunity) costs of holding alternative nominal assets (money and nominal bonds) equal to each other**

- **Makes CIA constraint "disappear"** $g=\beta-1 \Leftrightarrow i=0 \Leftrightarrow \mu=0$ **(examine consumer FOCs)**

Practical Relevance of Friedman Rule

- **A benchmark in the theory of monetary policy**
  - **Akin to the theoretical benchmark Ricardian Equivalence provides for fiscal policy…**

- **Do monetary authorities actually follow the Friedman Rule?**
  - **Japan for the past 10+ years: nominal interest rates virtually zero**
  - **U.S. right now: nominal interest rates virtually zero**

  - **But does this seem attributable to "good policy" in "normal times"**
  - **…or the need to run the best policy in bad times?**

- **Friedman Rule a very controversial result in monetary theory**
  - **Strikes many as simply not sensible or practical**
  - **Question: What (important) features of the economy are missing from the analysis on which the Friedman Rule is premised?**

- **Keynesian/New Keynesian answer: sticky prices need to be considered in any normative analysis of monetary policy**
  - **Next: New Keynesian policy analysis**

OPTIMAL MONETARY POLICY: THE STICKY-PRICE CASE

RELEVANT MARKET STRUCTURE(S)?

- Real business cycle (RBC) / neoclassical theory
  - All (goods) prices are determined in perfect competition
  - In both consumption-leisure and consumption-savings dimensions
  - Critical assumption: no firm is a price ***setter*** → no firm has any market power

- New Keynesian theory
  - Starting point: firms ***do*** wield (at least some) market power
  - Critical assumption: firms ***do*** set their (nominal) prices
  - Purposeful setting / re-setting of (nominal) prices may entail costs of some sort
    - "Menu costs," but soon interpret more broadly
    - Central issue in macro: how do "costs of adjusting prices" ("sticky prices") affect monetary policy insights and recommendations?

- Upcoming analysis
  - Step 1: Develop theory in which firms are purposeful price setters, not price takers
  - Step 2: Superimpose on the theory some "costs" of setting / re-setting nominal prices
  - NOW → Step 3: Study optimal monetary policy

BASICS OF OPTIMAL POLICY ANALYSIS

- **Describe the demand-side environment (i.e., consumers)**
  - **Arguments of utility function?**
  - **Which assets trade in private-sector financial markets?**
  - **Derive consumer optimality conditions**

- **Describe the supply-side environment (i.e., firms)**
  - **Which inputs are used in production process?**
  - **Derive firm profit-maximizing conditions**
    - **So far: simple factor price = marginal product conditions (i.e., wage = mpn, etc.)**
    - **Soon: New Keynesian firm analysis more involved (price-setting decisions)**

- **Describe actions/role of government**
  - **How is monetary policy conducted? How is fiscal policy conducted?**
  - **How do government policy choices affect private sector behavior?**

- **Describe resource constraint**

- **Describe private-sector equilibrium**
  - **For given policy choices by government, how does market equilibrium arise?**
  - **How does price adjustment/setting affect market clearing?**

- **Optimal policy analysis best thought of as picking a government policy that induces the "best" private-sector equilibrium**

"CASHLESS" NK FRAMEWORK: CONSUMERS

- **Basic NK tenet**
  - **Money demand issues (i.e., medium-of-exchange role of money) not very important in modern developed economies**
  - **"Cashless" analysis**

- **Implications for formal NK analysis – monetary policy…**
  - **…does not operate on/through demand-side of economy (consumers)**
  - **…operates on/through supply-side of economy (firms)**

- **Recent events: monetary policy operates on/through financial sector of economy?**
  - **Intermediation between demand-side and supply-side…more research coming…**

- **Representative consumer**
  - **Period-*t* utility function *u*(*c*, 1-*n*)**
  - **Subjective discount factor β**
  - **Period-*t* budget constraint identical to CIA or MIU model, except no money balances**

$$P_t c_t + P_t^b B_t + S_t a_t = P_t w_t n_t + B_{t-1} + (S_t + D_t) a_{t-1}$$

  …and so on for period *t*+1, *t*+2, etc.

  - **No MIU component or CIA constraint**

"CASHLESS" NK FRAMEWORK: CONSUMERS

- **Lagrangian**

λ is multiplier on budget constraint

$$u(c_t,1-n_t)+\beta u(c_{t+1},1-n_{t+1})+\beta^2 u(c_{t+2},1-n_{t+2})+\ldots$$
$$+\lambda_t\left[P_t w_t n_t + B_{t-1} + (S_t + D_t)a_{t-1} - P_t c_t - P_t^b B_t - S_t a_t\right]$$
$$+\beta\lambda_{t+1}\left[P_{t+1} w_{t+1} n_{t+1} + B_t + (S_{t+1} + D_{t+1})a_t - P_{t+1} c_{t+1} - P_{t+1}^b B_{t+1} - S_{t+1} a_{t+1}\right]$$
$$+\ldots.$$

- **FOCs**

***c_t*:**

***n_t*:**

***B_t*:**

***a_t*:**

KEY CONCEPTUAL DIFFERENCE BETWEEN NK ANALYSIS AND CIA ANALYSIS:

Instead, nom i.r. shows up only in consumption-savings optimality condition

- **Combine into "MRS = price ratio" type of optimality condition**

Consumption-leisure optimality condition

$$\frac{u_2(c_t,1-n_t)}{u_1(c_t,1-n_t)} = w_t$$

AND

$$\frac{u_1(c_t,1-n_t)}{\beta u_1(c_{t+1},1-n_{t+1})} = (1+i_t)\frac{P_t}{P_{t+1}}$$

Consumption-savings optimality condition (from bond first-order condition)

Retail Firms

- Representative retail firm's profit-maximization problem

$$\max_{\{y_{it}\}_{i=0..\infty}} P_t \left[\int_0^1 y_{it}^{1/\varepsilon} di \right]^{\varepsilon} - \int_0^1 P_{it} y_{it} di$$

Chooses profit-maximizing quantity of input of *each* wholesale good. Focus analysis on any arbitrary wholesale good – call it y_{jt}.

- FOC with respect to y_{jt} (for any j)

.

.

.

- …after several rearrangements

$$y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{\frac{\varepsilon}{1-\varepsilon}} y_t$$

DEMAND FUNCTION FOR GOOD j

- IDENTICAL TO BASIC (FLEXIBLE-PRICE) DIXIT-STIGLITZ FRAMEWORK!

WHOLESALE FIRMS

- **Representative wholesale firm's profit-maximization problem**

$$\max_{P_{jt}} P_{jt}y_{jt} - P_t mc_{jt}y_{jt} - \frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}} - 1\right)^2 P_t + \frac{\beta}{1+\pi_{t+1}}\left[P_{jt+1}y_{jt+1} - P_{t+1}mc_{t+1}y_{jt+1} - \frac{\psi}{2}\left(\frac{P_{jt+1}}{P_{jt}} - 1\right)^2 P_{t+1}\right]$$

$\downarrow$

Substitute in demand function for wholesale good *j* in both period t and t+1 (and t+2, t+3, t+4, …)

$\downarrow$

In period t, firm chooses P_{jt}

So FOC with respect to P_{jt}

$\downarrow$

Symmetric equilibrium

$\downarrow$

$$\frac{1}{1-\varepsilon}\left[1-\varepsilon mc_t\right]y_t - \psi\pi_t(1+\pi_t) + \beta\psi\pi_{t+1}(1+\pi_{t+1}) = 0$$

New Keynesian Phillips Curve

WHOLESALE FIRMS

- **So far haven't considered (explicitly) the inputs to a wholesale firm's production process**

- **Very simple model of production**
  - **Production technology in every period (for any wholesale firm *j*)**

$$y_{jt} = f(n_{jt}) = n_{jt}$$

  - **Can think of Cobb-Douglas with capital share = 0**

- **Labor hired by wholesale firm *j* taking market wage w_t as given**

- **Recall: CRS production technology → marginal cost of production is independent of quantity produced**

"CASHLESS" NK FRAMEWORK: GOVERNMENT

- **Money is a physical object in the "background"**
  - **DOES exist…**
  - **…so there IS a budget constraint for it**
  - **But not of direct importance for (routine) monetary policy issues**
    - **(Hence doesn't appear in consumers' budget constraints….where does it go?…)**

- **Central bank's budget constraint – in every period *t*,**

$$M_t = (1+g_t)M_{t-1} \Leftrightarrow \tau_t = g_t M_{t-1}$$

  - **g_t the growth rate of money supply in period *t***

NONETHELESS: $g = \pi$ in long run (i.e., steady state)

Even NK theory views basic monetarist quantity-theoretic link between money growth and inflation as being correct in the *long run*

- **Money-supply rule technically isomorphic to interest rate rule**

- **But interest rates explicitly the "policy tool" in New Keynesian analysis**
- **i.e., to *implement* an inflation target, what is the nominal interest rate the central bank should set?**
  - **Instead of focusing on what is the money growth rate the central bank should set?**

"CASHLESS" NK FRAMEWORK

- Resource constraint – in every period *t*,

$$c_t + \frac{\psi}{2}(\pi_t)^2 = n_t \left(= f(n_t)\right)$$

  - Total output used for private-sector consumption…
  - …and menu costs
    - Recall: menu costs are a REAL cost – hence absorb some of the economy's resources

- Summarize private-sector equilibrium conditions

Describes demand side

  - Consumption-leisure optimality condition $\frac{u_2(c_t, 1-n_t)}{u_1(c_t, 1-n_t)} = w_t$
  - Consumption-savings optimality condition $\frac{u_1(c_t, 1-n_t)}{\beta u_1(c_{t+1}, 1-n_{t+1})} = \frac{1+i_t}{1+\pi_{t+1}}$

Describes supply side

  - Labor-demand condition $w_t < f'(n_t) = 1$ Note wage is LESS THAN MP of labor
  - New Keynesian Phillips Curve $\frac{1}{1-\varepsilon}\left[1-\varepsilon mc_t\right]n_t - \psi\pi_t(1+\pi_t) + \beta\psi\pi_{t+1}(1+\pi_{t+1}) = 0$

Describes market clearing

  - Resource constraint $c_t + \frac{\psi}{2}(\pi_t)^2 = n_t$

"CASHLESS" NK FRAMEWORK

- **Resource constraint – in every period *t*,**

$$c_t + \frac{\psi}{2}\left(\pi_t\right)^2 = n_t \left(= f(n_t)\right)$$

  - **Total output used for private-sector consumption…**
  - **…and menu costs**
    - **Recall: menu costs are a REAL cost – hence absorb some of the economy's resources**

- **Summarize private-sector equilibrium conditions**

Describes demand side

  - **Consumption-leisure optimality condition** $\frac{u_2(c_t,1-n_t)}{u_1(c_t,1-n_t)} = w_t$
  - **Consumption-savings optimality condition** $\frac{u_1(c_t,1-n_t)}{\beta u_1(c_{t+1},1-n_{t+1})} = \frac{1+i_t}{1+\pi_{t+1}}$

Describes supply side

  - **Labor-demand condition** $mc_t < f'(n_t) = 1$ $\boxed{mc_t = w_t \text{ in every period}}$
  - **New Keynesian Phillips Curve** $\frac{1}{1-\varepsilon}\left[1-\varepsilon mc_t\right]n_t - \psi\pi_t(1+\pi_t) + \beta\psi\pi_{t+1}(1+\pi_{t+1}) = 0$

Describes market clearing

  - **Resource constraint** $c_t + \frac{\psi}{2}\left(\pi_t\right)^2 = n_t$

"CASHLESS" NK FRAMEWORK

- Condense private-sector equilibrium conditions….
  - …by imposing $w_t = mc_t$ and $n_t = c_t + \frac{\psi}{2}\left(\pi_t\right)^2$ everywhere

Consumption-leisure optimality condition

$$\frac{u_2\left(c_t, 1-c_t-\frac{\psi}{2}(\pi_t)^2\right)}{u_1\left(c_t, 1-c_t-\frac{\psi}{2}(\pi_t)^2\right)} = mc_t$$

Consumption-savings optimality condition

$$\frac{u_1\left(c_t, 1-c_t-\frac{\psi}{2}(\pi_t)^2\right)}{\beta u_1\left(c_{t+1}, 1-c_{t+1}-\frac{\psi}{2}(\pi_{t+1})^2\right)} = \frac{1+i_t}{1+\pi_{t+1}}$$

One final substitution to condense things….

- New Keynesian Phillips Curve

$$\frac{1}{1-\varepsilon}\left[1-\varepsilon mc_t\right]\left(1-c_t-\frac{\psi}{2}\pi_t^2\right)-\psi\pi_t(1+\pi_t)+\beta\psi\pi_{t+1}(1+\pi_{t+1})=0$$

"CASHLESS" NK FRAMEWORK

- **Condense private-sector equilibrium conditions….**
  - **…by imposing $w_t = mc_t$ and** $n_t = c_t + \frac{\psi}{2}(\pi_t)^2$ **everywhere**
  - **Consumption-savings optimality condition**

$$\frac{u_1\left(c_t, 1-c_t-\frac{\psi}{2}(\pi_t)^2\right)}{\beta u_1\left(c_{t+1}, 1-c_{t+1}-\frac{\psi}{2}(\pi_{t+1})^2\right)} = \frac{1+i_t}{1+\pi_{t+1}}$$

  - **New Keynesian Phillips Curve**

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c_t, 1-c_t-\frac{\psi}{2}(\pi_t)^2\right)}{u_1\left(c_t, 1-c_t-\frac{\psi}{2}(\pi_t)^2\right)}\right]\left(1-c_t-\frac{\psi}{2}\pi_t^2\right)-\psi\pi_t(1+\pi_t)+\beta\psi\pi_{t+1}(1+\pi_{t+1})=0$$

"CASHLESS" NK FRAMEWORK

- **Condense private-sector equilibrium conditions….**
  - **…by imposing $w_t = mc_t$ and $n_t = c_t + \frac{\psi}{2}(\pi_t)^2$ everywhere**
  - **Consumption-savings optimality condition in the steady state**

$$\frac{u_1\left(c, 1-c-\frac{\psi}{2}\pi^2\right)}{\beta u_1\left(c, 1-c-\frac{\psi}{2}\pi^2\right)} = \frac{1+i}{1+\pi}$$

  - **New Keynesian Phillips Curve in the steady state**

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c, 1-c-\frac{\psi}{2}\pi^2\right)}{u_1\left(c, 1-c-\frac{\psi}{2}\pi^2\right)}\right]\left(1-c-\frac{\psi}{2}\pi^2\right)-\psi\pi(1+\pi)+\beta\psi\pi(1+\pi)=0$$

- **Limit attention to steady-state (i.e., long run) policy questions**

"CASHLESS" NK FRAMEWORK

- **Condense private-sector equilibrium conditions….**
  - **…by imposing $w_t = mc_t$ and** $n_t = c_t + \frac{\psi}{2}(\pi_t)^2$ **everywhere**
  - **Consumption-savings optimality condition in the steady state**

$$\frac{u_1\left(c, 1-c-\frac{\psi}{2}g^2\right)}{\beta u_1\left(c, 1-c-\frac{\psi}{2}g^2\right)} = \frac{1+i}{1+g}$$

  - **New Keynesian Phillips Curve in the steady state**

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c, 1-c-\frac{\psi}{2}g^2\right)}{u_1\left(c, 1-c-\frac{\psi}{2}g^2\right)}\right]\left(1-c-\frac{\psi}{2}g^2\right)-\psi g(1+g)+\beta\psi g(1+g)=0$$

- **Limit attention to steady-state (i.e., long run) policy questions**
  - **And (finally!…) use long-run monetarist relationship $g = \pi$**

"CASHLESS" NK FRAMEWORK

- **Condense private-sector equilibrium conditions….**
  - **…by imposing $w_t = mc_t$ and $n_t = c_t + \frac{\psi}{2}(\pi_t)^2$ everywhere**
  - **Consumption-savings optimality condition in the steady state**

NK view:

Monetary policy does not operate through demand side…

$$\frac{u_1\left(c, 1-c-\frac{\psi}{2}g^2\right)}{\beta u_1\left(c, 1-c-\frac{\psi}{2}g^2\right)} = \frac{1+i}{1+g}$$

  - **New Keynesian Phillips Curve in the steady state**

Monetary policy operates through supply side

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c, 1-c-\frac{\psi}{2}g^2\right)}{u_1\left(c, 1-c-\frac{\psi}{2}g^2\right)}\right]\left(1-c-\frac{\psi}{2}g^2\right) - \psi g(1+g) + \beta\psi g(1+g) = 0$$

- **Limit attention to steady-state (i.e., long run) policy questions**
  - **And (finally!...) use long-run monetarist relationship $g = \pi$**

"CASHLESS" NK FRAMEWORK

- **Condense private-sector equilibrium conditions….**
  - **…by imposing $w_t = mc_t$ and** $n_t = c_t + \frac{\psi}{2}(\pi_t)^2$ **everywhere**
  - **Consumption-savings optimality condition in the steady state**

NK view:

Monetary policy does not operate through demand side…

$$\cancel{\frac{u_1\left(c,1-c-\frac{\psi}{2}g^2\right)}{\beta u_1\left(c,1-c-\frac{\psi}{2}g^2\right)}=\frac{1+i}{1+g}}$$

SO IGNORE C-S CONDITION IN ANALYSIS OF OPTIMAL POLICY PROBLEM….

ONLY TAKE INTO ACCOUNT NKPC!

  - **New Keynesian Phillips Curve in the steady state**

Monetary policy operates through supply side

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c,1-c-\frac{\psi}{2}g^2\right)}{u_1\left(c,1-c-\frac{\psi}{2}g^2\right)}\right]\left(1-c-\frac{\psi}{2}g^2\right)-\psi g(1+g)+\beta\psi g(1+g)=0$$

- **Limit attention to steady-state (i.e., long run) policy questions**
  - **And (finally!…) use long-run monetarist relationship $g = \pi$**

OPTIMAL POLICY ANALYSIS

- **Can express entire steady-state private-sector equilibrium as**

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c,1-c-\frac{\psi}{2}g^2\right)}{u_1\left(c,1-c-\frac{\psi}{2}g^2\right)}\right]\left(1-c-\frac{\psi}{2}g^2\right)-\psi g(1+g)+\beta\psi g(1+g)=0$$

- **Defines implicitly a reaction function $\bar{c}(g)$**
  - **A (potentially complicated..) summary description of how private-sector equilibrium quantities depend on any given choice of government policy *g***

- **Maintained assumption**
  - **Central bank knows/understand perfectly the private-sector reaction function**
  - **Realism? Impossible for a central bank to literally know this…**
  - **…but provides a starting point for analysis**

- **→ Central bank takes into account the reaction function $\bar{c}(g)$ when setting (optimal) policy**

OPTIMAL POLICY ANALYSIS

- **Goal of policy makers**
  - **Maximize welfare (utility) of representative consumer**
  - **In steady-state**

$$\sum_{s=0}^{\infty}\beta^s u\left(\bar{c},1-\bar{c}-\frac{\psi}{2}g^2\right)=\frac{u\left(\bar{c},1-\bar{c}-\frac{\psi}{2}g^2\right)}{1-\beta}$$

Recall infinite summation formula

$$\downarrow$$

$$\sum_{s=0}^{\infty}\beta^s u\left(\bar{c}(g),1-\bar{c}(g)-\frac{\psi}{2}g^2\right)=\frac{u\left(\bar{c}(g),1-\bar{c}(g)-\frac{\psi}{2}g^2\right)}{1-\beta}$$

Taking into account private-sector reaction function

- **The formal policy problem**
  - **Choose *g* that maximizes private-sector welfare**
  - **The *c*(*g*) function summarizes the behavior of private markets**

OPTIMAL POLICY ANALYSIS

- **Optimal monetary policy problem**

$$\sum_{s=0}^{\infty}\beta^s u\left(\bar{c}(g),1-\bar{c}(g)-\frac{\psi}{2}g^2\right)=\frac{u\left(\bar{c}(g),1-\bar{c}(g)-\frac{\psi}{2}g^2\right)}{1-\beta}$$

- **FOC with respect to *g***
  - **KEY: NOW need to take into account the dependence of private-market outcomes on the policy in place**

$\downarrow$ **Rearrange to MRS = … form**

- **If policy is set optimally**

$$\frac{u_2\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}{u_1\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}=\frac{c'(g)}{c'(g)+\psi g}$$

NOTE: If ψ = 0, exactly the same optimal-policy condition as in flexible-price analysis (Chapter 17) – i.e., RHS = 1

OPTIMAL POLICY ANALYSIS

- **If policy is set optimally**

$$\frac{u_2\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}{u_1\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}=\frac{c'(g)}{c'(g)+\psi g}$$

- **Compare with private-sector equilibrium**

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c,1-c-\frac{\psi}{2}g^2\right)}{u_1\left(c,1-c-\frac{\psi}{2}g^2\right)}\right]\left(1-c-\frac{\psi}{2}g^2\right)-\psi g(1+g)+\beta\psi g(1+g)=0$$

- **QUESTION: What money growth rate *g* achieves <u>this</u> outcome?...**

- **Impossible to solve for *g*!....(in general, *<u>no</u>* *g* can achieve this....)**

OPTIMAL POLICY ANALYSIS

- **If policy is set optimally**

$$\frac{u_2\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}{u_1\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}=\frac{c'(g)}{c'(g)+\psi g}$$

- **Compare with private-sector equilibrium**

$$\frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon u_2\left(c,1-c-\frac{\psi}{2}g^2\right)}{\varepsilon u_1\left(c,1-c-\frac{\psi}{2}g^2\right)}\right]\left(1-c-\frac{\psi}{2}g^2\right)-\psi g(1+g)+\beta\psi g(1+g)=0$$

Introduce *ε* factor here…

- **QUESTION: What money growth rate *g* achieves <u>this</u> outcome?…**
- **Impossible to solve for *g*!…. (in general, <u>*no*</u> *g* can achieve this….)**
- **…unless we slightly "modify" the condition (NKPC) describing private-sector equilibrium…**
  - **Interpretation: a corrective *fiscal policy* intervention**

OPTIMAL POLICY ANALYSIS

- **QUESTION: What money growth rate *g* aligns these two conditions?**

$$\frac{u_2\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}{u_1\left(c(g),1-c(g)-\frac{\psi}{2}g^2\right)}=\frac{c'(g)}{c'(g)+\psi g}$$

Summarizes outcome under *optimal policy*

$$\frac{1}{1-\varepsilon}\left[1-\frac{u_2\left(c,1-c-\frac{\psi}{2}g^2\right)}{u_1\left(c,1-c-\frac{\psi}{2}g^2\right)}\right]\left(1-c-\frac{\psi}{2}g^2\right)-\psi g(1+g)+\beta\psi g(1+g)=0$$

Summarizes private-sector equilibrium for *any arbitrary policy choice*

- **By inspection…**

OPTIMAL POLICY ANALYSIS

- **QUESTION: What money growth rate *g* aligns these two conditions?**

$$\frac{u_2\left(c(g),1-c(g)\right)}{u_1\left(c(g),1-c(g)\right)}=1$$

Summarizes outcome under *optimal policy*

$$\frac{1}{1-\varepsilon}\left[1-\frac{u_2\left(c,1-c\right)}{u_1\left(c,1-c\right)}\right]\left(1-c\right)=0$$

Summarizes private-sector equilibrium for *any arbitrary policy choice*

- **By inspection… *g* = 0 aligns the private-sector outcome with the policymaker's desired outcome**

OPTIMAL POLICY ANALYSIS

- **QUESTION: What money growth rate *g* aligns these two conditions?**

$$\frac{u_2\left(c(g),1-c(g)\right)}{u_1\left(c(g),1-c(g)\right)}=1$$

$$\frac{1}{1-\varepsilon}\left[1-\frac{u_2\left(c,1-c\right)}{u_1\left(c,1-c\right)}\right]\left(1-c\right)=0 \xrightarrow[\text{that the } \textit{equilibrium} \text{ features}]{\text{Solution requires}} \frac{u_2\left(c,1-c\right)}{u_1\left(c,1-c\right)}=1$$

- **By inspection… *g* = 0 aligns the private-sector outcome with the policymaker's desired outcome**

ALIGNING THESE IS THE GOAL!

Zero Inflation Policy

- Optimal long-run money growth rate is $g = 0$
- Money supply should remain constant in the long run!

- Optimal long-run inflation
  - Monetarist link

$$\pi = g = 0$$

A normative statement

  - Zero long-run inflation!
  - Central bank should seek to target zero inflation on average

- Optimal long-run nominal interest rate
  - $i > 0$ from Fisher relation (aka consumption-savings optimality condition)

- Quite different policy recommendation from Friedman Rule!
  - Zero inflation, not negative inflation
  - ←→ Positive nominal interest rate, not zero nominal interest rate

- Fundamental difference: sticky prices vs. flexible prices

UNDERSTANDING ZERO INFLATION POLICY

- What does $g = \pi = 0$ achieve?
- Eliminates the price adjustment costs in the resource constraint

RECALL: A social planner doesn't need to consider "prices" – hence would not care to incur "price adjustment costs"

$$c + \frac{\psi}{2}\pi^2 = n \xrightarrow{\text{Zero inflation}} c = n$$

  - Adjustment costs are a cost!...
  - There is no benefit of "sluggish" or "sticky" prices!...
  - Basic economics: an activity has positive costs but no benefits → optimally want none of that activity!

- Private-sector achieves efficiency along consumption-leisure margin

$$\frac{u_2(c, 1-c)}{u_1(c, 1-c)} = 1$$

  - Zero inflation allows policy makers to achieve economic efficiency in private markets – even though central bank is NOT a Social Planner
    - Notice very nuanced/precise statements/logic...

- HOWEVER: zero inflation only "works" if fiscal policy is also being set optimally – raises coordination issues, etc?...

Practical Relevance of Zero Inflation

- A benchmark in the theory of monetary policy
- Has been a practical guide for the conduct of monetary policy in last 30 years

- Zero inflation technically only a long-run optimal policy recommendation
- But has become the intellectual guidepost for central banks worldwide even for business-cycle-frequency inflation goals

- Encapsulated in the mantra "low and stable inflation"

- No central banks target (explicitly or implicitly) EXACTLY ZERO inflation
  - Rather, positive long-run inflation targets

- What can rationalize positive long-run inflation targets?
  - WAGE (as opposed to price) rigidity
  - Financial frictions?
  - Hitting zero-lower-bound on nom i.r. during recessions the central issue
    - i.e., Blanchard suggestion

Active areas of research

- Nature of and costs (and benefits?) associated with price adjustment still not well understood

ECON 401

Advanced Macroeconomics

The New Keynesian Model and Monetary Policy

Fabio Ghironi

University of Washington

Introduction

- Sanjay Chugh's textbook and slides highlight the central role of the supply-side of the economy—specifically, the New Keynesian Phillips curve—in the conduct of monetary policy in the New Keynesian framework.

- These slides expand and clarify the point he is making.

- Chugh explains New Keynesian macroeconomics and some of its important results without log-linearizing the model, but it turns out that using the log-linearized version of the New Keynesian framework allows us to highlight some points most transparently.

- The model I introduce in the following slides can be obtained from log-linearizing the model that Chugh explains augmented with the introduction of uncertainty.

  – You should have learned from the RBC model that the log-linear model with uncertainty is identical to the log-linear model with perfect foresight except for having the expectation operator applied to variables at $t+1$ and for the introduction of random shocks in the equations.

  – This is the reason why many refer to log-linearization as delivering certainty equivalence.

- I introduce shocks in all equations of the model below to allow for different sources of fluctuations.

The Basic New Keynesian Macroeconomic Model

- The basic, log-linearized New Keynesian macroeconomic model consists of three equations.

The Intertemporal IS

- There is an equation that describes how output today depends on the *ex ante* real interest rate and on expected future output:

$$y_t = -\sigma\left[i_t - E_t\left(\pi_{t+1}\right)\right] + E_t\left(y_{t+1}\right) + u_t, \tag{1}$$

  where, as usual, $\sigma > 0$ measures the responsiveness of today's output to the real interest rate, i_t is the nominal interest rate, π_{t+1} is inflation next period, $E_t\left(.\right)$ is the expectation of the variable inside the parentheses based on information available at time t, and u_t is an exogenous demand shock.

- This equation is an intertemporal IS equation.
  - Today's GDP contracts if the real interest rate increases; today's GDP rises if it is expected to rise tomorrow.
  - u_t is an exogenous demand shock.
- This equation follows from the log-linearized Euler equation for bond holdings once equilibrium conditions are imposed
- It describes the demand-side of the economy.

The Basic New Keynesian Macroeconomic Model, Continued

The New Keynesian Phillips Curve

- There is an equation that describes how today's inflation depends on today's output and on expected future inflation:

$$\pi_t = \lambda y_t + \beta E_t (\pi_{t+1}) + z_t, \quad (2)$$

  where $\lambda > 0$ and $\beta > 0$ are parameters (β is the households' discount factor, λ is a parameter that depends on the extent of nominal rigidity and on the extent of monopoly power), and z_t is an exogenous shock.

  – Current inflation rises if GDP rises and if inflation is expected to rise tomorrow.

- This equation is known as New Keynesian Phillips Curve (NKPC) and it describes the supply-side of the model.

- It can be obtained by log-linearizing the non-linear NKPC that Sanjay Chugh presents and imposing additional equilibrium conditions.

Monetary Policy

- Finally, there is an equation that describes monetary policy in terms of what is known as a Taylor-type rule for interest rate setting, from the 1993 article by John Taylor in the *Carnegie-Rochester Conference Series on Public Policy* that started the literature on Taylor rules.

- For instance:

$$i_t = \alpha_1 \pi_t + \alpha_2 y_t + x_t, \tag{3}$$

  where $\alpha_1 > 0$ and $\alpha_2 > 0$ are policy response parameters, and x_t is an exogenous policy shock.

- Equations (1)-(3) are a system of three dynamic equations for the three endogenous variables y_t, π_t, and i_t as functions of the exogenous shocks u_t, z_t, and x_t.

- Usually, it is assumed that these shocks follow so-called first–order autoregressive processes, in which the level of the shock today depends on its level last period and on an innovation in the current period.

Solving the Basic New Keynesian Macroeconomic Model

- If the policy response parameters α_1 and α_2 satisfy the following restriction:

$$(\alpha_1 - 1)\,\lambda + \alpha_2\,(1 - \beta) > 0,$$

  the log-linear system (1)-(3) has a unique solution.

  – This can be verified using a method described in an appendix to the slides on the RBC model.

- If the central bank is not responding to GDP (i.e., $\alpha_2 = 0$), the condition for a unique solution reduces to $\alpha_1 > 1$:

  – The central bank must respond to inflation more than proportionally.

- This is known as Taylor Principle and it captures the idea that, to stabilize the economy, the central bank should cause the real interest rate to rise (by having the nominal interest rate rise more than inflation) when the economy is “overheating” and inflation is increasing.

- Requiring that monetary policy be such that it ensures a unique equilibrium is important:

  – Doing no harm (i.e., not introducing sunspot fluctuations in the economy by creating indeterminacy) should be the minimum that is expected of policy!

Solving the Basic New Keynesian Macroeconomic Model, Continued

- Provided the condition for determinacy (uniqueness) of the solution is satisfied, the solution of the model can be written as:

$$\begin{aligned} y_t &= \eta_{yu} u_t + \eta_{yz} z_t + \eta_{yx} x_t, \\ \pi_t &= \eta_{\pi u} u_t + \eta_{\pi z} z_t + \eta_{\pi x} x_t, \\ i_r &= \eta_{iu} u_t + \eta_{iz} z_t + \eta_{ix} x_t, \end{aligned} \tag{4}$$

  where the η's are coefficients that can be found with the method of undetermined coefficients. (Suggested exercise: Do this.)

- Notice: We can solve fully for output, inflation, and the interest rate without any reference to money and money supply.

- This happens because of the implicit assumption that, if we had money in the model, we would have introduced it via money-in-the-utility function in a separable way.

- Under this assumption, once monetary policy is conducted through interest rate setting, we need not worry about money, and its only role is (in the background) to implement the open market operations through which the central bank affects the interest rate.

Some Properties of the Solution and Some Model Variants

- The shocks u_t, z_t, and x_t describe the minimum state vector of the model.
- There is no endogenous, predetermined state (such as capital in the RBC model).
- This means that the endogenous variables y_t, π_t, and i_t are only as persistent as the shock themselves, and they will return monotonically to the steady state after shocks, without displaying any hump.
- This is a well known weakness of the basic New Keynesian framework, as empirical evidence points to hump-shaped responses to shocks that this model cannot replicate.
- A solution to generate hump-shaped responses of inflation to shocks is to build models in which current inflation also depends on past inflation, so that the NKPC becomes:

$$\pi_t = \rho\pi_{t-1} + \lambda y_t + \beta E_t(\pi_{t+1}) + z_t,$$

  with $0 < \rho < 1$.
- In this case, π_{t-1} becomes part of the state vector, and the model can generate humps in inflation responses to shocks.

Some Properties of the Solution and Some Model Variants, Continued

- We could also assume that habits in consumption imply that output today depends also on output yesterday, so the intertemporal IS becomes:

$$y_t = \varkappa y_{t-1} - \sigma\left[i_t - E_t\left(\pi_{t+1}\right)\right] + E_t\left(y_{t+1}\right) + u_t,$$

  with $0 < \varkappa < 1$.

- In this case, y_{t-1} becomes part of the state vector.

- Another variant of the model assumes that central bank policy is characterized by interest rate smoothing, so that the interest rate today depends also on the interest rate yesterday:

$$i_t = \alpha_1 \pi_t + \alpha_2 y_t + \alpha_3 i_{t-1} + x_t,$$

  with $\alpha_3 > 0$.

- In this case, i_{t-1} becomes part of the state vector.

- If no other change to the model is made (i.e., equations (1) and (2) continue to hold) the condition for a unique solution becomes:

$$\left(\alpha_1 + \alpha_3 - 1\right)\lambda + \alpha_2\left(1 - \beta\right) > 0.$$

- But let us return to the basic model (1)-(3).
- Suppose the central bank commits to a policy of zero inflation, so that $\pi_t = E_t(\pi_{t+1}) = 0$.
- We can use the NKPC equation (2) to back out the implied path of output:

$$y_t = -\frac{1}{\lambda} z_t.$$

- And we could then use the intertemporal IS equation (1) to back out the path of the interest rate that would be consistent with this outcome:

$$-\frac{1}{\lambda} z_t = -\sigma i_t - \frac{1}{\lambda} E_t \left(z_{t+1} \right) + u_t,$$

  or,

$$i_t = \frac{1}{\sigma} u_t + \frac{1}{\sigma\lambda} \left[z_t - E_t \left(z_{t+1} \right) \right].$$

- If we assume that the shock z_t is such that $z_t = \phi_z z_{t-1} + \varepsilon_{z,t}$, where $\varepsilon_{z,t}$ is a zero-mean innovation, $E_t \left(z_{t+1} \right) = \phi_z z_t$, and:

$$i_t = \frac{1}{\sigma} u_t + \frac{1}{\sigma\lambda} \left(1 - \phi_z \right) z_t. \tag{5}$$

Monetary Policy and the New Keynesian Phillips Curve, Continued

- This is the sense in which Sanjay Chugh says that monetary policy in the New Keynesian model works through the NKPC:
  - Once the central bank has chosen the path of inflation it wants to implement, the NKPC delivers the implied path of output, and the intertemporal IS then delivers the interest rate path that will correspond to the desired path of inflation and the implied path of output.
- Does this imply that if the central bank wants to pursue a policy of zero inflation there is no role for the policy rule (3)?
- No! In fact, we could verify that if the policy rule (3) were replaced by equation (5) the model would not have a unique solution.
  - The equilibrium would be indeterminate!
- It turns out that having the interest rate respond to endogenous variables of the model (such as inflation and output) and not just to exogenous shocks is crucial to deliver uniqueness of the solution.
  - This is a point that Michael Woodford (Columbia University) showed in his 2003 book on *Interest and Prices*.

Implementing Zero Inflation

- If the central bank wants to deliver zero inflation (and ultimately the interest rate path (5)), it must commit to the policy:

$$i_t = \alpha_1 \pi_t + \alpha_2 y_t,$$

  i.e., no exogenous monetary policy shock ($x_t = 0$), with a very high value of the response coefficient α_1.

  - In fact, you could verify from the solution equations in (4), once you have found the η's, that $\pi_t = 0$ when $\alpha_1 \to \infty$. (You should check this as an exercise.)

- Of course, this is a policy that works well to deliver $\pi_t = 0$ without problems in this simple model.

- In reality, a huge coefficient α_1 would cause problems from large volatility of the interest rate in response to even minuscule deviations of inflation from 0 that could happen for many reasons.

- This explains why the recommendation of a huge response coefficient can be good for a model, but not for reality.

ECON 401

Advanced Macroeconomics

Monetary Policy Commitment Versus Discretion

Fabio Ghironi

University of Washington

Introduction

- In solving for the optimal steady-state growth rate of money supply (and therefore the optimal steady-state inflation rate) in the sticky-price New Keynesian model, Sanjay Chugh finds the apparently odd result that the problem has no solution unless a “convenient” ε is added to the key equation in a specific spot.
- He interprets this ε as “help from the government through a fiscal policy action.”
- These slides clarify this point and relate it to the issue of monetary policy under commitment versus discretion.

A Distorted Flexible-Price Equilibrium

- Consider the New Keynesian model with flexible prices and use the notation $\varepsilon = \theta / (\theta - 1)$, where $\theta > 1$ is the elasticity of substitution between differentiated wholesale products.
- Assume that output of each differentiated product is produced using only labor according to the production function $y_t = Z_t N_t$, where Z_t is exogenous productivity.
  - I am dropping the individual firm identifier because we know that all firms make identical choices in equilibrium.
- Letting p_t be the price of an individual wholesale product and P_t be the price of the retail bundle, we know that optimal price setting implies:

$$\frac{p_t}{P_t} = \mu \frac{w_t}{Z_t},$$

  where $\mu \equiv \theta / (\theta - 1) = \varepsilon$ is the markup, w_t is the real wage, and w_t / Z_t is marginal cost.
- Since $P_t = p_t$ in equilibrium, this implies that:

$$w_t = \frac{1}{\mu} Z_t.$$

- The real wage is lower than productivity (since $\mu < 1$).

A Distorted Flexible-Price Equilibrium, Continued

- When we impose equalization of the real wage to the marginal rate of substitution between leisure and consumption in an environment of endogenous labor supply, we have the familiar result that monopoly power distorts the amount of labor employed in equilibrium:

$$\frac{U_{1-N}(C_t, 1-N_t)}{U_C(C_t, 1-N_t)} = \frac{1}{\mu} Z_t.$$

- Because $\mu > 1$, too little labor is demanded (and supplied) and too little output is produced.
- Firms with monopoly power have an incentive to reduce output supply in order to extract a higher price than under perfect competition.
- And workers who are comparing the benefit of consumption (priced at a markup) over leisure (priced at no markup) have an incentive to over-demand leisure relative to consumption.
- The result is under-production of output and under-employment of labor relative to the perfectly competitive outcome.
- The problem would be removed if we had $\theta \to \infty$, i.e., if wholesale goods were perfectly substitutable:
  - In this case, wholesale firms would have no monopoly power, and it would be $\mu = 1$.

Sticky Prices and Monetary Policy

- Now, when changing prices is costly, optimal price setting implies:

$$\frac{p_t}{P_t} = \mu_t \frac{w_t}{Z_t}.$$

- The markup becomes time-varying!

- In particular, it becomes a function of output and of current and expected future inflation.

- Let us write this compactly as:

$$\mu_t = \mu\left(y_t, \pi_t, E_t\left(\pi_{t+1}\right)\right).$$

- If log-linearized around a steady state with zero inflation and steady-state output normalized to 1, this boils down to:

$$\hat{\mu}_t = -\frac{\psi}{\theta - 1}\left[\hat{\pi}_t - E_t(\hat{\pi}_{t+1})\right],$$

  where hats denote percentage deviations from steady state and ψ is the scale parameter for the cost of adjusting prices.

Sticky Prices and Monetary Policy, Continued

- This is important: The markup falls if inflation rises.

- Since costs of adjusting prices give firms an incentive to smooth price changes over time, firms will absorb the impact of rising inflation by letting the markup component of their prices fall when inflation becomes higher.

- But this has implications for monetary policy:

  – The central bank knows that an increase in inflation will erode the markup, causing it to become lower than its flexible-price level $\theta / (\theta - 1)$.

- This creates a temptation for the central bank to use monetary expansion to boost output above the inefficient level that prevails under flexible prices!

Temptation to Expand

- When we introduced the issue of optimal monetary policy, we were careful to frame it in the following terms:
  - If you were the central bank and you could *commit* to a choice of inflation rate, what inflation rate would you want to *commit* to?
- The answer was $\pi_t = 0$ because the central bank recognizes that it cannot address the effect of monopoly power directly, and so it better just focus on removing the effect of the sticky-price distortion by choosing a policy of zero inflation.
- The word “*commitment*” was important.
- What if the central bank cannot commit? What about the temptation to erode the impact of the markup by having $\pi_t > 0$?
- This is the problem underlying the inability to solve the optimal monetary problem in Sanjay Chugh’s slides without the help from fiscal policy.
- Appropriate help from fiscal policy removes the issue.

Help from Fiscal Policy

- Return to optimal wholesaler price setting under flexible prices, but now assume that wholesale firm revenues are taxed at a rate $\tau > 0$.

- If you make this assumption, optimal price setting implies:

$$\frac{p_t}{P_t} = \frac{\mu}{1-\tau}\frac{w_t}{Z_t} = \frac{\theta}{(1-\tau)(\theta-1)}\frac{w_t}{Z_t}.$$

  - The markup is now adjusted for the rate of revenue taxation.

- This implies that the government can choose the τ such that:

$$\frac{\theta}{(1-\tau)(\theta-1)} = 1.$$

Help from Fiscal Policy, Continued

- If the government does that, we have

$$\frac{p_t}{P_t} = \frac{w_t}{Z_t},$$

or, in equilibrium, $w_t = Z_t$.

- The government has chosen the τ that removes the impact of monopoly power on the economy (an application of the idea that optimal policy is about minimizing—in this case, removing completely—the effect of distortions), and the flexible-price economy has become efficient, because we again have that:

$$\frac{U_{1-N}(C_t, 1-N_t)}{U_C(C_t, 1-N_t)} = Z_t,$$

as in the RBC model.

- If the government does that, once we introduce sticky prices, the central bank no longer faces the temptation to try to use inflation to erode the markup relative to its flexible-price level, and the optimal thing for it to do is to pursue a policy of zero inflation (in and outside the steady state) regardless of whether it behaves under commitment or in a discretionary fashion.

Help from Fiscal Policy, Continued

- This is the "trick" that Sanjay Chugh is using when he is adding that ε to the policy problem of the central bank (in his case, focusing only on steady-state inflation):
- He is implicitly assuming that a smart government has set τ appropriately to make things such that the flexible-price equilibrium is not distorted, making it unambiguously optimal to have zero inflation to remove the effect of the only remaining distortion (sticky prices).
- Now let us spend a bit longer understanding the issue of commitment versus discretion.

Monetary Expansion in the AS-AD Diagram

- Consider a standard aggregate supply-aggregate demand (AS-AD) diagram, and suppose the central bank increases nominal money supply from M to $M\prime$.
- The following figure describes the short-run effect of this policy action.
- Assume that, before the change in M, AS intersected AD at point A: $Y = Y_n$ (the "trend" level of output) and $P = P^e$ (the price level P was where it was expected to be when all price and wage contracts in the economy had been signed).
  - We are using a diagram with P on the vertical axis, but we could make the same arguments below with inflation instead of the price level on the vertical axis.
- Think of point A as the pre-shock flexible-price equilibrium of the economy.
- The monetary police expansion causes the AD curve to shift to the right to $AD\prime$.
  - Output increases to $Y\prime$ and the price level increases to $P\prime$.

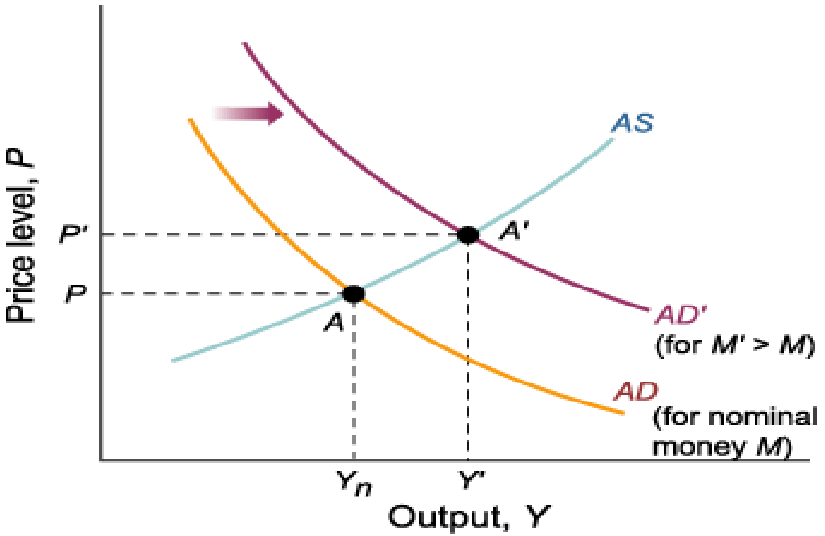

AS
P'
P
A'
A
AD'
(for M' > M)
AD
(for nominal
money M)
Price level, P
Yn
Y'
Output, Y

Figure 1

Monetary Expansion in the AS-AD Diagram, Continued

- Over time, adjustment of expectations comes into play: $Y > Y_n \Rightarrow P > P^e \Rightarrow$ Wage and price setters that did not get to adjust to the shock immediately revise the expectations incorporated in their contracts, and the AS curve shifts up.
- The economy moves up along $AD\prime$.
- Adjustment stops when $Y = Y_n$ again and price level = $P'' = P^{e\prime}$ ($> P^e$).

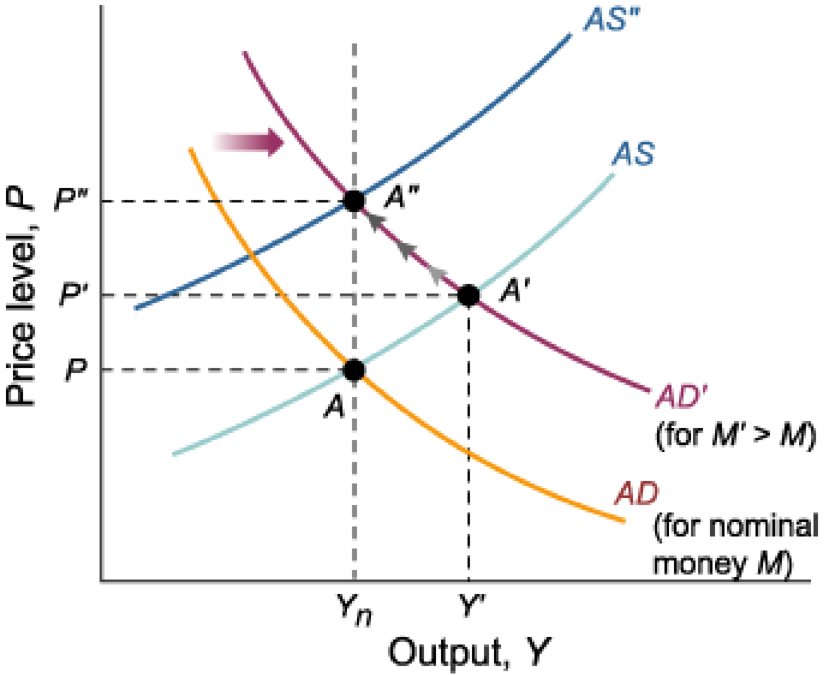

AS"
AS
P"
P'
P
A"
A'
A
AD'
(for M' > M)
AD
(for nominal
money M)
Price level, P
Yn
Y'
Output, Y

Figure 2

Monetary Expansion in the AS-AD Diagram, Continued

- In the long run, AS is given by AS'', and the economy is at A'', with output back at trend and the effect of the monetary expansion fully reflected into prices.
  - We are implicitly assuming that long-run neutrality of money holds in the diagram, i.e., that the economy behaves as if there is a vertical long-run AS curve at $Y = Y_n$.
  - This notion has become a subject of discussion since the Global Financial Crisis of 2007-08 and the Great Recession that followed, with scholars arguing that the trend position of the economy is itself endogenous to the management of aggregate demand in the short to medium run.
  - The sticky price model in the slides is such that monetary policy is not neutral in the long run: You could verify that a non-zero long-run inflation rate would have long-run real effects.
  - But these would be small under plausible assumptions and it would still be the case that a welfare-maximizing policymaker acting under commitment would choose a zero inflation rate because of the costs that inflation would impose on the economy.

Expectations

- Now let us think about expectation formation.
- If all agents in the economy of the figure form expectations in a forward looking manner, use the information at their disposal optimally, and can renegotiate their contracts at the same time after the initial surprise without incurring additional costs to implementing price and wage changes, the transition from the short-run equilibrium to the long-run position following a monetary policy expansion will take just one round of expectation and contract revision.
- Why is that? Because if all agents are forward-looking and they know the structure of the model, as soon as they observe the policy shock and find themselves in the short-run equilibrium, they know that the economy must eventually converge to the long-run position.
- As soon as they get a chance to renegotiate contracts based on their revised expectations, they will immediately set the expected price level at the long-run equilibrium level.
- This will cause the economy to move from the short-run equilibrium to the long-run equilibrium in just one round of expectation and contract revision, thus shortening the amount of time during which monetary policy causes output to be above trend.

Expectations, Continued

- If all agents are backward or current-looking, expectation revisions reset expectations (at best) to the currently observed price level, resulting in a transition to the medium run equilibrium that may take several rounds of expectation and contract revision.

- If forward- and backward-looking agents coexist, as it is plausible, the transition will be faster than in the fully backward-looking case, but slower than in the fully rational one.

- Now, monetary policy is effective in terms of generating a level of output that differs from trend to the extent that it generates a price level that differs from the expected price level embedded in contracts—either because of a full surprise effect at the time of the policy action or because, even if agents knew that a policy change was coming, something (like costs of adjusting prices or wages) prevented them from resetting prices fully before the policy change.

- Unexpected policy has a larger impact because no agent will have had a chance to renegotiate her/his contract.

- Fully credible, anticipated policy will have no real effect (no effect on output) and will only affect prices and wages if all agents have a chance adjust prices and wages fully between the announcement of the policy and the time when the policy change actually happens.

Expectations, Continued

- Now suppose that, once Y has returned to Y_n, the central bank increases M again.
- The process will be repeated (taking the long-run equilibrium following the previous monetary expansion as the new initial position):
- Y will rise above Y_n for some time but eventually return to Y_n, with P increasing to match the further increase in M over time.
- If the central bank plays this game repeatedly, even if agents are not fully forward-looking and optimizing to begin with, they will eventually recognize the central bank's behavior and incorporate it in their own.
- Expectation and contract revisions will become more and more frequent, so that the deviations of Y from Y_n caused by monetary expansion will become shorter and shorter lived.
- Eventually the AS curve will become de facto vertical at Y_n, and all the policymaker will accomplish by increasing M will be to increase P immediately.
- Think of this as an environment in which the scale parameter of costs of price adjustment is becoming smaller and smaller, as frequent, sizable price changes have become the norm.

Commitment Versus Discretion

- Now let us use the AS-AD model and our discussion of the role of expectations to talk about a theory of inflationary consequences of policymaker incentives and agents' expectations that was proposed by Finn Kydland and Edward Prescott in 1977 (*Journal of Political Economy*) to explain how high inflation can arise as outcome of the interaction (the "game") between the central bank and private agents.
  - The theory was then reformulated by Robert Barro and David Gordon (1983, *Journal of Political Economy*) and has become known as the Barro-Gordon model.
  - However, Kydland and Prescott were the first to propose it. Kydland and Prescott received the Nobel Prize in 2004 for their work in macroeconomics.
- Suppose that the policymaker has a target of output Y^* above the natural level Y_n because the latter is inefficiently low (for instance, as a consequence of monopoly power combined with endogenous supply of labor as in the basic New Keynesian model of the slides).
- Suppose that, when price and wage contracts are being negotiated, the policymaker—who knows that she/he can only drive Y above Y_n for some time—announces that she/he will follow a rigorous monetary policy aimed at preserving price stability.
- Assume however that the policymaker has no way of actually precommitting herself/himself in a credibly binding way to implementing the announcement.

Commitment Versus Discretion, Continued

- Suppose that private agents have forward-looking, rational expectations.
- Moreover, they know the structure of the economy (the AS-AD model), the fact that the policymaker has an objective $Y^* > Y_n$, and that the policymaker cannot precommit herself/himself in a binding way to actually implementing her/his announced policy.
- If private agents believe the central bank's announcement that monetary policy will preserve a stable price, they will embed this in their expectations and set prices and wages accordingly.
- Now, once private agents have signed their contracts believing the policymaker's announcement, it is no longer optimal for her/him to actually implement it.
- She/he has an incentive to expand monetary policy, exploiting nominal rigidity, to drive Y to Y^* at least for some time.

Commitment Versus Discretion, Continued

- However, private agents know that the policymaker has this incentive.
- They know that, if they sign contracts based on expectations of a stable price, since the policymaker is not bound by any credible precommitment device to implementing her/his announced rigorous policy, she/he will fool them with a monetary expansion that erodes markups and real wages.
- Rational, optimizing private agents recognize this, do not believe the policymaker's announcement, and incorporate her/his expected behavior in their contract negotiation, setting price expectations at the level that corresponds to the long-run equilibrium to which the economy would have eventually converged if the policymaker had actually managed to fool private agents for some time.
- Optimal behavior by rational private agents will thus cause the AS curve to shift up even before the policymaker actually takes any action.

Commitment Versus Discretion, Continued

- At this point, if the policymaker sticks to her/his announcement of stable monetary policy, the short run equilibrium will be not only below Y^*, the policymaker's target, but also below Y_n!

- What is the optimal thing to do for the policymaker if she/he wants at least to avoid having $Y < Y_n$ for some time?

- It is to validate agents' expectations and do what they expected her/him to do in the first place: expand monetary policy, shift the AD curve to the right, and drive the equilibrium where agents expected it to be: at Y_n, but with a higher price level.

  – If the policymaker does not validate agents' expectations and surprises them by sticking to her/his prior announcement, Y will be below Y_n for as long as it takes for expectations and contracts to be revised and the AS curve to return to its original position.

  – This is how the central banker would establish her/his credibility for being "tough": at a potentially significant cost for the economy.

- Given our assumptions, high-price expectation and monetary policy expansion are the best responses of the private sector and the central bank, respectively, to each other's behavior.

Commitment Versus Discretion, Continued

- Thus, as a consequence of the inability of the policymaker to precommit to a stable-price policy, her/his incentive to generate Y above Y_n, and private agents' recognition of the policymaker's lack of commitment and incentives, all that we see is no deviation of Y from Y_n and a higher price level.
- This is the Nash equilibrium of the game between the central bank and the private sector.
- Monetary expansion is the only possible equilibrium policy in this game.
- Inability to commit credibly to the stable-price policy—i.e., the fact that policy is conducted under discretion—results in a high-price-level (or high-inflation) outcome for the economy.

The Time Inconsistency of Optimal Policy

- Central to the theory is the concept of time inconsistency of the optimal policy:
  - Ex ante, when private sector expectations embedded in wage and price contracts are being formed, the rigorous, stable-price policy is optimal.
  - Announcing anything different would simply result in an immediate change in price expectations.
  - Ex post, once private agents have committed to contracts based on believing the policymaker's announcement of a stable-price policy, the policy is no longer optimal.
  - It is optimal for the policymaker to fool agents with a monetary expansion.
- Note that the policymaker is benevolent:
  - If effective on the real economy, the monetary expansion increases Y above the inefficient level Y_n for some time.
  - But the agents' recognition of the policymaker's incentive and their desire to protect themselves from inflation prevent the policymaker from accomplishing this goal.

The Time Inconsistency of Optimal Policy, Continued

- The ex ante optimal policy is thus time inconsistent.
- The time consistent policy (expected by rational agents and implemented in equilibrium) is to expand monetary policy, which ends up only resulting in high prices without any output gain.
- The “help from fiscal policy” that Sanjay Chugh introduces in his analysis of optimal monetary policy removes this problem by making the trend level of output efficient, eliminating the temptation to expand that the central bank succumbs to under discretion.

A Repeated Interaction

- The fact that the interaction between the central bank and the private sector is repeated over time rather than just played once in a one-shot game such as the one described above provides a possible mechanism that would support a stable-price equilibrium.
- Suppose that the game between the policymaker and the private sector is played each period over an infinite horizon.
- Suppose that the private sector "tells" the policymaker today: "Ok, we believe your stable-price announcement. But if you cheat on us later on, we will never believe you again, and we will always set expectations consistent with the one-shot-game, monetary expansion behavior."
- In this case, when deciding what to do, the policymaker is trading off the gains from surprising the private sector in the short run against the losses from high price expectations (and high actual price since, as we discussed, it will be optimal to validate those expectations) for the infinite future.
- If the policymaker cares enough about the future, she/he will stick to the announced policy of monetary rigor.

A Repeated Interaction, Continued

- The problem with this mechanism is that policymakers definitely do not have an infinite horizon.
  - Note that the fact that central bankers are generally appointed for long periods is another way to strengthen their independence, and thus their credibility.
- If the game is played a finite number of times, the mechanism that sustains the stable-price policy in the infinite horizon case unravels, and the only possible policy equilibrium is the monetary expansion of the one-shot game.
- Why?
- Because, in the last period of the game (call it period T), the interaction reverts to the single-period interaction in which the policymaker has a clear incentive to cheat on the announcement of rigorous policy (since there is no future game to be played next period involving that policymaker).
- Thus, monetary expansion is the only equilibrium policy in period T.

A Repeated Interaction, Continued

- But in period $T-1$, agents know that this will happen in period T, i.e., they know that the policymaker will expand monetary policy in period T.
- They will respond to this by setting high price expectations already at $T-1$, which the policymaker will find it optimal to validate to avoid a recession then.
- The same mechanism will apply at $T-2$, $T-3$, and so on.
- By backward induction, the only equilibrium policy outcome in all periods from T back to the current time will be monetary expansion and a high price level.

Commitment Mechanisms

- Bottom line: The ability (or inability) of the policymaker to precommit credibly to a course of policy is crucial for observed outcomes.
  - Think of Ulysses and the sirens: It was only through the precommitment devices of filling his sailors' ears with wax and having himself tied to the mast of the ship that Ulysses was able to escape the sirens.
- Policies that act directly on Y_n (for instance, enforcement of anti-trust legislation or appropriate tax setting by the government as in the "trick" used by Sanjay Chugh) are better suited than monetary policy to resolve the problem of an inefficiently low Y_n.
- Absent such policies, precommitment mechanisms that bind policymakers to implement announced policies are a remedy for the inflationary consequences of time inconsistency, by giving the policymaker a way to "resist the sirens" of an output target above Y_n.
- The theory proposed by Kydland and Prescott and Barro and Gordon has been used to explain several high inflation episodes across countries and over time due to lack of precommitment, including high inflation in the U.S. in the 1970s.
- The argument fit stylized facts in several countries, and it has been (and still is) central to the focus of monetary policy and the design of monetary institutions to establish and preserve anti-inflationary credibility.

Commitment Mechanisms, Continued

Inflation Targeting

- A commitment mechanism that several countries have implemented beginning in the early 1990s is the adoption of an inflation targeting regime that clearly specifies the central bank's target and responsibilities with respect to inflation and puts an institutional setup in place to ensure central bank independence in the pursuit of this target and accountability in case of failure.

- Inflation targeting regimes are generally characterized by much transparency in communication between the central bank and the public to help the establishment of necessary credibility.

- Several countries are operating variants of such regime, including industrial countries and emerging markets.

Commitment Mechanisms, Continued

- As Michael McLeay and Silvana Tenreyro have argued in an *NBER Macroeconomics Annual 2019* article on “Optimal Inflation and the Identification of the Phillips Curve,” the pursuit of inflation targeting by central banks over a number of years may be responsible for the inability to easily identify the Phillips Curve (or its New Keynesian variant) in the data that has led some to argue that it is no longer a relevant concept.
- The Phillips Curve relation may be alive and well as part of the mechanisms of the economy.
- But if the central bank has successfully kept inflation low and stable over a number of years, the data will not show it “transparently.”

Commitment Mechanisms, Continued

- Now, some view the failure of central banks like the Federal Reserve or the European Central Bank to accomplish their 2 percent inflation targets consistently since the Global Financial Crisis of 2007-08 as an indictment of the failure of inflation targeting.
- Remembering the times when inflation in the U.S. was well into double digits and, in Italy, it was above 20 percent, I was (and still am) much more lenient than these colleagues toward an inflation rate stubbornly at, say, 1.7 percent when the target is 2 percent.
- Central banks have had to contend with a variety of pressures toward low-flation or even de-flation since 2007-08 (and some, like the Bank of Japan, since the early 1990s).
- The impact of the current COVID-19 crisis on inflation remains to be seen, but I certainly expect lively debates on the topic among policymakers and academics.

Commitment Mechanisms, Continued

Fixed Exchange Rate

- Adoption of a fixed exchange rate regime is another possible precommitment device—provided the commitment to pegging the currency is credible—when the domestic institutional setup does not support a country's independent, stable policies.

- For many countries, the strategy of adopting a fixed exchange rate that tied their monetary policies to those of established, low-inflationary central banks was indeed the precommitment device that would otherwise have been missing.

- By pegging the exchange rate, a country can "import" the low inflation policy of the country to which it pegs.

Commitment Mechanisms, Continued

- This argument is the core of Francesco Giavazzi and Marco Pagano's (*European Economic Review*, 1988) analysis of the European Monetary System (EMS) in place between 1979 and the advent of the euro:
  - European central banks that could not implement a credible commitment to low inflation policies of their own found it better to commit to shadowing the Bundesbank through the EMS constraint rather than finding themselves in the high-inflation, discretionary-policy outcome often generated by the inflationary pressures from their governments.
- In sum, a commitment device in the form of an explicit inflation targeting regime (if credible institutions for it are in place) or an exchange rate peg can be key to delivering price stability.
  - Nominal GDP targeting and price-level targeting have also been receiving attention in debates.

Rules

- Inflation targeting and exchange rate pegs are examples of policy rules.
- Kydland and Prescott's and Barro and Gordon's work started a huge literature on the advantages of rules versus discretion.
- The policymaker we looked at in our discussion of the time inconsistency problem operates under discretion.
- She/he is free to reoptimize her/his behavior at each point in time.
- The fact that what is optimal ex ante is no longer optimal ex post and the private sector's forward-looking behavior then result in the unfavorable equilibrium with monetary expansion and no output gain.
- Thus, it would be better if the policymaker could commit to a rule that forces her/him to implement the ex ante optimal policy by removing discretion.

Rules, Continued

- As we mentioned, inflation targeting and fixed exchange rate regimes are examples of policy rules to which the policymaker can precommit in order to establish independent domestic anti-inflationary credibility (under a properly designed inflation targeting regime) or to import the monetary policy credibility of the center country (under an exchange rate peg).

- The so-called Taylor rule (a rule for interest rate setting in response to inflation and output movement first studied by John Taylor in a 1993 article in the *Carnegie-Rochester Conference Series on Public Policy*) is another example of a rule to which the policymaker could be committed in some binding form.

  - Note an important difference: Inflation targeting or an exchange rate peg are targeting rules, the Taylor rule is an instrument rule.

- Even if some macroeconomists went as far as suggesting that, say, the Federal Reserve should be committed to automatically implementing the Taylor rule in each period, Taylor himself in his 1993 article interprets it more as a benchmark guideline for policymaking in normal conditions, from which departures should be allowed in special circumstances.

Rules, Continued

- He views the rule as constrained discretion, whereby the policymaker operates under discretion, but subject to the constraint of a benchmark guideline for policymaking in normal times.
- Clearly, such constrained discretion is feasible for institutions that do not lack credibility in the pursuit of a stable monetary environment.
- It is much less feasible for central banks that lack credibility, for which a more binding commitment can be the best way to bring inflation under control.
- Having said this, large crises—such as the Global Financial Crisis of 2007-08 or the current crisis created by COVID-19—are moments in which any rule designed for policymaking under normal economic conditions must be abandoned.

ECON 401

Advanced Macroeconomics

Topic 4

Unemployment

Fabio Ghironi
University of Washington

UNEMPLOYMENT: OVERVIEW

Basic Labor Market Issues

- Labor fluctuations at extensive margin (number of people working) larger than at intensive margin (hours worked per employee)

- Labor markets perhaps the important macro market to understand/model more deeply

- **Theoretical interest:** Many results from existing frameworks point to it

- **Empirical interest:** Labor-market outcomes the most important economic aspect of many (most?) people's lives

BASIC LABOR MARKET ISSUES

- How can production resources sit idle even when there is "high aggregate demand?"

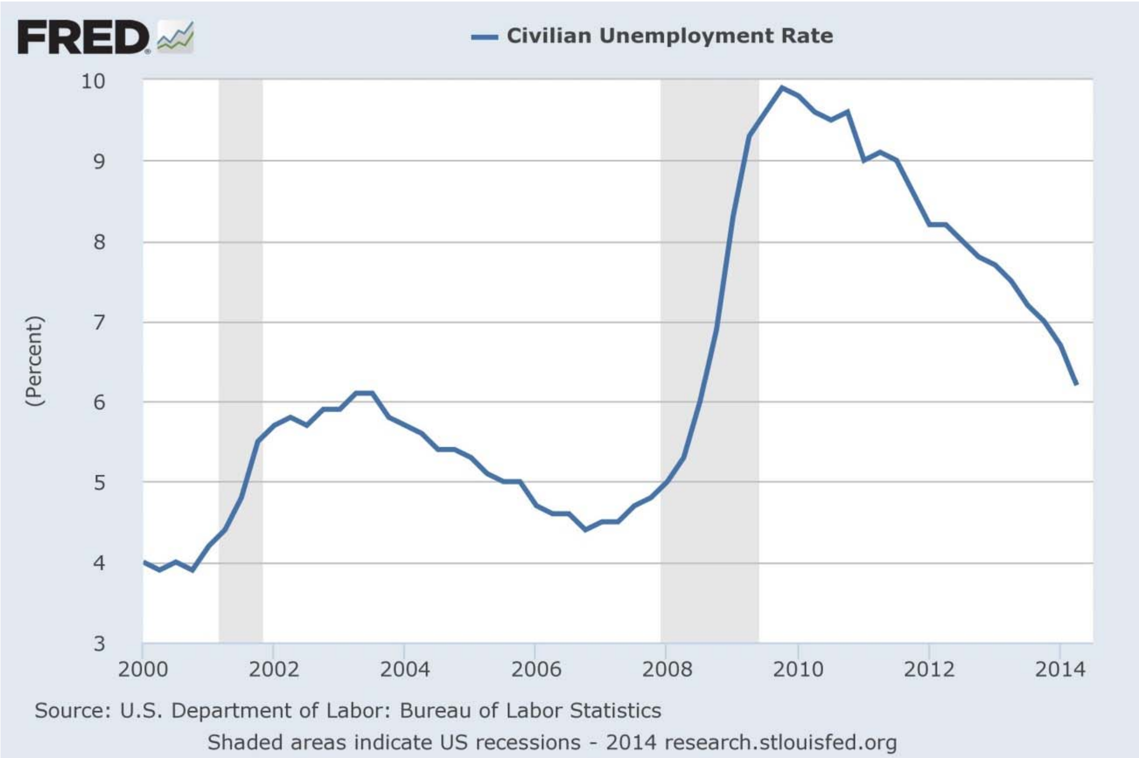
- Coordination frictions in labor markets
  - Finding a job or an employee takes time and/or resources
  - Not articulated in basic neoclassical/RBC-based frameworks

- Are labor market transactions "spot" transactions?
  - Or do they occur in the context of ongoing relationships?
  - The answer implies quite different roles for prices (wages)

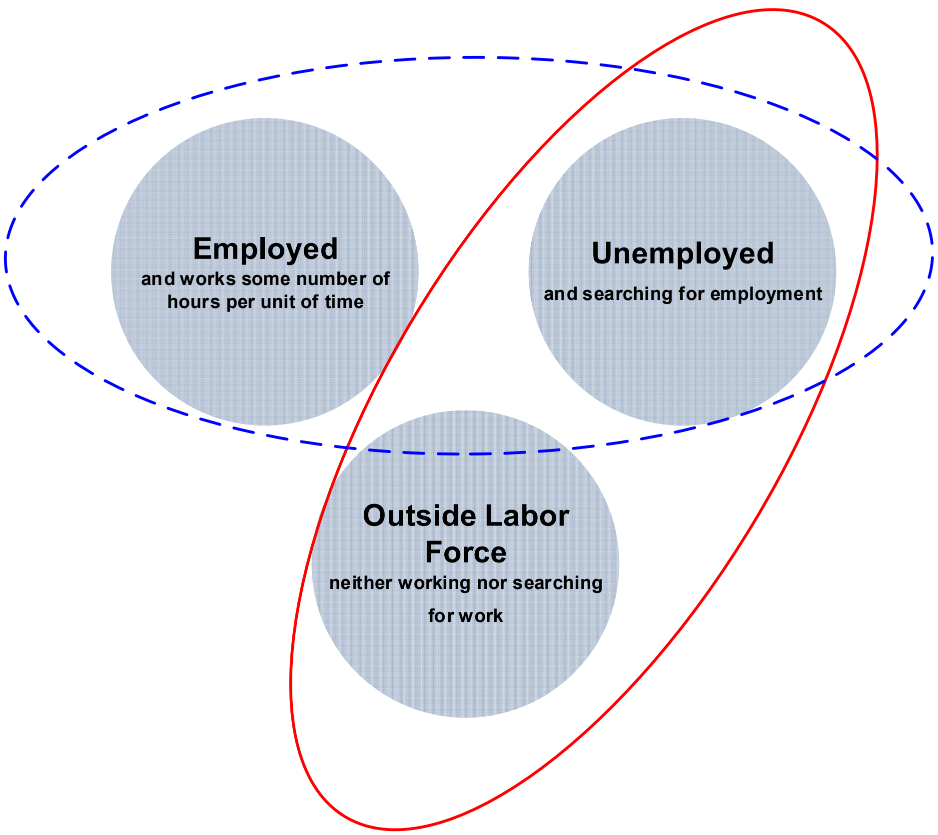
- "Structural" vs. "frictional" unemployment
  - Structural: unemployment induced by fundamental changes in technology, etc – dislocations due to insufficient job training, changing technical/educational needs of workforce, etc.
  - Frictional: temporarily unemployed as workers and jobs shuffle from one partner to another

UNEMPLOYMENT RATE

- Various categories of unemployment
  - Not working but actively searching for a job



BLS CATEGORIES



Red-circled categories labeled as “leisure”

But some portion of that “leisure time” was actually spent looking for a job

Circled-blue categories are “inside the labor force”

SEARCH AND MATCHING FRAMEWORK

- **Search and Matching Framework developed by**
  - **Diamond (2010 Nobel Prize)**
  - **Mortensen (2010 Nobel Prize)**
    - **(*dec- 2014*)**
  - **Pissarides (2010 Nobel Prize)**

SEARCH AND MATCHING: BUILDING BLOCKS

BUILDING BLOCKS OF MATCHING FRAMEWORK

1. **Aggregate matching technology**
   - **Brings together unemployed individuals (*s*) searching for work and job vacancies (*v*) seeking workers**
   - **A technology from the perspective of the economy (just like aggregate production function)**
   - **Allows for the *possibility* that an individual actively seeking a job cannot find a job**

UNEMPLOYMENT!

2. **If match successfully occurs, "surplus" arises between employer and employee…**
3. **Relationship-based "surplus" is split between employer and employee through wage**
   - **Important: wage is NOT NECESSARILY "market clearing"**
4. **Representative Consumer Analysis + Representative Firm Analysis**
5. **Employer-employee RELATIONSHIPS**
   - **Individuals with jobs work for "many periods" in the same job**

DEFINITIONS AND NOTATION

- **Notation**
  - ***c*: consumption ("all stuff")**
  - **n^s: number of units of time spent working**
  - **s: number of units of time spent searching for a job**

  - **p^{FIND}: *probability* that a unit of time spent in *s* leads to a successful job match**

$$\boxed{p^{FIND} \in (0,1)}$$

No guarantee that sending out resumes or interviewing for jobs leads to an ACTUAL job

  - **v: number of jobs ("job vacancies") available**
  - **q^{FIND}: *probability* that an available job gets filled**
  - **b^{ue}: unemployment benefit for unsuccessful search**
  - **w: real wage per unit of time spent working**

MATCHING + PROBABILITIES

- **Aggregate matching function**

$$m(s, v)$$

Evidence shows Cobb-Douglas describes aggregates well

  - **Brings together individuals looking for work (s) and employers looking for workers (v)**
  - **A technology from the perspective of the economy (just like aggregate production function)**
  - **Black box that describes all the possible coordination, matching, informational, temporal, geographic, etc. frictions in finding workers and jobs**

- **Probability of successfully getting a job**

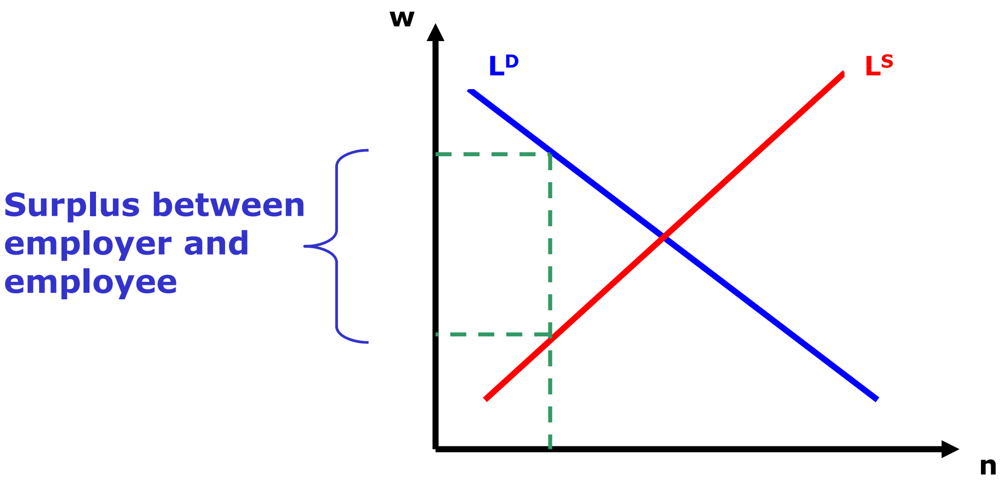
$$p^{\text{FIND}} \equiv \frac{m(s, v)}{s}$$

SURPLUS

- Recall
  - Optimal "labor supply" described by $MRS_{c,l} = w$ (C-L model)
  - Optimal "labor demand" described by $w = MPL$ (firm profit model)

- NOT the case in search and matching framework!
- Nonetheless, a good way to view matching framework…

Question 1: which wage is equilibrium?…



Answer: any wage inside the "surplus!"

Question 2: methodology to determine wage within the surplus set?

WAGE DETERMINATION

- Bargaining (Nash – "A Beautiful Mind")

Bargaining powers η and 1-η measure "strength" of each party in negotiations

$$\max_{w} \underbrace{\left(\mathbf{W}(w)-\mathbf{U}(w)\right)^{\eta}}_{\text{Payoff to an individual of agreeing to wage } w \text{ and beginning his/her employment}} \underbrace{\left(\mathbf{J}(w)-\mathbf{V}(w)\right)^{1-\eta}}_{\text{Payoff to a firm of agreeing to pay } w \text{ and employing the worker}}$$

- The unique problem whose solution satisfies three axioms (Nash 1950)
  - "Pareto optimal" between potential employee and potential employer
  - Independent of outside job offers

- But ***MANY*** other "wage" determination systems within surplus region!

WAGE DETERMINATION

- Bargaining (Nash)
- Bargaining (Proportional)

- Are health benefits heavily sponsored by employer?
  - If so, doesn’t count as “wage” but is an important “perk” of job

- Are life insurance benefits heavily sponsored by employer?
  - If so, doesn’t count as “wage” but is an important “perk” of job

- Firm-provided benefits became widespread in 1970’s
  - Many price and wage controls set during Nixon era to tame inflation….
  - …so firms provided “non-wage” benefits to entice workers

SEARCH AND MATCHING: "SUPPLY" AND "DEMAND"

BUILDING BLOCKS OF MATCHING FRAMEWORK

1. Aggregate matching technology
   - Brings together unemployed individuals (s) searching for work and job vacancies (v) seeking workers
   - A technology from the perspective of the economy (just like aggregate production function)
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$\boxed{\text{UNEMPLOYMENT!}}$

2. If match successfully occurs, "surplus" arises between employer and employee…
3. Relationship-based "surplus" is split between employer and employee through wage
   - Important: wage is NOT NECESSARILY "market clearing"
4. **Representative Consumer Analysis + Representative Firm Analysis – NOW**
5. Employer-employee RELATIONSHIPS
   - Individuals with jobs work for "many periods" in the same job

Later…

CONSUMER ANALYSIS

- **Static framework**
- **Extension of basic consumption-labor framework**

$$u(c)-h\left(\left(1-p^{FIND}\right)s+n^{s}\right)$$

- **If p^{FIND} = 1 and s = 0 →** $u(c)-h\left(n^{s}\right)$ **(the C-L model)**

- **Budget constraint**

$$c=(1-t)\cdot w\cdot n^{s}+\underbrace{\left(1-p^{FIND}\right)\cdot s}\cdot b^{ue}$$

"Unemployment" defined as actively-searching job-seekers who did not find a job.

Each receives an ue benefit b^{ue}

CONSUMER ANALYSIS

- **Budget constraint**

$$c = (1-t)\cdot w\cdot n^s + \left(1-p^{FIND}\right)\cdot s\cdot b^{ue}$$

- **Job-finding constraint**

$$n^s = p^{FIND} s$$

  - **Each unit of time in search leads has a probability p^{FIND} of contacting an open job vacancy**
  - **A second constraint on consumer optimization**

CONSUMER ANALYSIS

- **Lagrangian**

$$u(c)-h\left(\left(1-p^{FIND}\right)s+n^{s}\right)+\lambda^{h}\cdot\left[(1-t)\cdot w\cdot n^{s}+\left(1-p^{FIND}\right)\cdot s\cdot b^{ue}-c\right]+\mu^{h}\left[p^{FIND}s-n^{s}\right]$$

- **FOCs with respect to *c*, *n^h*, and *s*:**

$$u'(c)-\lambda^{h}=0$$

$$-h'((1-p^{FIND})\cdot s+n^{h})+\lambda^{h}\cdot(1-t)\cdot w-\mu^{h}=0$$

$$-h'((1-p^{FIND})\cdot s+n^{h})\cdot\underbrace{(1-p^{FIND})}_{\text{Chain Rule required here!}}+\lambda^{h}\cdot(1-p^{FIND})\cdot b^{ue}+\mu^{h}\cdot p^{FIND}=0$$

- **Combine FOCs (by eliminating λ^h and μ^h) into "labor supply" function…**

Consumer Analysis

❒ Labor Force Participation (aka Labor Supply)

$$\frac{h'(lfp)}{u'(c)} = p^{FIND} \cdot (1-t) \cdot w + (1-p^{FIND}) \cdot b^{ue}$$

MRS between "*HOPING TO* work" in labor-market and consumption

Suppose $b^{ue} = 0$

(After-tax *AND AFTER UE BENEFIT*) payoff of a *CHANCE* to work in markets

$$\frac{h'(lfp)}{u'(c)} = p^{FIND} \cdot (1-t) \cdot w$$

$p^{FIND} = 1$

MRS between work in labor-market and consumption

$$\frac{h'(lfp)}{u'(c)} = (1-t) \cdot w$$

(After-tax) payoff of unit of work in markets

FIRM ANALYSIS

- Firm must pay "recruiting costs" to look for workers
- Think of as
  - Career fairs
  - Posting job openings on Monster.com

- Firm production requires only labor (no physical capital)
  - Output = $Af(n^D)$ (Note: $f(.)$ production function need not be linear in n^D)

- Profit function

Number of new jobs advertised

$$A \cdot f(n^D) - w \cdot n^D - \omega \cdot v$$

Cost *per* job advertisement

- Each job vacancy advertised has probability q^{FIND} of successfully filling a position

MATCHING + PROBABILITIES

- **Aggregate matching function**

$$m(s, v)$$

Evidence shows Cobb-Douglas describes aggregates well

  - **Brings together individuals looking for work (s) and employers looking for workers (v)**
  - **A technology from the perspective of the economy (just like aggregate production function)**
  - **Black box that describes all the possible coordination, matching, informational, temporal, geographic, etc. frictions in finding workers and jobs**

- **Probability of successfully getting a job or hiring a new worker**

$$\boxed{p^{FIND} \equiv \frac{m(s, v)}{s}} \qquad \boxed{q^{FIND} \equiv \frac{m(s, v)}{v}}$$

FIRM ANALYSIS

- Firm must pay "recruiting costs" to look for workers
- Think of as
  - Career fairs
  - Posting job openings on Monster.com

- Firm production requires only labor (no physical capital)
  - Output = $Af(n^D)$ (Note: $f(.)$ production function need not be linear in n^D)

- Profit function

Number of new jobs advertised

$$A \cdot f(n^D) - w \cdot n^D - \omega \cdot v$$

Cost *per* job advertisement

- Each job vacancy advertised has probability q^{FIND} of successfully filling a position

- Number of new employees hired is $n^D = q^{FIND} v$

FIRM ANALYSIS

- **Lagrangian**

$$A\cdot f(n^D)-w\cdot n^D-\omega\cdot v+\mu^f(q^{FIND}\cdot v-n^D)$$

- **FOCs with respect to n^D and v:**

$$A\cdot f'(n^D)-w-\mu^f=0$$

$$-\omega+\mu^f\cdot q^{FIND}=0$$

- **Combine FOCs (by eliminating μ^f) into "labor demand" function….**

$$\omega=q^{FIND}\cdot\left(A\cdot f'(n^D)-w\right)$$

Firm Analysis

❒ Vacancy Posting (aka Labor Demand)

$$\omega = q^{FIND} \cdot \left(A \cdot f'(n^D) - w \right)$$

Cost of attempting to attract a job application

Marginal profit ***IF*** someone is hired

$q^{FIND} = 1$

$$\omega = A \cdot f'(n^D) - w$$

$\omega = 0$

Marginal product of labor (*mpn*)

$$A \cdot f'(n^D) = w$$

real wage

MATCHING ANALYSIS VS. "CLASSICAL" ANALYSIS

Matching

$$\frac{h'(lfp)}{u'(c)} = p^{FIND} \cdot (1-t) \cdot w + (1-p^{FIND}) \cdot b^{ue}$$

$$\omega = q^{FIND} \cdot \left(A \cdot f'(n^D) - w\right)$$

Classical

$$\frac{h'(lfp)}{u'(c)} = (1-t) \cdot w$$

$$A \cdot f'(n^D) = w$$

Question: How does labor market clear?

SEARCH AND MATCHING: MATCHING EQUILIBRIUM

Building Blocks of Matching Framework

1. **Aggregate matching technology**
   - **Brings together unemployed individuals (*s*) searching for work and job vacancies (*v*) seeking workers**
   - **A technology from the perspective of the economy (just like aggregate production function)**
   - **Allows for the *possibility* that an individual actively seeking a job cannot find a job**

UNEMPLOYMENT!

2. **If match successfully occurs, "surplus" arises between employer and employee…**

3. **Relationship-based "surplus" is split between employer and employee through wage**
   - **Important: wage is NOT NECESSARILY "market clearing"**

4. **Representative Consumer Analysis + Representative Firm Analysis**

5. **Employer-employee RELATIONSHIPS**
   - **Individuals with jobs work for "many periods" in the same job**

MATCHING ANALYSIS VS. "CLASSICAL" ANALYSIS

Matching

$$\frac{h'(lfp)}{u'(c)} = p^{FIND} \cdot (1-t) \cdot w + (1-p^{FIND}) \cdot b^{ue}$$

$$\omega = q^{FIND} \cdot \left(A \cdot f'(n^D) - w \right)$$

Classical

$$\frac{h'(lfp)}{u'(c)} = (1-t) \cdot w$$

$$A \cdot f'(n^D) = w$$

Question: How does labor market clear?

MATCHING ANALYSIS VS. "CLASSICAL" ANALYSIS

- **Let b^{ue} = 0 and t = 0 (simplifies analysis)**

Matching

$$\frac{h'(lfp)}{u'(c)} = p^{FIND} \cdot w$$

$$\omega = q^{FIND} \cdot \left(A \cdot f'(n^D) - w\right)$$

"Market-clearing" through real wage adjustment?...

Isolate the w terms...

Classical

$$\frac{h'(lfp)}{u'(c)} = w$$

$$A \cdot f'(n^D) = w$$

"Market-clearing" through real wage adjustment

$$\boxed{\frac{h'(lfp)}{u'(c)} = A \cdot f'\left(n^D\right)}$$

(NOTE: simply *MRS* = *MPN*)

MATCHING ANALYSIS VS. "CLASSICAL" ANALYSIS

- **Let b^{ue} = 0 and t = 0 (simplifies analysis)**

- $$\frac{h'(lfp)}{u'(c)} = p^{FIND} \cdot w$$ **Isolate *w*** $$\left(\frac{h'(lfp)}{u'(c)}\right) \cdot \frac{1}{p^{FIND}} = w$$

- $$\omega = q^{FIND} \cdot \left(A \cdot f'(n^D) - w\right)$$ **Isolate *w*** $$w = A \cdot f'(n^D) - \frac{\omega}{q^{FIND}}$$

"Market-clearing" through real wage adjustment?...

...if so...

$$\left(\frac{h'(lfp)}{u'(c)}\right) \cdot \frac{1}{p^{FIND}} = A \cdot f'\left(n^D\right) - \frac{\omega}{q^{FIND}}$$

A bit more algebra to shed economic light...

MATCHING ANALYSIS VS. "CLASSICAL" ANALYSIS

- **Let b^{ue} = 0 and t = 0 (simplifies analysis)**

$$\left(\frac{h'(lfp)}{u'(c)}\right)\cdot\frac{1}{p^{FIND}}=A\cdot f'\left(n^D\right)-\frac{\omega}{q^{FIND}}$$

A bit more algebra to shed economic light…

$$\frac{h'(lfp)}{u'(c)}=p^{FIND}\cdot A\cdot f'\left(n^D\right)-\omega\left(\frac{p^{FIND}}{q^{FIND}}\right)$$

Recall in "classical" equilibrium

$$\frac{h'(lfp)}{u'(c)}=A\cdot f'\left(n^D\right)\quad\Rightarrow\quad\frac{h'(lfp)/u'(c)}{A\cdot f'\left(n^D\right)}=1$$

(NOTE: simply *MRS* = *MPN*…)

Even more algebra to try to re-arrange to "look like" "classical" equilibrium…

$$\frac{h'(lfp)/u'(c)}{A\cdot f'\left(n^D\right)}=p^{FIND}-\frac{\omega\left(p^{FIND}/q^{FIND}\right)}{A\cdot f'\left(n^D\right)}$$

MATCHING ANALYSIS VS. "CLASSICAL" ANALYSIS

- Let b^{ue} = 0 and t = 0 (simplifies analysis)
- Compare the two

Matching

$$\underbrace{\frac{h'(lfp)/u'(c)}{A\cdot f'\left(n^D\right)}}_{=1\ ?\ldots} = \underbrace{p^{FIND} - \frac{\omega\left(p^{FIND}/q^{FIND}\right)}{A\cdot f'\left(n^D\right)}}_{\text{Does this }=1\ ?\ldots}$$

Classical

$$\underbrace{\frac{h'(lfp)/u'(c)}{A\cdot f'\left(n^D\right)}}_{=1} = 1$$

- Tedious rigamarole of algebra hasn't gotten us anywhere…
- Consider "equilibrium" in different way

MATCHING EQUILIBRIUM

- Focus on p^{FIND} or q^{FIND}

$$p^{FIND} \equiv \frac{m(s, v)}{s}$$

$$q^{FIND} \equiv \frac{m(s, v)}{v}$$

- If matching function is

$$m(s, v) = s^{\gamma} v^{1-\gamma}$$

Cobb-Douglas matching function, $0 < \gamma < 1$

- then

$$p^{FIND} = \frac{s^{\gamma} v^{1-\gamma}}{s} = \frac{v^{1-\gamma}}{s^{1-\gamma}} = \left(\frac{v}{s}\right)^{1-\gamma}$$

$$q^{FIND} = \frac{s^{\gamma} v^{1-\gamma}}{v} = \frac{s^{\gamma}}{v^{\gamma}} = \left(\frac{v}{s}\right)^{-\gamma}$$

Lots of algebra…
but informative algebra…

MATCHING EQUILIBRIUM

- Both $p^{FIND} = (v/s)^{1-\gamma}$ and $q^{FIND} = (v/s)^{-\gamma}$ depend on

$$\boxed{\theta \equiv \frac{v}{s}}$$

- Aggregate labor market tightness θ

- If θ increases, p^{FIND} increases
  - Easier for actively searching unemployed individual to find a job
  - Higher probability of successfully finding work

- If θ increases, q^{FIND} decreases
  - More difficult for a business to hire a new employee
  - Lower probability of successfully hiring a suitable job candidate

- Market tightness θ measures "congestion effects"

MATCHING EQUILIBRIUM

- Use $p^{FIND} = (\theta)^{1-\gamma}$ in labor force participation condition

$$\frac{h'(lfp)}{u'(c)} = p^{FIND} \cdot w \longrightarrow \frac{h'(lfp)}{u'(c)} = \theta^{1-\gamma} \cdot w \xrightarrow{\text{Solve for } \theta} \theta = \left[\frac{h'(lfp)}{u'(c)} \cdot \frac{1}{w}\right]^{\frac{1}{1-\gamma}}$$

- Use $q^{FIND} = (\theta)^{-\gamma}$ in job-creation condition

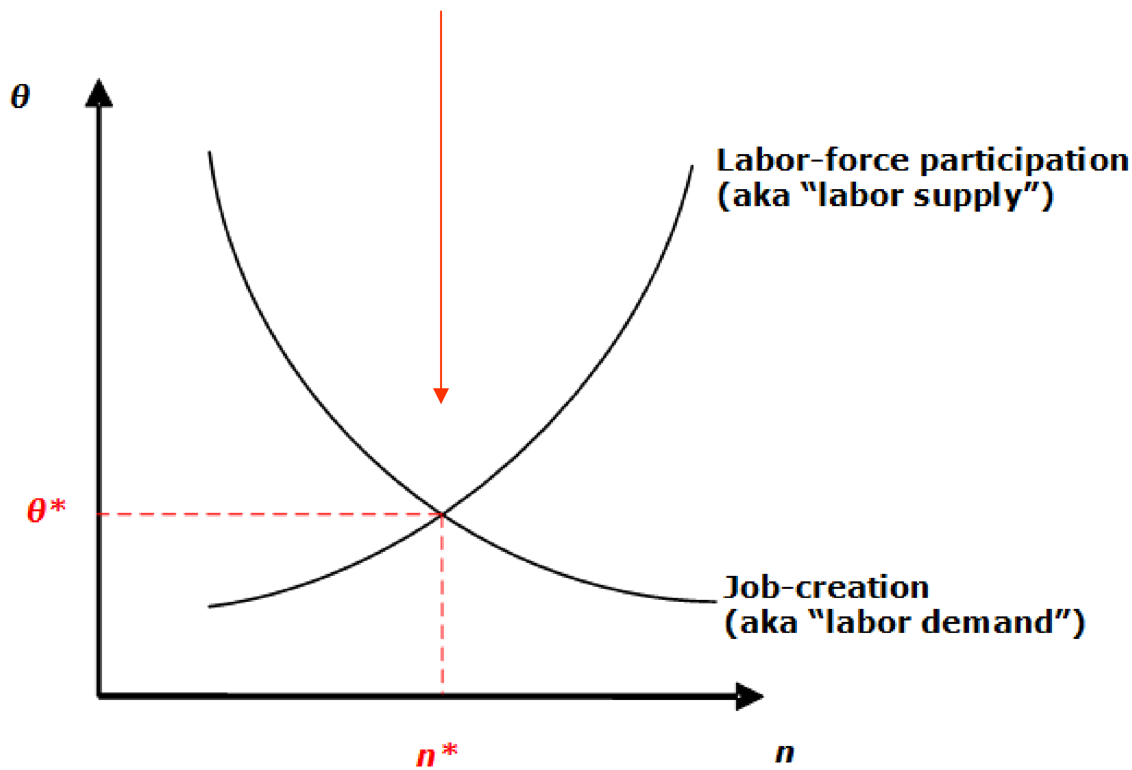
$$\omega = q^{FIND} \cdot \left(A \cdot f'\left(n^D\right) - w\right) \longrightarrow \omega = \theta^{-\gamma} \cdot \left(A \cdot f'\left(n^D\right) - w\right)$$

$$\xrightarrow{\text{Solve for } \theta} \theta = \left[\frac{\omega}{A \cdot f'\left(n^D\right) - w}\right]^{-\frac{1}{\gamma}}$$

MATCHING EQUILIBRIUM

- Labor-force participation function (plotted as function of θ)
- Job-creation function (plotted as function of θ)

- Matching Market Clearing: $n^* = q^{FIND}v = p^{FIND}s$



WHITHER WAGES?

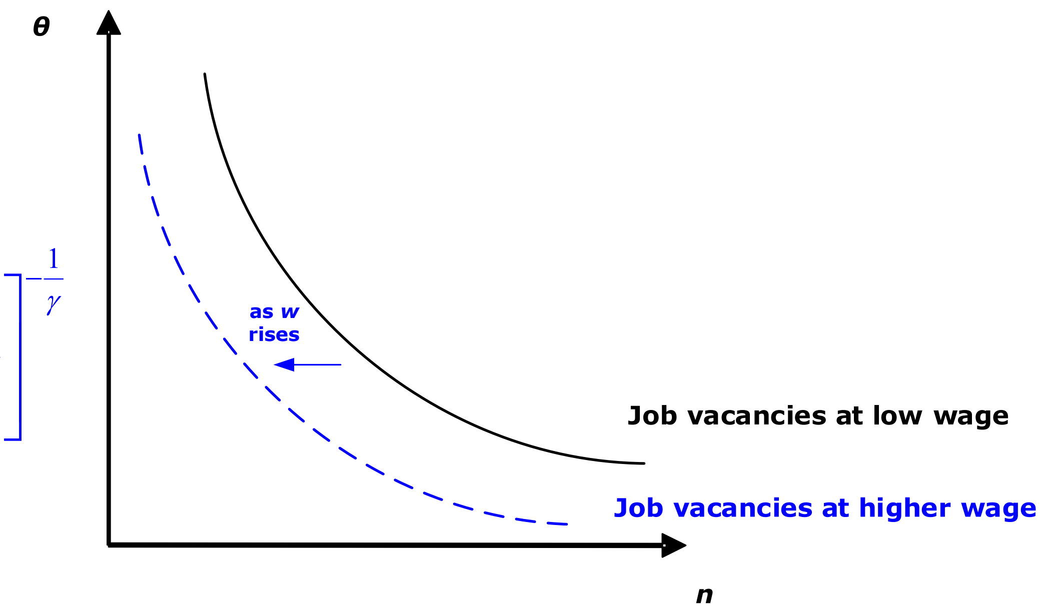
- **Matching market clears via adjustment of market tightness θ**
- **Where did the real wage w go?...**

WHITHER WAGES?

- **Matching market clears via adjustment of market tightness *θ***
- **Where did the real wage *w* go?...**

- **It's a shift factor...**
- **E.g., if *w* rises**

$$\theta = \left[\frac{\omega}{A \cdot f'\left(n^D\right) - w} \right]^{-\frac{1}{\gamma}}$$

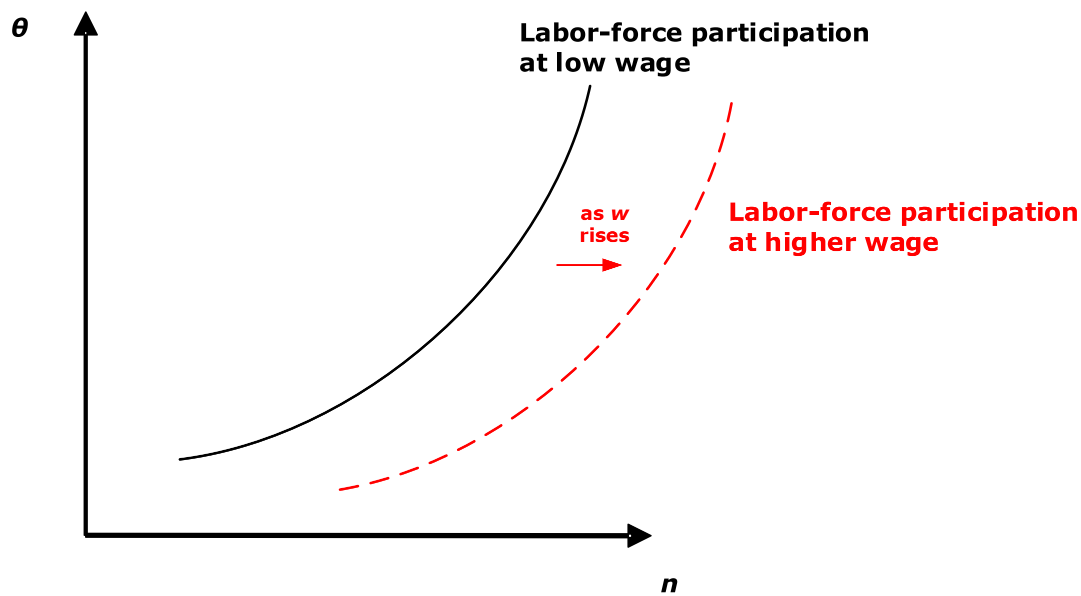


WHITHER WAGES?

- Matching market clears via adjustment of market tightness *θ*
- Where did the real wage *w* go?...

- It's a shift factor...
- E.g., if *w* rises

$$\theta = \left[\frac{h'(lfp)}{u'(c)} \cdot \frac{1}{w} \right]^{\frac{1}{1-\gamma}}$$



WHITHER WAGES?

- **From tedious rigamarole of algebra…**

$$\frac{h'(lfp)/u'(c)}{A\cdot f'\left(n^D\right)} = p^{FIND} - \frac{\omega\left(p^{FIND}/q^{FIND}\right)}{A\cdot f'\left(n^D\right)}$$

$\downarrow$ **Simplify the economics by using *mrs*, *mpn*, and set *A* = 1**

$$\frac{mrs}{mpn} = \underbrace{p^{FIND} - \frac{\omega\left(p^{FIND}/q^{FIND}\right)}{mpn}}_{}$$

Is this = 1 ?… Is this > 1 ?… Is this < 1 ? …

$\downarrow$ **Several steps of algebra**
Main idea: use $p^{FIND} = (\theta)^{1-\gamma}$ and $q^{FIND} = (\theta)^{-\gamma}$

$\boxed{\textbf{SMALLER THAN 1!}}$

$\downarrow$

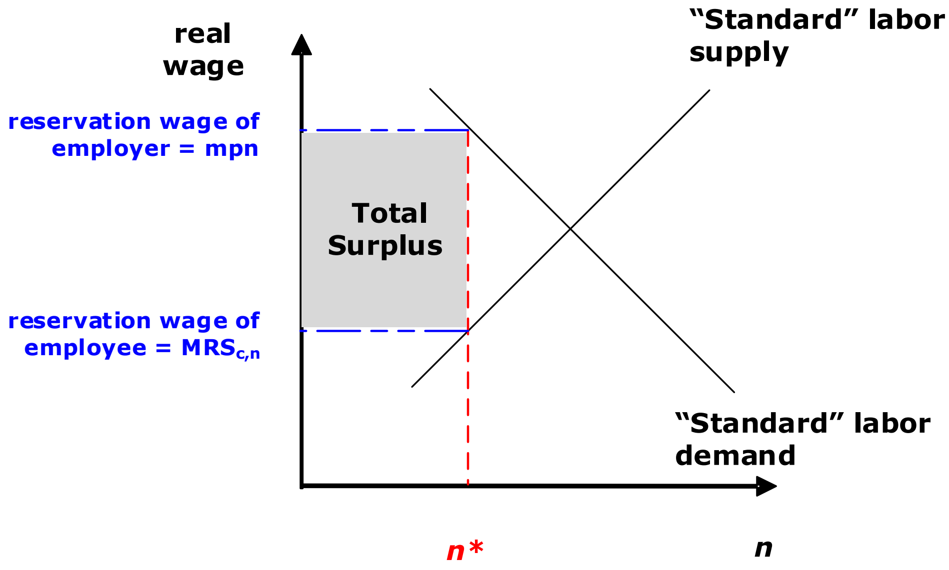
$\boxed{\text{Therefore, } mrs < mpn \text{ at } n^*}$

WHITHER WAGES?

- $mrs_{c,n} < mpn$ at n^*
- "Equilibrium wage" = …?…

- Anywhere below the reservation wage of employer (= *mpn*)
- Anywhere above the $mrs_{c,n}$ of job-searcher

- Marginal surplus at n^*
- Total surplus – shaded box

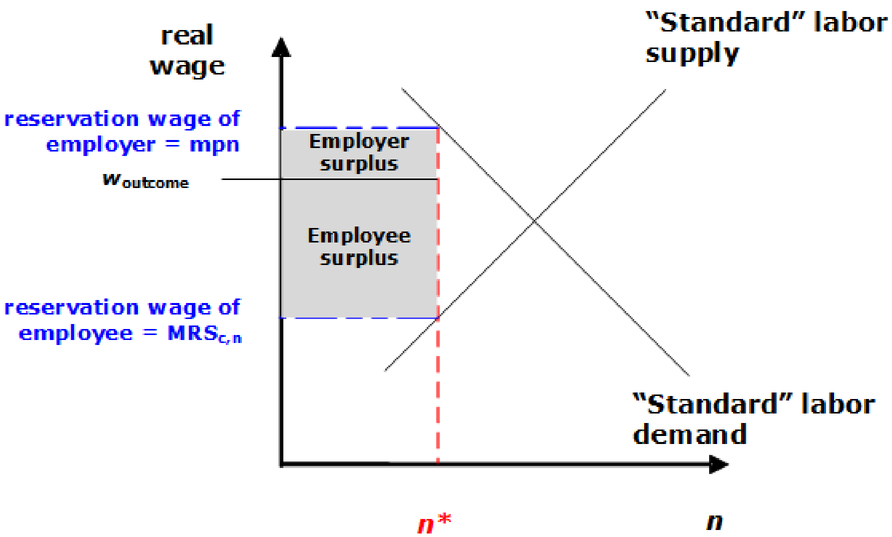


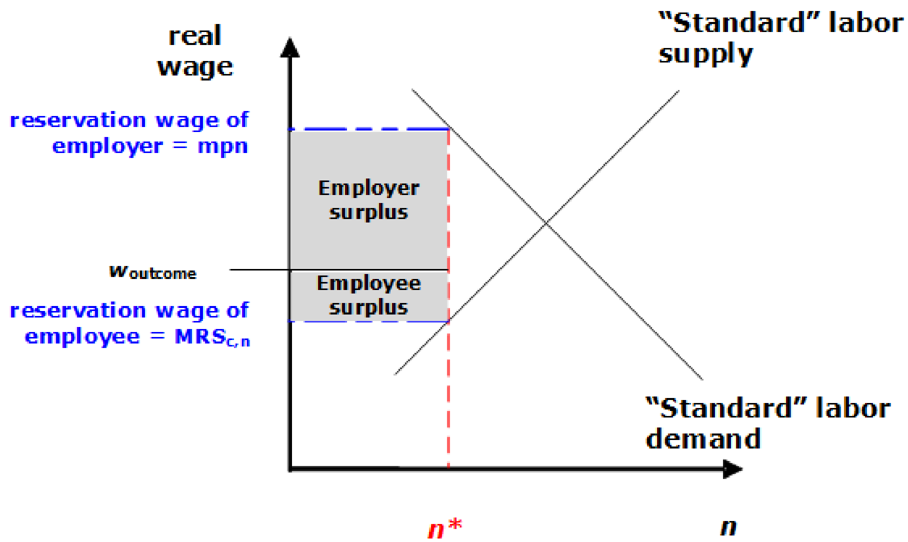
WHITHER WAGES?

- *$mrs_{c,n}$* < *mpn* at *n**
- "Equilibrium wage" = …?…

- Anywhere below the reservation wage of employer (= *mpn*)
- Anywhere above the *$mrs_{c,n}$* of job-searcher

- Two "equilibrium wage outcomes"
  - Out of an infinite number of "equilibrium wage outcomes" inside surplus set





LONG-LASTING JOBS

BUILDING BLOCKS OF MATCHING FRAMEWORK

1. Aggregate matching technology
   - Brings together unemployed individuals (*s*) searching for work and job vacancies (*v*) seeking workers
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   - Allows for the ***possibility*** that an individual actively seeking a job cannot find a job

UNEMPLOYMENT!

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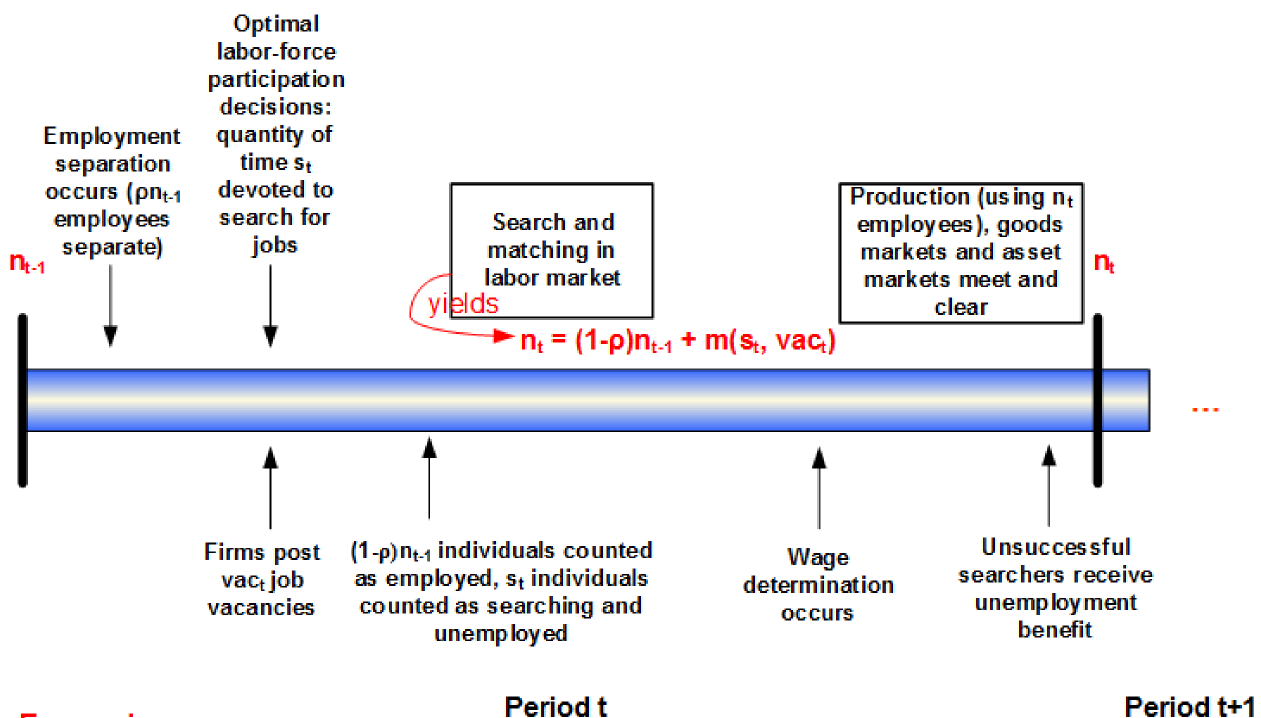
   - **Important: wage is NOT NECESSARILY "market clearing"**
4. Representative Consumer Analysis + Representative Firm Analysis
5. **Employer-employee RELATIONSHIPS**
   - **Individuals with jobs work for "many periods" in the same job**

TIMING OF EVENTS

- Extend one-period framework to infinite-period framework
- Long-lasting jobs: $n_t = (1-\rho)n_{t-1} + m(s_t, v_t)$



NOTE: Economic planning occurs for the ENTIRE remaining lifetime.

LABOR MARKETS

- Extend one-period framework to infinite-period framework
- Long-lasting jobs: $n_t = (1-\rho)n_{t-1} + m(s_t, v_t)$
- Typical employee works for many time periods
- Economy-wide job turnover rate $\rho \in (0,1]$
- One-period framework has $\rho = 1$
- U.S. quarterly job turnover percentage $\rho \approx 0.10$
- Modify job-finding constraint in representative consumer (now lifetime…) utility maximization

$$n_t^s = (1-\rho)n_{t-1}^s + p_t^{FIND} s_t$$

- Modify job-hiring constraint in representative firm (now lifetime…) profit maximization

$$n_t^D = (1-\rho)n_{t-1}^D + q_t^{FIND} v_t$$

CONSUMER ANALYSIS

- Infinite sequence of budget constraints

$$c_t = (1-t_t)w_t n_t^s + \left(1 - p_t^{FIND}\right) s_t \cdot b$$

- Infinite sequence of job-finding constraints

$$n_t^s = (1-\rho)n_{t-1}^s + p_t^{FIND} s_t$$

- (Rest of details of infinite-horizon representative consumer optimization as per usual by now, Lagrangian, FOCs, etc…)

- Consumption-LFP optimality condition

$$\frac{h'(lfp_t)}{u'(c_t)} = p_t^{FIND} \cdot \left[(1-t_t)w_t + (1-\rho)\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right) \cdot \left(\frac{h'(lfp_{t+1})}{u'(c_{t+1})}\right)\right] + \left(1 - p_t^{FIND}\right) b$$

FIRM ANALYSIS

- **Infinite sequence of job-finding constraints**

$$n_t^D = (1-\rho)n_{t-1}^D + q_t^{FIND} v_t$$

- **(Rest of details of infinite-horizon representative firm optimization as per usual by now, Lagrangian, FOCs, etc…)**

- **Vacancy Posting Condition**

$$\frac{\omega}{q_t^{FIND}} = A_t \cdot f'\left(n_t^D\right) - w_t + \left(\frac{1-\rho}{1+r_t}\right) \cdot \frac{\omega}{q_{t+1}^{FIND}}$$

JOBS AS ASSETS

- **Consumption-LFP optimality condition**

$$\frac{h'(lfp_t)}{u'(c_t)} = p_t^{FIND} \cdot \left[(1-t_t)w_t + (1-\rho)\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right) \cdot \left(\frac{h'(lfp_{t+1})}{u'(c_{t+1})}\right) \right] + \left(1-p_t^{FIND}\right)b$$

- **Vacancy Posting Condition**

$$\frac{\omega}{q_t^{FIND}} = A_t \cdot f'\left(n_t^D\right) - w_t + \left(\frac{1-\rho}{1+r_t}\right) \cdot \frac{\omega}{q_{t+1}^{FIND}}$$

- **Recall**

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1+r_t}$$

True for ANY asset, whether tangible (physical *k*) or intangible

- **"Our employees are our most valuable asset"**
- **(Vacancy Posting Condition can be rewritten as)**

$$\frac{\omega}{q_t^{FIND}} = A_t \cdot f'\left(n_t^D\right) - w_t + (1-\rho)\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right) \cdot \frac{\omega}{q_{t+1}^{FIND}}$$

JOBS AS LONG-LASTING INTERACTIONS

- **Consumption-LFP optimality condition**

$$\frac{h'(lfp_t)}{u'(c_t)} = p_t^{FIND} \cdot \left[(1-t_t)w_t + (1-\rho)\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right) \cdot \left(\frac{h'(lfp_{t+1})}{u'(c_{t+1})}\right) \right] + \left(1 - p_t^{FIND}\right) b$$

- **Vacancy Posting Condition**

$$\frac{\omega}{q_t^{FIND}} = A_t \cdot f'\left(n_t^D\right) - w_t + (1-\rho)\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right) \cdot \frac{\omega}{q_{t+1}^{FIND}}$$

- **With $\rho < 1$**
- **Employees know their employers**
- **Employers know their employees**

- **Will experience economy-wide aggregate shocks together**

- **Provides implicit "insurance" over time for both employees and employers…**

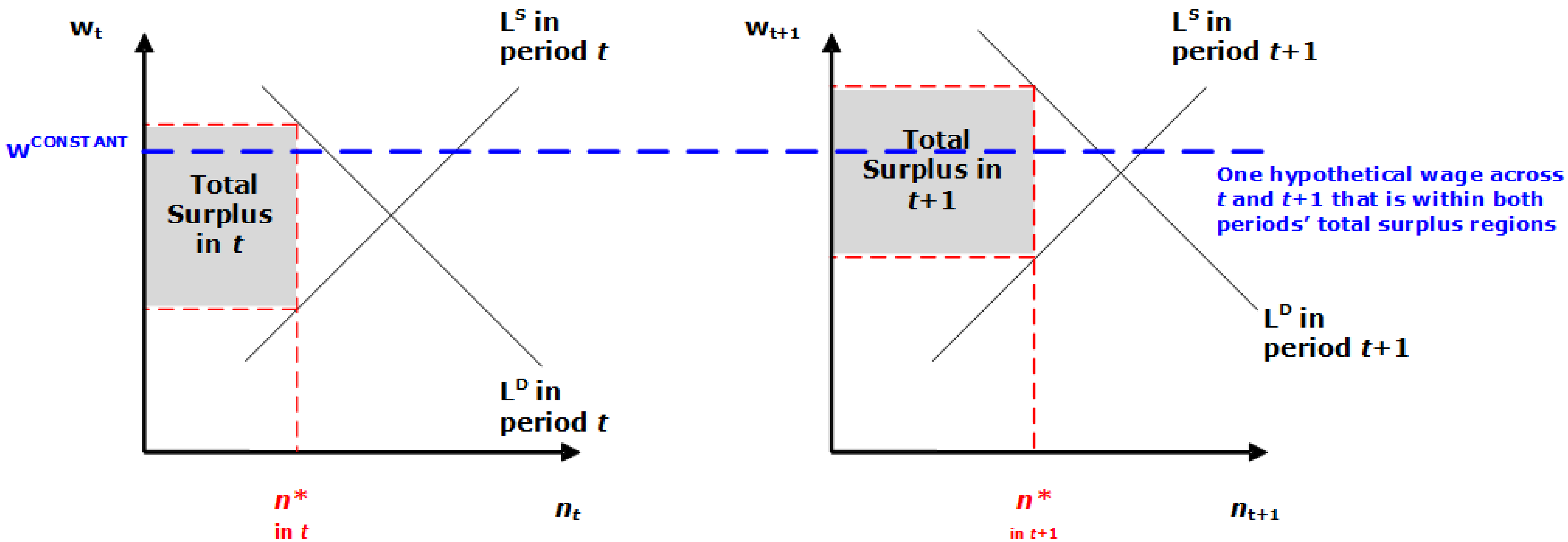
- **…by allowing for "rigid real wages" across time periods**

WHITHER WAGES?

- Suppose **two-period** job relationship
- A positive TFP shocks occurs between period t and period $t+1$
  - Shifts outwards "classical" labor demand curve

- w_t = …?… w_{t+1} = …?…

- Scenario 1: constant w across both periods



WHITHER WAGES?

- Suppose two-period job relationship
- A positive TFP shocks occurs between period t and period $t+1$
  - Shifts outwards "classical" labor demand curve

- w_t = …?… w_{t+1} = …?…

- Scenario 1: constant w across both periods

- Scenario 2: $w_t = mpn_t$ and $w_{t+1} = mrs_{t+1}$

- Scenario 3: $w_t = mrs_t$ and $w_{t+1} = mpn_{t+1}$

- Scenario 4:

$$w_t = mrs_t + 0.5*(mpn_t - mrs_t) \text{ and } w_{t+1} = mrs_{t+1} + 0.5*(mpn_{t+1} - mrs_{t+1})$$

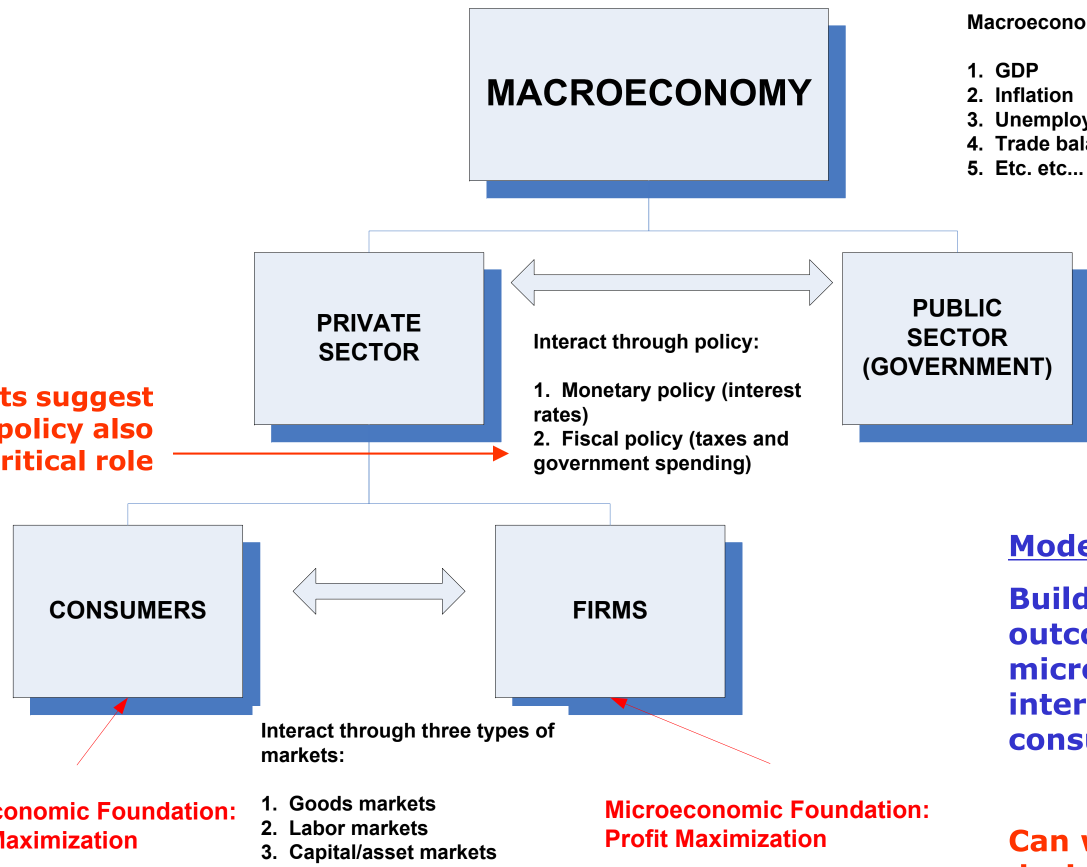
- All scenarios are equilibrium wages!
- As are an infinitely-more combinations of wages that lie within the surplus regions of both time periods!

THE FINANCIAL ACCELERATOR: FINANCIAL MARKETS AND THE MACROECONOMY

FINANCIAL ACCELERATOR

- "Financial accelerator" framework
  - The most widely-used and applied framework in macroeconomic theory and policy for thinking about financial markets
  - Developed in series of studies by Bernanke, Gertler, and Gilchrist in 1980's and 1990's
- Popular-press language
  - "Financial accelerator"
  - "Financial feedback loops"
  - "Loan spirals"
- Describes well many of the financial-macroeconomic linkages underpinning the dynamics of
  - Great Depression
  - Great Recession
- Will develop idea in context of firm theory
- Can also develop idea in context of consumer theory.
  - "Credit constraint" analysis of consumption/savings decisions.

Building Blocks of an Economy



Macroeconomic Outcomes:

1. GDP
2. Inflation
3. Unemployment
4. Trade balances
5. Etc. etc...

AGGREGATE OUTCOMES

Recent events suggest regulatory policy also plays a critical role

Microeconomic Foundation: Utility Maximization

Microeconomic Foundation: Profit Maximization

Modern Macroeconomic Theory:

Build a theory of aggregate outcomes by studying microeconomic decisions and interactions between firms and consumers

Can we feasibly model the decisions of *every* consumer and firm and *every* interaction between them?...

OUTLINE OF FRAMEWORK

Major ideas underlying Financial Accelerator Framework

1. **Firms' financial assets (i.e., stocks and bonds) matter for their ability to purchase physical assets (i.e., machines and equipment)**

2. **Market prices of financial assets matter for firm financing constraints**

3. **Government regulation affects the linkage between financial markets and real (i.e., goods and physical capital) markets *through* financing constraints**

OUTLINE OF FRAMEWORK

Four Building Blocks of the Financial Accelerator Framework

1. **Two-Period Model of Firm Profit Maximization**
   - ❑
   - ❑ **Enriched to allow for both physical assets (machines and equipment) and financial assets (stocks and bonds)**

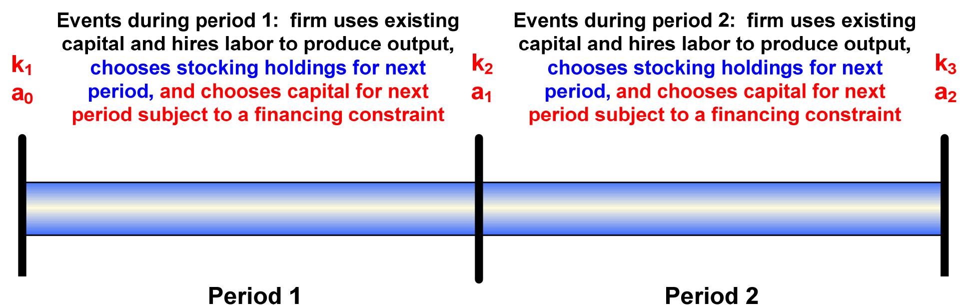
2. **Financing Constraint – conceptually, the key building block**
   - ❑ **Quantity of physical capital firms can purchase depends on the market value (i.e., price x quantity) of their financial assets**
   - ❑ **Reflects market and regulatory structures designed to mitigate informational asymmetries**
   - ❑ **(Basic theory of firms features no constraints of this type on firm profit maximization.)**

3. **Government Regulation/Oversight of Financial Relationships**

4. **Relationship between Firm Profits and Dividends**

ENRICHING THE BASIC FIRM THEORY

- ## Timeline of events



- ## Notation
  - k_2: capital used for production in period 2 (decided upon in period 1)
  - n_2: labor used for production in period 2
  - w_2: real wage rate for labor in period 2 ($w_2 = W_2/P_2$)
  - i: nominal interest rate (between period 1 and period 2)
  - P_2: nominal price of output produced and sold by firm in period 2
    AND nominal price of one unit of capital bought by the firm in period 2 for use in period 3
  - a_1: real wealth (stock) holdings at beginning of period 2/end of period 1
  - S_2: nominal price of a unit of stock in period 2
  - D_2: nominal dividend paid in period 2 by each unit of stock held at the start of period 2
  - π_2: net inflation rate between period 1 and period 2 (recall: $\pi_2 = P_2/P_1 - 1$)

The "definining features" of stock

RATES OF RETURN

- "Interest rates" can be defined for any type of asset
  - There is no ***single*** interest rate in the economy

- Interpret/understand the ***two*** types of "interest rates" that co-exist in this richer theory of firm profit maximization
  - ***i***: nominal interest rate on bonds

    REAL INTEREST RATE ON GOVERNMENT BONDS: A "RISKLESS" ASSET

    - Recall

      $$1+i=\frac{1}{P_1^b} \quad \xleftrightarrow[\text{as}]{\text{can rewrite}} \quad i=\frac{1}{P_1^b}-1 \quad \xleftrightarrow[\textit{real}\text{ interest rate}]{\text{express as}} \quad 1+r=\frac{1+i}{1+\pi}$$

    - Thus can think of bonds (one ***type*** of financial asset) as being in the background of the analysis

  - ***i***^{STOCK}: nominal return on stock – i.e., "interest rate on stock" (though bad terminology)

    REAL INTEREST RATE ON STOCKS: A "RISKY" ASSET

    - Define according to

      $$1+i^{STOCK}=\frac{S_2+D_2}{S_1} \quad \xleftrightarrow[\text{as}]{\text{can rewrite}} \quad i^{STOCK}=\frac{S_2+D_2}{S_1}-1 \quad \xleftrightarrow[\textit{real}\text{ interest rate}]{\text{express as}} \quad 1+r^{STOCK}=\frac{1+i^{STOCK}}{1+\pi}$$

    - Measures the net dollar return (in period 2) on one share of stock (whose purchase price was S_1 in period 1)

- Can distinguish two measures of real interest rates in this framework

FIRM PROFIT FUNCTION

- A ***dynamic*** **profit maximization problem**
  - **Because firm exists for both periods**
  - **All analysis conducted from the perspective of the very beginning of period 1**
  - **→ Must consider present-discounted-value (PDV) of lifetime (i.e., two-period) profits**

- **Dynamic profit function**
  - **(specified in nominal terms – could specify in real terms…)**

As usual: no physical or financial assets needed for "period 3"

$$\overbrace{P_1 f(k_1,n_1) + P_1 k_1 + (S_1 + D_1)a_0 - P_1 w_1 n_1 - P_1 k_2 - S_1 a_1}^{\text{Period-1 profits}} + \overbrace{\frac{P_2 f(k_2,n_2)}{1+i} + \frac{P_2 k_2}{1+i} + \frac{(S_2 + D_2)a_1}{1+i} - \frac{P_2 w_2 n_2}{1+i} - \frac{P_2 \overbrace{k_3}^{=0}}{1+i} - \frac{S_2 \overbrace{a_2}^{=0}}{1+i}}^{\text{(PDV of) period-2 profits}}$$

- $P_1 f(k_1,n_1)$: **Total revenue in period 1 (price x output)**
- $P_1 k_1$: **Value of pre-existing <u>physical</u> capital (an asset for firms)**
- $(S_1 + D_1)a_0$: **Value (inclusive of dividends) of pre-existing <u>financial</u> assets (i.e., stock-holdings in *other* firms)**
- $P_1 w_1 n_1$: **Total labor cost in period 1**
- $P_1 k_2$: **Total cost of buying <u>physical</u> capital for period 2 (time to build → must purchase period-2 capital in period 1)**
- $S_1 a_1$: **Total cost of buying <u>financial</u> assets (i.e., stock-holdings in *other* firms) for period 2**
- $\frac{P_2 f(k_2,n_2)}{1+i}$: **Total revenue in period 2 (price x output)**
- $\frac{P_2 k_2}{1+i}$: **Value of pre-existing <u>physical</u> capital (an asset for firms)**
- $\frac{(S_2 + D_2)a_1}{1+i}$: **Value (inclusive of dividends) of pre-existing <u>financial</u> assets (i.e., stock-holdings in *other* firms)**
- $\frac{P_2 w_2 n_2}{1+i}$: **Total labor cost in period 2**
- $\frac{P_2 k_3}{1+i}$: **Total cost of buying <u>physical</u> capital for period 3 (time to build → must purchase period-3 capital in period 2)**
- $\frac{S_2 a_2}{1+i}$: **Total cost of buying <u>financial</u> assets (i.e., stock-holdings in *other* firms) for period 3**

Firm Profit Function

- **A *dynamic* profit maximization problem**
  - **Because firm exists for both periods**
  - **All analysis conducted from the perspective of the very beginning of period 1**
  - **→ Must consider present-discounted-value (PDV) of lifetime (i.e., two-period) profits**

- **Dynamic profit function**
  - **(specified in nominal terms – could specify in real terms…)**

As usual: no physical or financial assets needed for "period 3"

$$\overbrace{\underbrace{P_1 f(k_1,n_1)}_{\substack{\text{Total revenue}\\ \text{in period 1}\\ \text{(price x}\\ \text{output)}}} - \underbrace{P_1(k_2-k_1)}_{\substack{\text{Total}\\ \text{investment}\\ \text{costs (time}\\ \text{to build)}}} - \underbrace{P_1 w_1 n_1}_{\substack{\text{Total labor}\\ \text{cost in}\\ \text{period 1}}} - \underbrace{S_1(a_1-a_0)}_{\substack{\text{Total costs of}\\ \text{purchasing new shares}}} + \underbrace{D_1 a_0}_{\substack{\text{Total dividend}\\ \text{receipts}}}}^{\text{Period-1 profits}} + \overbrace{\underbrace{\frac{P_2 f(k_2,n_2)}{1+i}}_{\substack{\text{Total revenue}\\ \text{in period 2}\\ \text{(price x}\\ \text{output)}}} - \underbrace{\frac{P_2(\overset{=0}{k_3}-k_2)}{1+i}}_{\substack{\text{Total}\\ \text{investment}\\ \text{costs (time to}\\ \text{build)}}} - \underbrace{\frac{P_2 w_2 n_2}{1+i}}_{\substack{\text{Total labor}\\ \text{cost in}\\ \text{period 2}}} - \underbrace{\frac{S_2(\overset{=0}{a_2}-a_1)}{1+i}}_{\text{Total costs of purchasing new shares}} + \underbrace{\frac{D_2 a_1}{1+i}}_{\text{Total dividend receipts}}}^{\text{(PDV of) period-2 profits}}$$

INFORMATIONAL ASYMMETRIES

- "Informational asymmetries" pervasive in borrowing/lending relationships
- Borrower (whether consumer, firm, or financial institution) ***much*** more likely to know his own ability/willingness to repay a loan
  - Lenders only know little about the "quality" or "trustworthiness" of a borrower
  - Asymmetry of information – cannot be eliminated
- To mitigate consequences of informational asymmetries, lenders often require borrower to have a stake in "succeeding" in the project/purpose for which funds are being borrowed
  - Consumers
    - e.g., down payment on house purchase
    - e.g., down payment on car purchase
    - If stop making payments on house or car
      - Borrower loses down payment (in addition to the car or house…)…
      - Affects individual's incentives *before* borrowing

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    - Total amount of loan (typically) depends on individual's collateral
  - Firms
    - Capital investment (factories, technology upgrades, etc) outlays
    - Payroll outlays
    - Financing inventories
    - Total amount of loan (often) depends on firm's collateral

"Working capital"

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    - Payroll outlays
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    - Total amount of loan (often) depends on firm's collateral

  "Working capital"
  - Financial institutions: borrow in order to make (big) loans
    - By raising "small" quantities of funds from many different sources

FINANCING CONSTRAINT

- Capture this idea through a financing constraint on firm's ability to purchase capital between period 1 and period 2

- Financing constraint
  - Total expenditures on period-1 physical investment must be equal to market value of firm's financial (stock) holdings
  - (Technically, smaller than or equal to, so an inequality constraint…but will only analyze constraint with equality)

$$P_1 \cdot inv_1 = \quad S_1 \cdot a_1$$

$inv_1 = k_2 - k_1$ (investment is ***change in*** quantity of physical capital)

$$P_1 \cdot (k_2 - k_1) = \quad S_1 \cdot a_1$$

  - Important: a_1 appears in the financing constraint, ***not*** a_0
    - Idea this assumption captures: firm will purposefully change the value of *financial* assets it holds in order to affect the quantity of *physical* investment in which it can engage
    - (From the perspective of beginning of period 1, a_1 has not yet been chosen, whereas a_0 is pre-determined)

GOVERNMENT OVERSIGHT OF FINANCIAL MARKETS

- **Government oversight of informational asymmetries in borrower/lender relationships**
  - **Filing of proper documentation**
  - **Full disclosure ("truth-in-lending") laws**
  - **Direct lending in some markets**
  - **…**

- **Capture government Regulation of financial dealings in our framework in very simple way**
  - **Firm can borrow up to a multiple *R* of the market value of its financial assets for physical investment purposes**
  - **e.g., if government regulates that expenditures on investment cannot be larger than 5 times market value of financial assets, *R* = 5 is the leverage ratio**

- **Will think of *R* as government regulation…**
  - **…but can and does also reflect market and institutional arrangements**

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Government regulation *R*

$$\boxed{P_1 \cdot (k_2 - k_1) = R \cdot S_1 \cdot a_1}$$

Impose this financing constraint on firm profit maximization problem

FINANCIAL ACCELERATOR FRAMEWORK

- **Four Building Blocks of the Financial Accelerator Framework**

1. **Firm Profit Function**

$$P_1 f(k_1, n_1) + P_1 k_1 + (S_1 + D_1) a_0 - P_1 w_1 n_1 - P_1 k_2 - S_1 a_1 + \frac{P_2 f(k_2, n_2)}{1+i} + \frac{P_2 k_2}{1+i} + \frac{(S_2 + D_2) a_1}{1+i} - \frac{P_2 w_2 n_2}{1+i} - \overset{=0}{\frac{P_2 k_3}{1+i}} - \overset{=0}{\frac{S_2 a_2}{1+i}}$$

2. **Financing Constraint**

$$P_1 \cdot (k_2 - k_1) = S_1 \cdot a_1$$

3. **Government Regulation of Financial Relationships (imposition of *R* on financing constraint)**

$$P_1 \cdot (k_2 - k_1) = R \cdot S_1 \cdot a_1$$

4. **Relationship between firm profits and dividends**

LATER

FIRM PROFIT MAXIMIZATION

Maximize two-period profits

$$P_1 f(k_1,n_1)+P_1k_1+(S_1+D_1)a_0-P_1w_1n_1-P_1k_2-S_1a_1+\frac{P_2 f(k_2,n_2)}{1+i}+\frac{P_2k_2}{1+i}+\frac{(S_2+D_2)a_1}{1+i}-\frac{P_2w_2n_2}{1+i}-\overbrace{\frac{P_2k_3}{1+i}}^{=0}-\overbrace{\frac{S_2a_2}{1+i}}^{=0}$$

Subject to financing constraint

$$P_1\cdot(k_2-k_1)=R\cdot S_1\cdot a_1$$

Construct Lagrangian

$$P_1 f(k_1,n_1)+P_1k_1+(S_1+D_1)a_0-P_1w_1n_1-P_1k_2-S_1a_1+\frac{P_2 f(k_2,n_2)}{1+i}+\frac{P_2k_2}{1+i}+\frac{(S_2+D_2)a_1}{1+i}-\frac{P_2w_2n_2}{1+i}$$
$$+\lambda\left[R\cdot S_1\cdot a_1-P_1\cdot(k_2-k_1)\right]$$

Lagrange multiplier on financing constraint

CRUCIAL OBSERVATION: in basic firm theory, value of this multiplier is….

$\lambda = 0$ i.e., there was no financing constraint!

SOON: will think about what regulatory and/or market features make the financing constraint effectively "disappear" (i.e., cause $\lambda = 0$)

FIRM PROFIT MAXIMIZATION

$$P_1 f(k_1,n_1) + P_1 k_1 + (S_1 + D_1)a_0 - P_1 w_1 n_1 - P_1 k_2 - S_1 a_1 + \frac{P_2 f(k_2,n_2)}{1+i} + \frac{P_2 k_2}{1+i} + \frac{(S_2 + D_2)a_1}{1+i} - \frac{P_2 w_2 n_2}{1+i}$$
$$+ \lambda\left[R \cdot S_1 \cdot a_1 - P_1 \cdot (k_2 - k_1)\right]$$

- **FOCs with respect to *n_1*, *n_2***

Identical except for time subscripts

with respect to n_1: $$\cancel{P_1} f_n(k_1,n_1) - \cancel{P_1} w_1 = 0$$ **Equation 1**

with respect to n_2: $$\frac{\cancel{P_2} f_n(k_2,n_2)}{\cancel{1+i}} - \frac{\cancel{P_2} w_2}{\cancel{1+i}} = 0$$ **Equation 2**

  - **Financing constraint does not affect profit-maximizing choices of labor hiring…**
  - **…thus same analysis from basic theory of labor demand curve, etc, applies**
  - **GIVEN the PARTICULAR components of spending that financing constraints affect!**

- **FOCs with respect to *k_2*, *a_1***
  - **The interesting aspects of this framework**
  - **The heart of the accelerator mechanism**

FIRM PROFIT MAXIMIZATION

$$P_1 f(k_1,n_1) + P_1 k_1 + (S_1 + D_1)a_0 - P_1 w_1 n_1 - P_1 k_2 - S_1 a_1 + \frac{P_2 f(k_2,n_2)}{1+i} + \frac{P_2 k_2}{1+i} + \frac{(S_2 + D_2)a_1}{1+i} - \frac{P_2 w_2 n_2}{1+i}$$
$$+ \lambda \left[R \cdot S_1 \cdot a_1 - P_1 \cdot (k_2 - k_1) \right]$$

- **FOCs with respect to *k_2*, *a_1***

with respect to *k_2*: $$-P_1 + \frac{P_2 f_k(k_2,n_2)}{1+i} + \frac{P_2}{1+i} - \lambda P_1 = 0$$ **Equation 3**

with respect to *a_1*: $$-S_1 + \frac{S_2 + D_2}{1+i} + \lambda \cdot R \cdot S_1 = 0$$ **Equation 4**

- **Analysis of Equation 4 in isolation**
  - **Answers the central question: under what conditions does *λ* = 0?**
  - **Reveals how stock market returns affect financing constraints**
  - **Reveals how government regulation affects financing constraints**
- **Analysis of Equation 3 and Equation 4 jointly**
  - **Demonstrates how/why financial market prices (i.e., stock prices/returns) matter for macroeconomic activity**
  - **The financial accelerator effect**

WHY IS FINANCING A *CONSTRAINT*?

$$-S_1 + \frac{S_2 + D_2}{1+i} + \lambda \cdot R \cdot S_1 = 0$$

Equation 4

Solve for λ

$$\lambda = \left[S_1 - \frac{S_2 + D_2}{1+i} \right] \cdot \frac{1}{R \cdot S_1}$$

Pull $1/S_1$ term inside

$$\lambda = \left[1 - \frac{S_2 + D_2}{S_1} \cdot \frac{1}{1+i} \right] \cdot \frac{1}{R}$$

Multiply *and* divide second term in parentheses by P_1 *and* P_2

$$\lambda = \left[1 - \frac{S_2 + D_2}{S_1} \cdot \frac{P_1}{P_2} \cdot \frac{P_2}{P_1} \cdot \frac{1}{1+i} \right] \cdot \frac{1}{R}$$

Use definition of inflation, $1 + \pi_2 = P_2 / P_1$, and regroup terms

$$\lambda = \left[1 - \frac{S_2 + D_2}{S_1} \cdot \frac{P_1}{P_2} \cdot \frac{1+\pi_2}{1+i} \right] \cdot \frac{1}{R}$$

WHY IS FINANCING A *CONSTRAINT*?

$$\lambda = \left[1 - \frac{S_2 + D_2}{S_1} \cdot \frac{P_1}{P_2} \cdot \frac{1+\pi_2}{1+i}\right] \cdot \frac{1}{R}$$

(from previous page)

Use definition of "nominal interest rate on stock", $1 + i^{STOCK} = (S_2 + D_2)/\ S_1$

Use definition of inflation, $1 + \pi_2 = P_2\ /\ P_1$

$$\lambda = \left[1 - \frac{1+i^{STOCK}}{1+\pi_2} \cdot \frac{1+\pi_2}{1+i}\right] \cdot \frac{1}{R}$$

Fisher equation for stock: $1 + r^{STOCK} = (1 + i^{STOCK})/\ (1 + \pi_2)$

Fisher equation for bonds: $1 + r = (1 + i)\ /\ (1 + \pi_2)$

$$\lambda = \left[1 - \frac{1+r^{STOCK}}{1+r}\right] \cdot \frac{1}{R}$$

Final rewrite!

$$\lambda = \left[\frac{r - r^{STOCK}}{1+r}\right] \cdot \frac{1}{R}$$

The Lagrange multiplier on firm's financing constraint

WHY IS FINANCING A *CONSTRAINT*?

$$\lambda = \left[\frac{r - r^{STOCK}}{1+r} \right] \cdot \frac{1}{R}$$

The Lagrange multiplier on firm's financing constraint

- **Basic firm theory:**
  - **No financing constraint**
  - **Can interpret basic firm theory analysis as featuring *λ* = 0**
    - **Interpretation: under "normal market conditions," financing constraints don't matter (much…)**
    - **Interpret "normal market conditions" as steady state**

$$P_1 f(k_1, n_1) + P_1 k_1 + (S_1 + D_1)a_0 - P_1 w_1 n_1 - P_1 k_2 - S_1 a_1 + \frac{P_2 f(k_2, n_2)}{1+i} + \frac{P_2 k_2}{1+i} + \frac{(S_2 + D_2)a_1}{1+i} - \frac{P_2 w_2 n_2}{1+i}$$

$$+ \underbrace{\lambda}_{= 0} \left[R \cdot S_1 \cdot a_1 - P_1 \cdot (k_2 - k_1) \right]$$

- **If *λ* = 0 (i.e., "normal market conditions," aka steady state)**
  - **Labor demand decisions unaffected by financial market conditions**
  - **Capital demand decisions unaffected by financial market conditions**

- **Key question: what causes *λ* = 0?**

WHY IS FINANCING A *CONSTRAINT*?

$$\lambda = \left[\frac{r - r^{STOCK}}{1+r}\right] \cdot \frac{1}{R}$$

The Lagrange multiplier on firm's financing constraint

- **Two conditions for λ = 0**
  - **Market returns on risky assets equal returns on riskless assets**
    - **Risky assets: stocks**
    - **Riskless assets**
      - **Bonds (financial)**
      - **Machines and equipment (physical) – most directly relevant for firms' production and sales activity**
        - **Basic firm theory prediction: *r = mpk***

Can think of both government bonds (financial assets) and machines & equipment (physical assets) as "riskless": you (pretty much…) know what you're going to get from them.

$$r = r^{STOCK} \longrightarrow \lambda = 0$$

Interpretation: if returns on financial assets are aligned with returns on physical assets, financing constraints "don't matter"

  - **Government oversight of borrowing/lending relationships very lax**
    - **The larger is *R*, the lower is λ**
    - **Financing constraint:** $P_1 \cdot (k_2 - k_1) = R \cdot S_1 \cdot a_1$
      - **Holding constant market value of financial assets, higher *R* allows higher k_2**

Market value of financial assets

In practice, not literally infinity…

$$R = \infty \longrightarrow \lambda = 0$$

Interpretation: if government regulations allow high borrowing with little assets, financing constraints "don't matter"

FINANCING CONSTRAINT AND CAPITAL DEMAND

- **Suppose R = 1 in "steady state" (but keep R in rest of analysis)**
  - **$R > 1$ is "lax regulation" (because it lowers λ, all else constant)**
  - **$R < 1$ is "tight regulation" (because it increases λ, all else constant)**
  - **→ Whether or not financing constraint matters (i.e., whether or not $\lambda = 0$) all depends on whether or not $r^{STOCK} = r$ or not**

KEY IDEA: if returns on riskless assets = returns on risky assets → financing constraints "don't matter" for firm production decisions

$$-P_1 + \frac{P_2 f_k(k_2, n_2)}{1+i} + \frac{P_2}{1+i} - \lambda P_1 = 0$$

Equation 3 (FOC on k_2)

$$\lambda = \left[\frac{r - r^{STOCK}}{1+r}\right] \cdot \frac{1}{R}$$

Equation 4

(FOC on a_1)

- **Basic firm theory:**
  - **Capital demand function derived from Equation 3**
  - **Idea same as in basic theory…but now complicated by the financing constraint**

Substitute λ from Equation 4 into Equation 3

$$-P_1 + \frac{P_2 f_k(k_2, n_2)}{1+i} + \frac{P_2}{1+i} - \left[\frac{r - r^{STOCK}}{1+r}\right]\frac{1}{R} P_1 = 0$$

Rearrange

FINANCING CONSTRAINT AND CAPITAL DEMAND

$$-P_1 + \frac{P_2 f_k(k_2,n_2)}{1+i} + \frac{P_2}{1+i} - \left[\frac{r - r^{STOCK}}{1+r}\right]\frac{1}{R}P_1 = 0$$

(from previous page)

Divide by P_1

$$\frac{P_2 f_k(k_2,n_2)}{P_1(1+i)} + \frac{P_2}{P_1(1+i)} - \left[\frac{r - r^{STOCK}}{1+r}\right]\frac{1}{R} = 1$$

Use definition of inflation, $1 + \pi_2 = P_2 / P_1$

$$\left(\frac{1+\pi_2}{1+i}\right) f_k(k_2,n_2) + \frac{1+\pi_2}{1+i} - \left[\frac{r - r^{STOCK}}{1+r}\right]\frac{1}{R} = 1$$

Apply Fisher relation for “riskless” assets

$$\frac{f_k(k_2,n_2)}{1+r} + \frac{1}{1+r} - \left[\frac{r - r^{STOCK}}{1+r}\right]\frac{1}{R} = 1$$

Multiply by (1+r)

Marginal product of capital, *mpk*

$$\underline{f_k(k_2,n_2)} + \cancel{1} - \frac{r - r^{STOCK}}{R} = \cancel{1} + r$$

Suppose R = 1 in “steady state” but keep R in the analysis

COBB-DOUGLAS PRODUCTION FUNCTION

- Commonly-used functional form in quantitative macroeconomic analysis

$$f(k,n) = k^{\alpha} n^{1-\alpha}$$

- Describes the empirical relationship between aggregate GDP, aggregate capital, and aggregate labor quite well
- $\alpha \in (0,1)$ measures capital's share of output
  - Hence $(1-\alpha) \in (0,1)$ measures labor's share of output
  - Interpretation
    - The relative importance of (either) capital (or labor) in the production process
  - Estimates for U.S. economy: $\alpha \approx 0.3$
  - Estimates for Chinese economy: $\alpha \approx 0.15$ (not (yet) a very capital-rich economy)
- Cobb-Douglas form useful for illustrating factor demands
  - $mpn = f_n(k,n) = (1-\alpha)k^{\alpha} n^{-\alpha}$
  - $mpk = f_k(k,n) = \alpha k^{\alpha-1} n^{1-\alpha}$

Financing Constraint and Capital Demand

- **Firm-level demand for capital defined by the relation**

$$r = \alpha k^{\alpha-1} n^{1-\alpha} - \left[\frac{r - r^{STOCK}}{R}\right] \left(= mpk - \left[\frac{r - r^{STOCK}}{R}\right]\right)$$

$$r = \alpha k^{\alpha-1} n^{1-\alpha} - \frac{r}{R} + \frac{r^{STOCK}}{R}$$

$$\left[1 + \frac{1}{R}\right] r = \alpha k^{\alpha-1} n^{1-\alpha} + \frac{r^{STOCK}}{R}$$

$$\left[\frac{R+1}{R}\right] r = \alpha k^{\alpha-1} n^{1-\alpha} + \frac{r^{STOCK}}{R}$$

$$r = \left(\frac{R}{R+1}\right) \alpha k^{\alpha-1} n^{1-\alpha} + \frac{r^{STOCK}}{R+1}$$

Solve for *r* (return on "riskless" physical assets)

FINANCING CONSTRAINT AND CAPITAL DEMAND

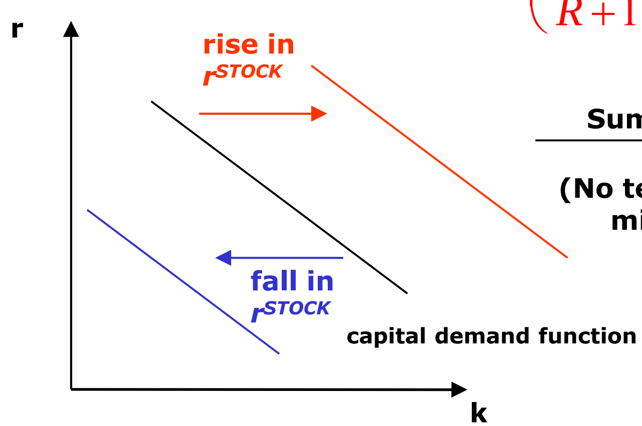
- **Firm-level demand for capital defined by the relation**

$$r = \left(\frac{R}{R+1}\right)\alpha k^{\alpha-1} n^{1-\alpha} + \frac{r^{STOCK}}{R+1}$$

Because exponent (α – 1) is a negative number, can move to denominator

$$r = \left(\frac{R}{R+1}\right)\alpha \left(\frac{n_t}{k_t}\right)^{1-\alpha} + \frac{r^{STOCK}}{R+1}$$

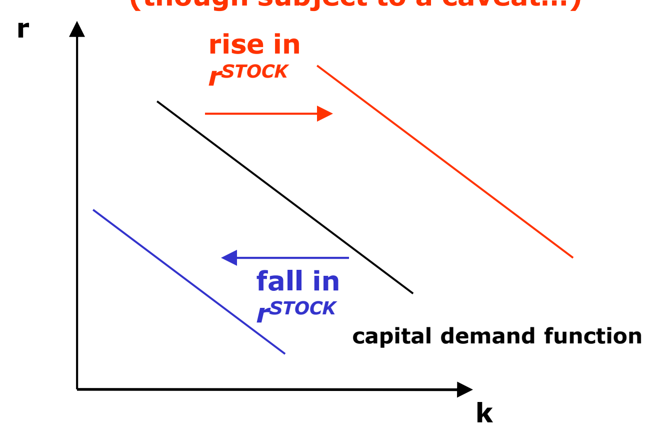
Rise (fall) in return on stock leads to shift out (in) of capital demand function (though subject to a caveat…)



Firm-level capital demand function

Sum over all firms

(No tension between the micro and macro)



Aggregate-level capital demand function

- **<u>IMPORTANT</u>: changes in financial market returns shift capital demand (and hence investment demand – recall $inv_t = k_{t+1} - k_t$)**

FINANCING CONSTRAINT AND CAPITAL DEMAND

- **Firm-level demand for capital defined by the relation**

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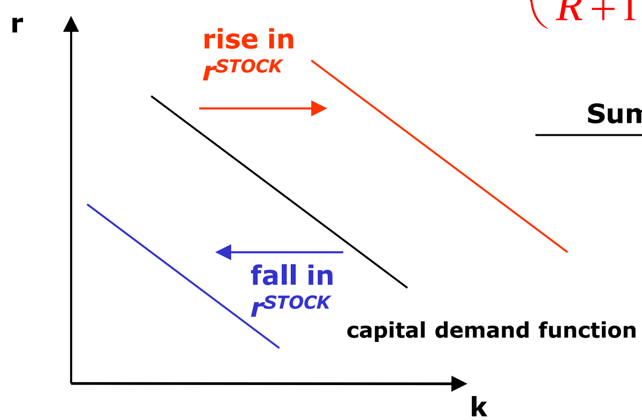
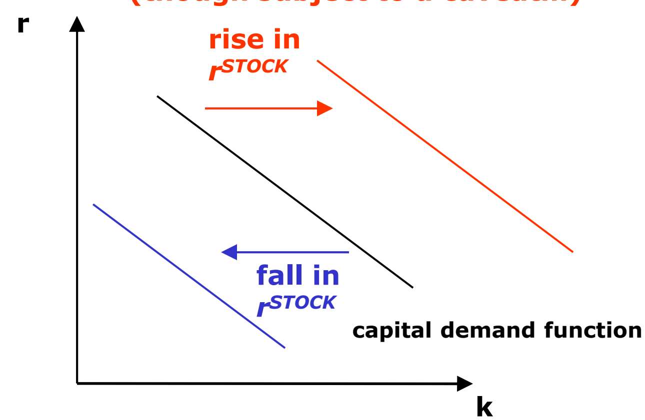


Firm-level capital demand function

Sum over all firms



Aggregate-level capital demand function

- **Basis for the financial accelerator effect**
- **Basis for understanding the role of financial oversight**

FINANCIAL ACCELERATOR FRAMEWORK

- **Four Building Blocks of the Financial Accelerator Framework**

1. Firm Profit Function

$$P_1 f(k_1, n_1) + P_1 k_1 + (S_1 + D_1)a_0 - P_1 w_1 n_1 - P_1 k_2 - S_1 a_1 + \frac{P_2 f(k_2, n_2)}{1+i} + \frac{P_2 k_2}{1+i} + \frac{(S_2 + D_2)a_1}{1+i} - \frac{P_2 w_2 n_2}{1+i} - \cancelto{0}{\frac{P_2 k_3}{1+i}} - \cancelto{0}{\frac{S_2 a_2}{1+i}}$$

2. Financing Constraint

$$P_1 \cdot (k_2 - k_1) = S_1 \cdot a_1$$

3. Government Regulation of Financial Relationships (imposition of *R* on financing constraint)

$$P_1 \cdot (k_2 - k_1) = R \cdot S_1 \cdot a_1$$

4. Relationship between firm profits and dividends

NOW

DIVIDENDS AND PROFITS

- **Dividend: payment made by a corporation to its shareholders; the portion of corporate profits paid out to stockholders**

- **Corporate dividend policies differ widely across industries and companies**
  - **Some companies retain most of their profits (to re-invest in ongoing projects)**
  - **Some industries' dividend policies subject to government regulation**

- **Recent average: ≈ 35 percent of profits disbursed as dividends**
  - **Based on recent data collected by U.S. Bureau of Economic Analysis for corporations listed on S&P 500**

- **Simplifying assumption for our analytical framework**
  - **All (100 percent) firm profits distributed as dividends**
  - **In period t, D_t = nominal profits$_t$**

- **Building Block 4: Relationship between firm profits and dividends**

$$D_t = P_t \cdot \underline{profit_t}$$

REAL profits of firm in period t

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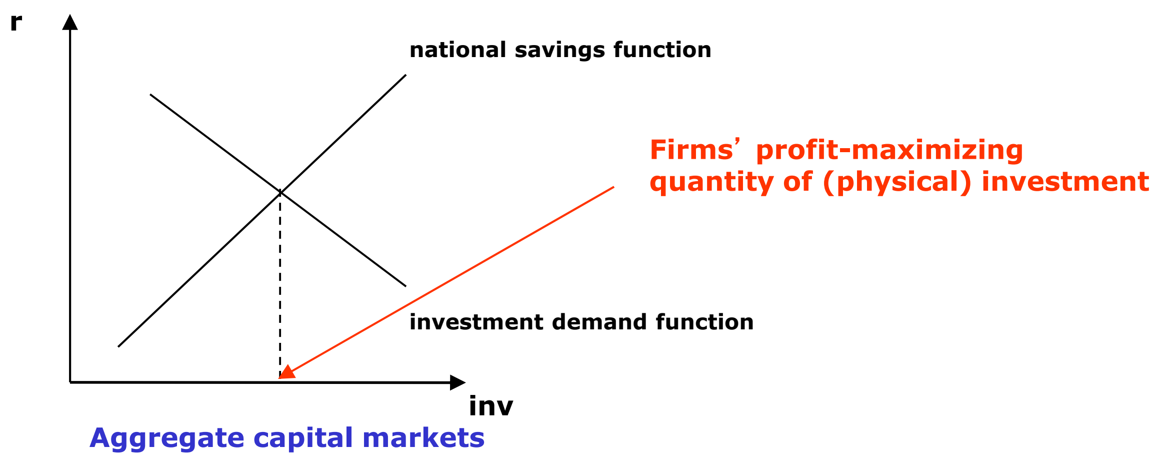
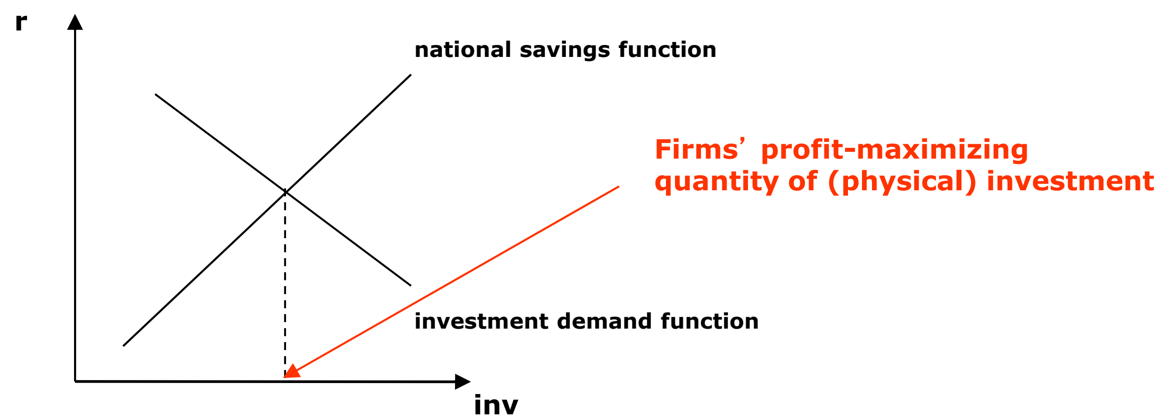


FINANCIAL ACCELERATOR IN ACTION

- Technically,
  - Riskless return $1+r=\frac{1+i}{1+\pi}=\frac{1}{P_1^b}\cdot\frac{1}{1+\pi}$ and risky return $1+r^{STOCK}=\frac{1+i^{STOCK}}{1+\pi}=\frac{S_2+D_2}{S_1}\cdot\frac{1}{1+\pi}$
  - Abuse of notation…



Aggregate capital markets

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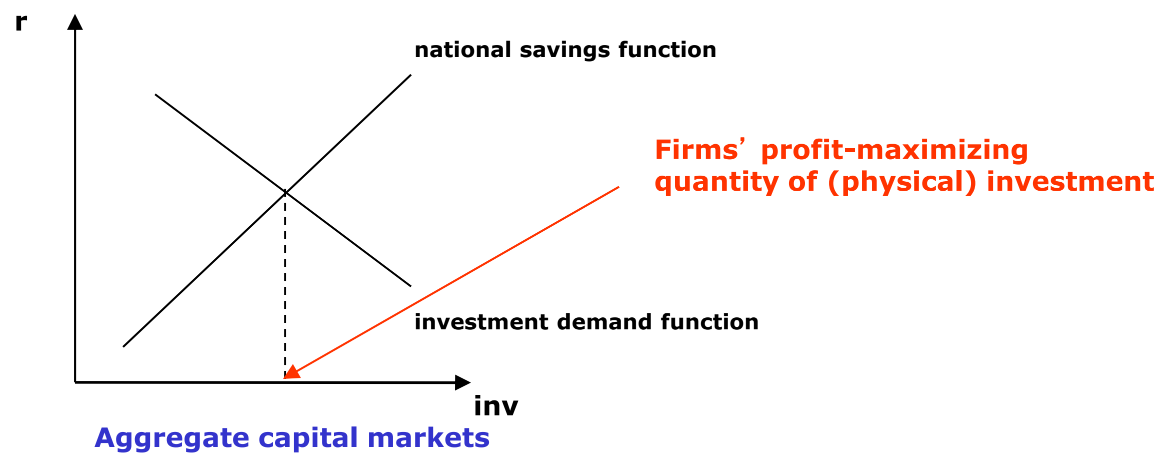

FINANCIAL ACCELERATOR IN ACTION

- **Suppose economy is in a "steady-state" in which $r = r^{STOCK}$…**
- **…then a shock causes r^{STOCK} to decline**
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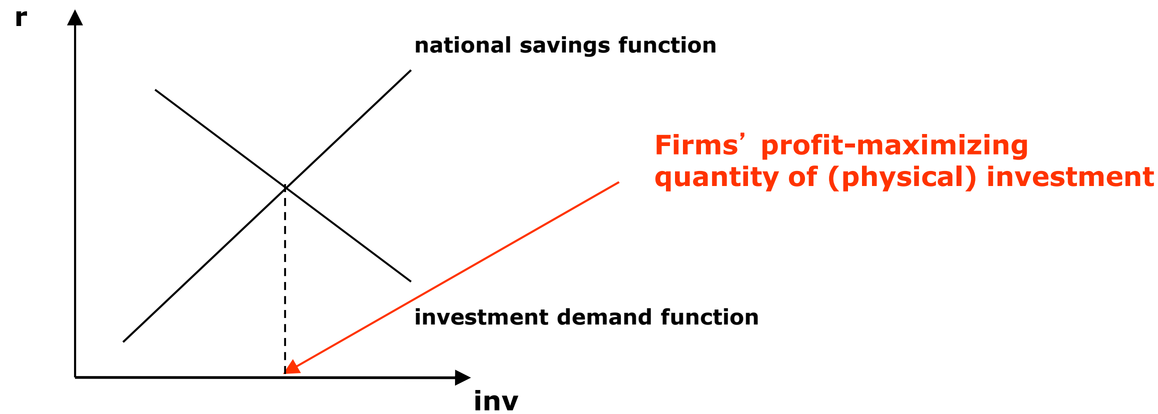
  - **…perhaps because of problems stemming from one or a few classes of financial assets (i.e., mortgage-backed bonds)**



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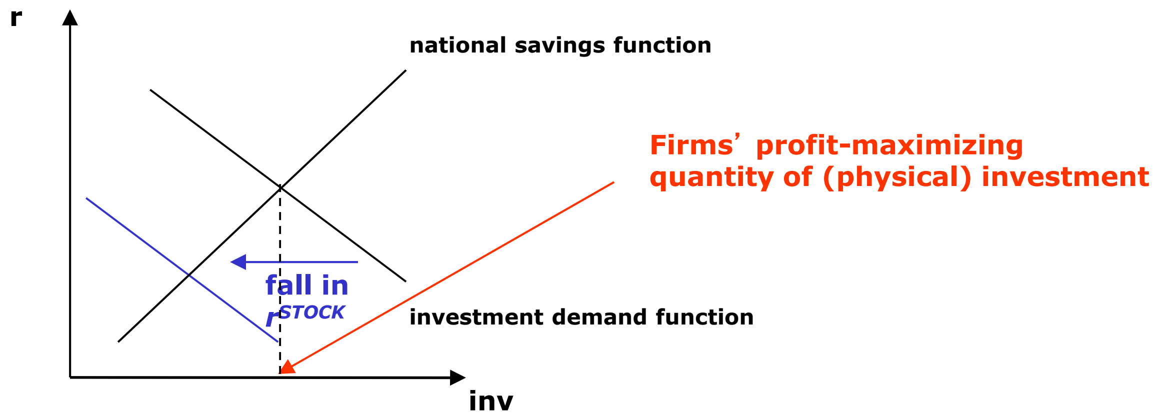

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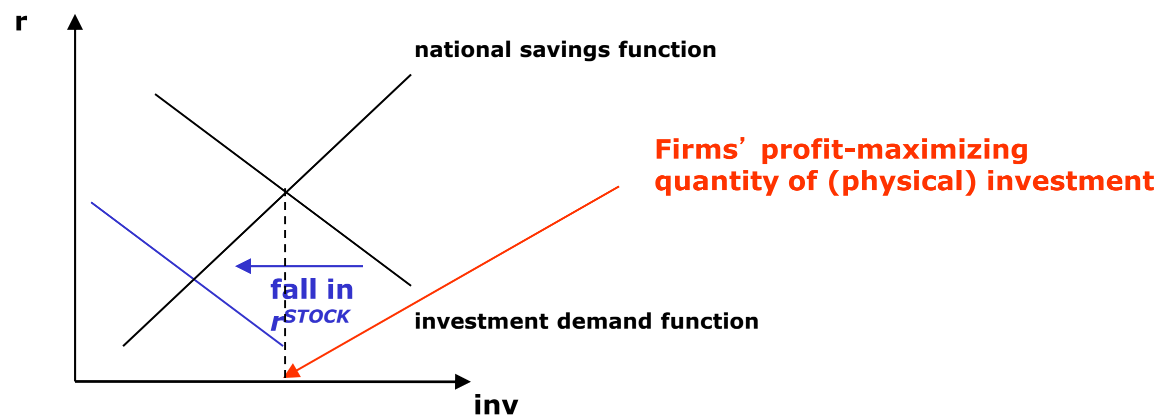

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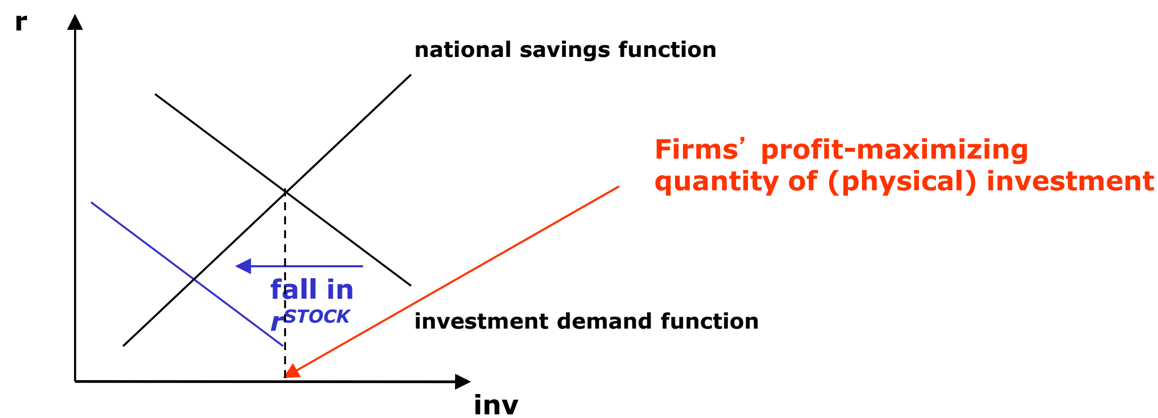

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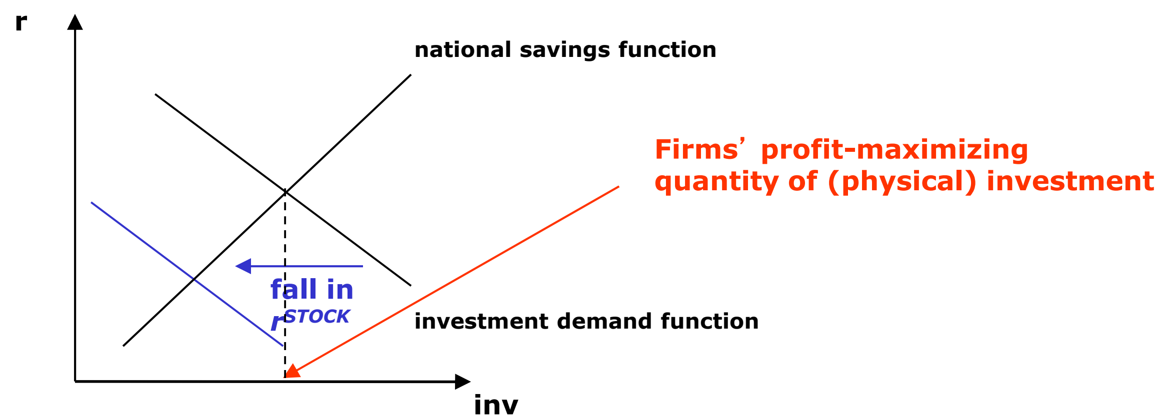

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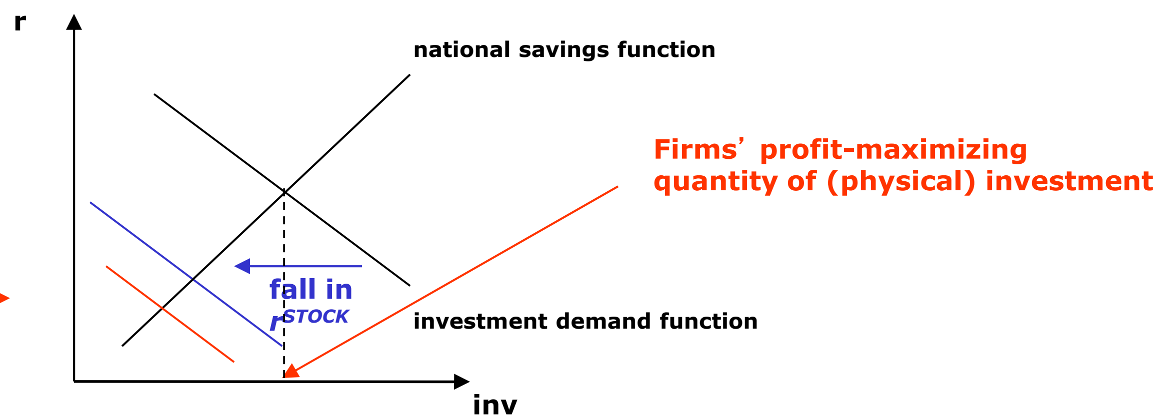

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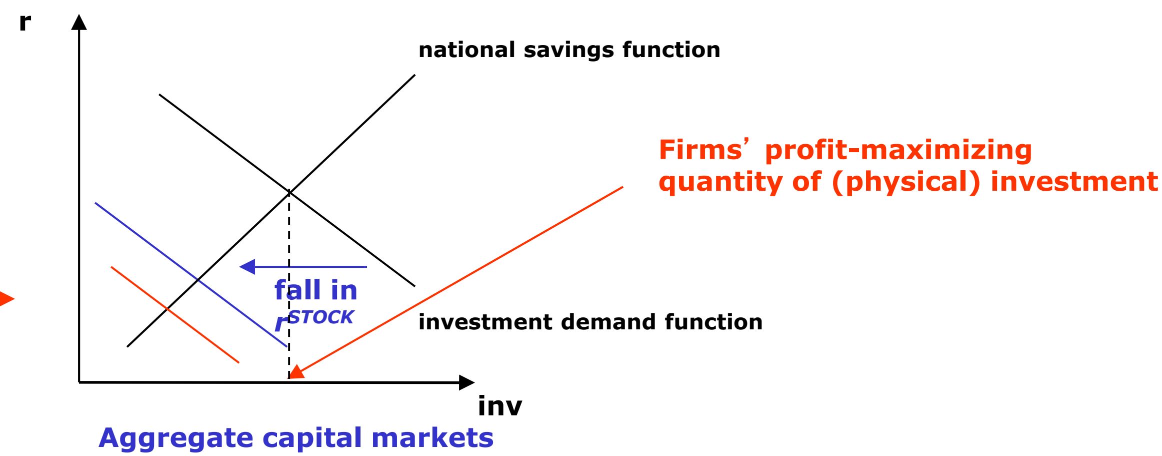

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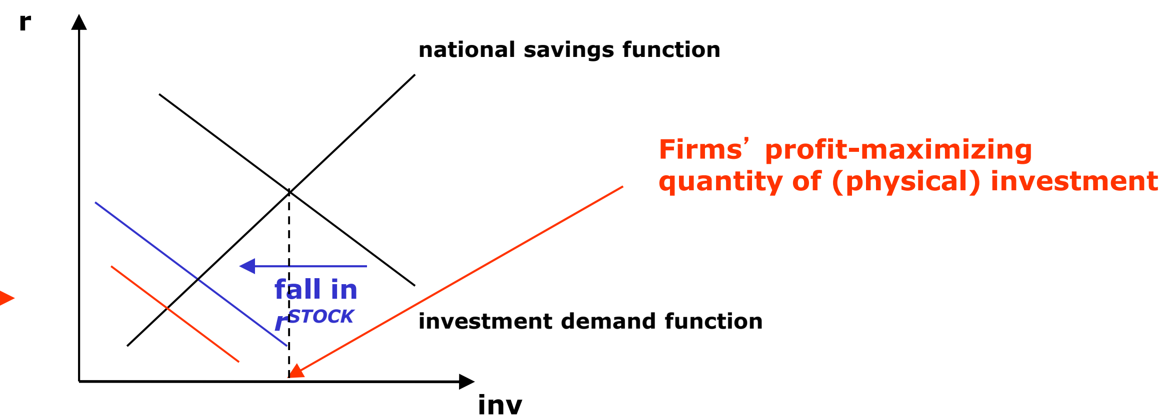

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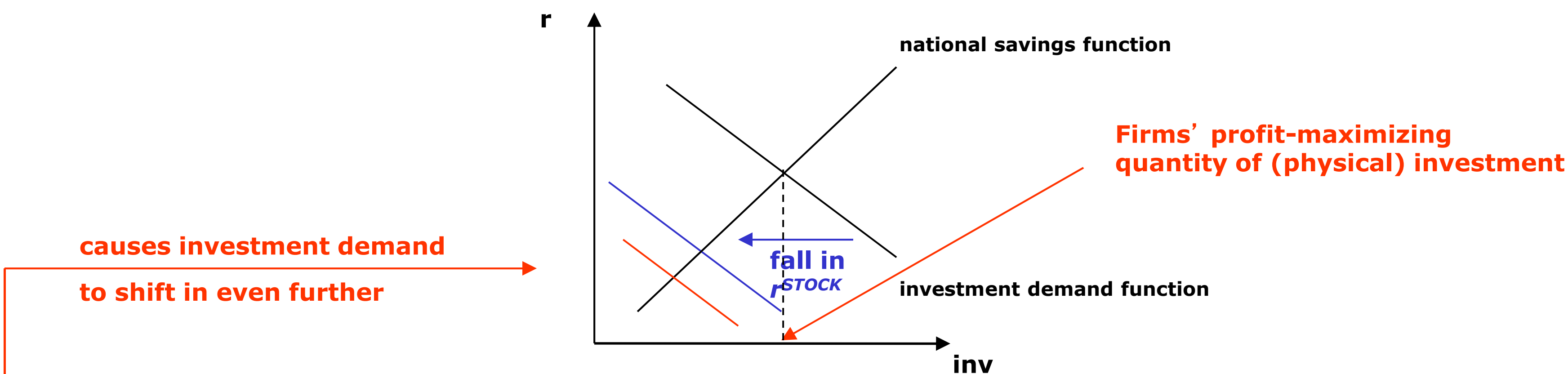

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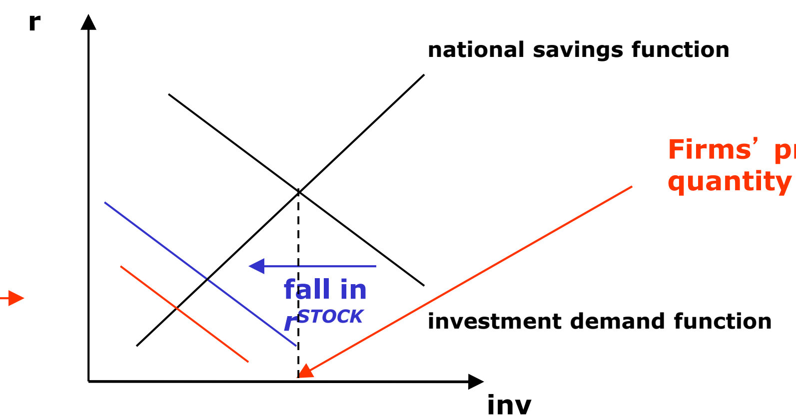

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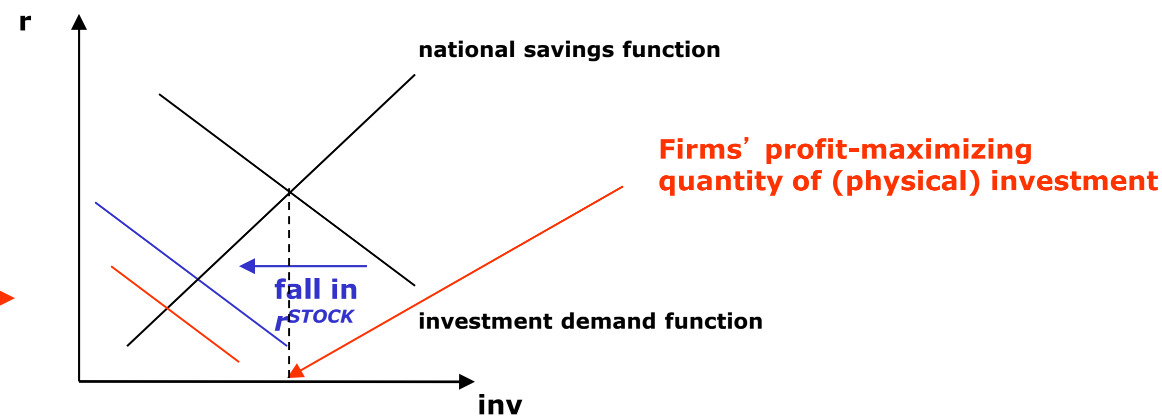

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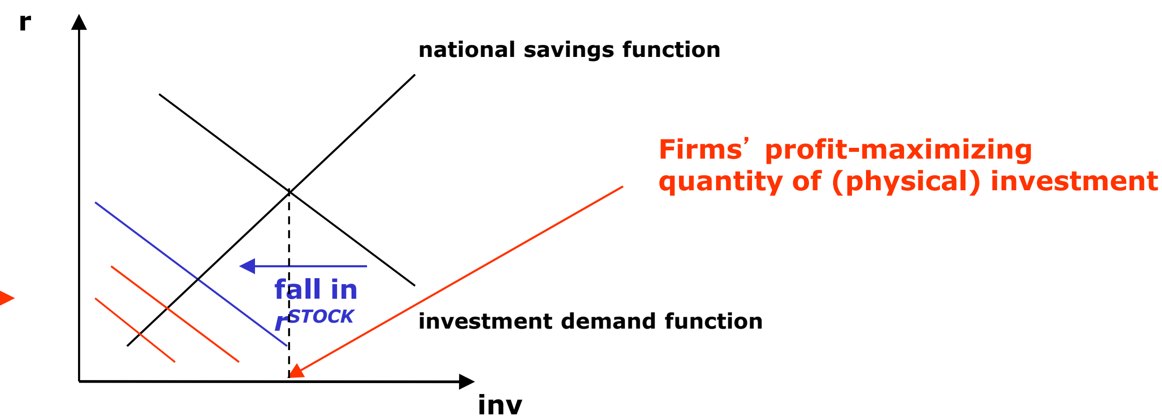

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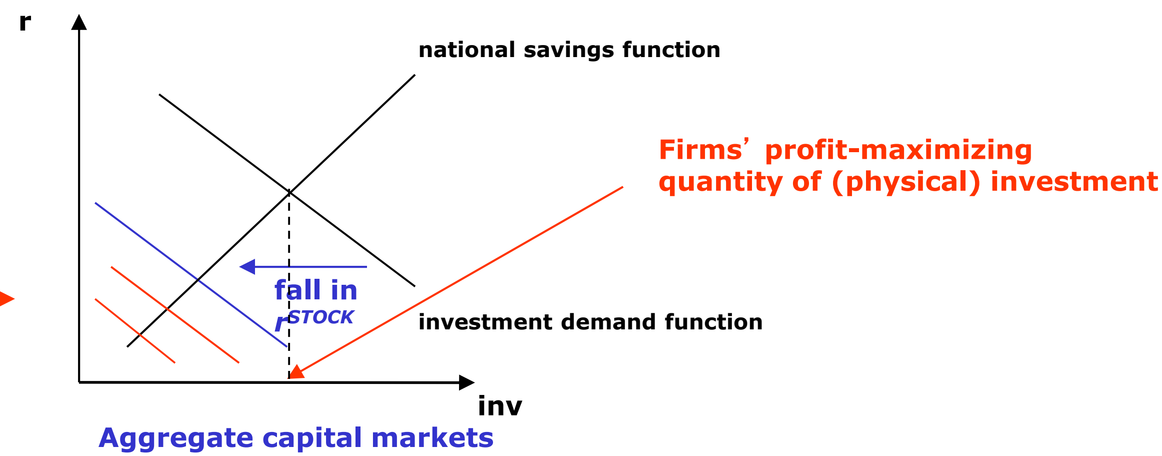

AND ON AND ON…

FINANCIAL ACCELERATOR

r^{STOCK} declines

Initial decline a shock (i.e., unpredictable)

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$$1 + r^{STOCK} = \left(\frac{S_2 + D_2}{S_1}\right) \cdot \frac{1}{1+\pi}$$

Terminology:

Financial Accelerator

Financial Feedback Loops

Loan Spirals

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If this link didn't exist or its severity could be mitigated, financial-macro spiral would NOT occur

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Recall basic reason for existence of this link: asymmetric information in financial dealings

Asymmetric information probably cannot be eliminated....

But can government regulation/interventions mitigate consequences?

POLICY AND REGULATORY RESPONSES

- **Entire accelerator mechanism due to financing constraint**

$$P_1 \cdot (k_2 - k_1) = R \cdot S_1 \cdot a_1$$

- **Lagrange multiplier related to asset returns and government regulation by**

$$\lambda = \left[\frac{r - r^{STOCK}}{1 + r}\right] \cdot \frac{1}{R}$$

- **If r^{STOCK} falls below r (which causes accelerator mechanism to begin)**
  - **λ increases**
  - **Optimal regulatory response: raise R, which would cause λ to decline!**
  - **If designed properly, a rise in R can perfectly offset the fall in r^{STOCK}, thus choking off the damaging effects of the accelerator**

- **Interpretation of rise in R**
  - **For a given market value of financial assets, S_1a_1, a higher R allows firms to borrow more from private lenders, in turn allowing them to purchase more (physical) capital**
  - **One interpretation: government "guarantees" private loans**
  - **Allows firms to produce more for the same level of financial resources**

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- Interpretation of rise in *R*
  - For a given market value of financial assets, $S_1 a_1$, a higher *R* allows firms to borrow more in order to purchase more (physical) capital
  - Allows firms to produce more for the same exact financial resources

- Changes in *R* can be time-consuming to implement
  - Simultaneously controlled by Federal Reserve, Treasury, Securities and Exchange Commission (SEC), Comptroller of the Currency, and several other regulatory agencies – huge coordination delays!

- Another "policy action" that has the same effect as raising *R*
  - Design policies to raise financial asset prices (i.e., S_1) directly!
  - Exactly the intention of U.S. Troubled Asset Relief Program (TARP)
    - Direct purchases by Treasury of a wide variety of financial assets
    - The increased demand for these assets would lift their price
  - Exactly the intention of Federal Reserve's programs to buy a wide variety of financial assets – increased demand would lift prices

Have these programs work as intended?

Yes and no?...

REAL INTEREST RATE

- ❑ ***r* a key variable for macroeconomic analysis**

REAL INTEREST RATE

- *r the key variable for macroeconomic analysis*

REAL INTEREST RATE

- ***r the key variable for macroeconomic analysis***

- ***r*** **measures the price of period-1 consumption in terms of period- 2 consumption**
- ***r*** **reflects degree of impatience**
- ***r*** **often reflects rate of consumption growth between periods**

- ***r*** **measures the price/return of physical assets (i.e., machines and equipment) of firms**
  - **"Riskless" assets**

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- **Now: *r also* measures price/return of risky assets (i.e., stock) in "steady state"**
  - **If $r = r^{STOCK}$, financing issues don't affect (very much) macroeconomic outcomes**
  - **If r and r^{STOCK} deviate significantly**
    - **Financial conditions of firms matter for investment/output**
    - **And can matter very importantly!**

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- **Can also think of *λ itself* as a type of real interest rate – an interest *SPREAD***
  - **The price of bringing funds from "outside sources" (i.e., lenders) "inside" the firm (i.e., the borrower) to finance operations**
    - **If $r = r^{STOCK}$, this price equals zero**
    - **Cost of "external funding sources" vs. "internal funding sources" due to info. asymmetry**

ECON 401

Advanced Macroeconomics

The Macroeconomics of Epidemics

Fabio Ghironi
University of Washington

Introduction

- The COVID-19 crisis has resulted in the rapid production of much research aimed at understanding the impact of the pandemics and of public health policy responses on the economy.
- A lively branch of this research has been focusing on the macroeconomic effects, combining tools and insights from macroeconomics and epidemiology.
- The tools we studied allow us to learn about this work.
- We focus on a recent NBER Working Paper on “The Macroeconomics of Epidemics” by Martin Eichenbaum (Northwestern University), Sergio Rebelo (Northwestern University), and Mathias Trabandt (Freie Universität Berlin).
  - I will refer to Eichenbaum, Rebelo, and Trabandt as ERT below.
- They integrate a standard epidemiology model into a similarly standard macroeconomic framework to study a number of scenarios.
- These slides are based on their paper.

The Pre-Infection Economy

Households

- ERT assume that the economy is populated by a measure-one continuum of ex-ante identical agents.

- Prior to the start of the epidemic, all agents are identical and maximize the objective function:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \left(\ln c_t - \frac{\varphi}{2} n_t^2 \right),$$

  where $\beta \in (0, 1)$ is the discount factor, c_t is consumption, n_t is hours worked, and $\varphi > 0$ is the weight attached to the disutility of labor effort.

The SIR-Macro Model, Continued

- The budget constraint of the representative agent is:

$$(1+\tau_t)\, c_t = w_t n_t + \Gamma_t,$$

where w_t is the real wage, Γ_t is a lump-sum transfer from the government, and τ_t is a tax on consumption.

- ERT think of this tax as a proxy for containment measures aimed at reducing social interactions in the post-infection economy.
  - For this reason, they also refer to τ_t as the containment rate.

- The first-order condition (FOC) for the representative agent's problem of choosing labor effort to maximize intertemporal utility is:

$$(1+\tau_t)\, \varphi n_t = c_t^{-1} w_{t.}$$

The SIR-Macro Model, Continued

Firms

- There is a unit-mass continuum of perfectly competitive identical firms that produce consumption output (Y_t) using hours worked (N_t) according to the technology:

$$Y_t = AN_t,$$

where $A > 0$.

- The representative firm chooses hours worked to maximize its time-t profits Π_t:

$$\Pi_t = AN_t - w_t N_t,$$

resulting in the (FOC):

$$w_t = A.$$

The SIR-Macro Model, Continued

The Government

- The government's budget constraint is given by:

$$\tau_t c_t = \Gamma_t.$$

Equilibrium

- In equilibrium, $n_t = N_t$ and $c_t = C_t = Y_t$, where C_t is aggregate consumption.

- These two conditions, the representative consumer's budget constraint, and the government budget constraint together imply:

$$c_t = C_t = Y_t = AN_t.$$

- The resource constraint of the economy coincides with the production function in this simple model.

The SIR-Macro Model, Continued

- Substituting the latter result into the optimal labor supply equation, recalling that optimal labor demand implies $w_t = A$, and that $n_t = N_t$ yields:

$$(1 + \tau_t)\,\varphi N_t = (AN_t)^{-1} A,$$

which can be solved to obtain:

$$N_t = \frac{1}{\sqrt{\varphi\,(1 + \tau_t)}}.$$

  - We can obviously discard the negative root.
  - The higher the consumption tax, the lower employment (and, therefore, output and consumption).

The SIR-Macro Model, Continued

The Outbreak of an Epidemic

- Epidemiology models generally assume that the probabilities governing the transition between different states of health are exogenous with respect to economic decisions.

- ERT modify the classic SIR (susceptible-infected-recovered) model proposed in 1927 by William Kermack and Anderson McKendrick of the Royal College of Physicians of Edinburgh so that these transition probabilities depend on people's economic decisions.

- Since purchasing consumption goods or working brings people into contact with each other, ERT assume that the probability of becoming infected depends on these activities.

The SIR-Macro Model, Continued

- The population is divided into four groups:

  a. susceptible (people who have not yet been exposed to the disease),

  b. infected (people who contracted the disease),

  c. recovered (people who survived the disease and acquired immunity),

  d. and deceased (people who died from the disease).

- The fractions of people in these four groups are denoted by S_t, I_t, R_t, and D_t, respectively.

The SIR-Macro Model, Continued

- The number of newly infected people in period t is denoted by T_t.
- Susceptible people can become infected in three ways.
- First, they can meet infected people while purchasing consumption goods.
- The number of newly infected people that results from these interactions is given by $\pi_1(S_t C_t^S)\left(I_t C_t^I\right)$.
- The terms $S_t C_t^S$ and $I_t C_t^I$ represent total consumption expenditures by susceptible and infected people, respectively.
- The parameter $\pi_1 > 0$ reflects both the amount of time spent shopping and the probability of becoming infected as a result of that activity.
  - In reality, different types of consumption involve different amounts of contact with other people.
  - For example, attending a rock concert is much more contact intensive than going to a grocery store.
  - For simplicity ERT abstract from this type of heterogeneity.

The SIR-Macro Model, Continued

- Second, susceptible and infected people can meet at work.
- The number of newly infected people that results from interactions at work is given by $\pi_2(S_t N_t^S)\left(I_t N_t^I\right)$.
- The terms $S_t N_t^S$ and $I_t N_t^I$ represent total hours worked by susceptible and infected people, respectively.
- The parameter $\pi_2 > 0$ reflects the probability of becoming infected as a result of work interactions.
  - Different jobs require different amounts of contact with people.
  - For example, working as a dentist or a waiter is much more contact intensive than writing software.
  - Again, for simplicity, ERT abstract from this source of heterogeneity.

The SIR-Macro Model, Continued

- Third, susceptible and infected people can meet in ways not directly related to consuming or working, for example meeting a neighbor or touching a contaminated surface.
- The number of random meetings between infected and susceptible people is $S_t I_t$.
- These meetings result in $\pi_3 S_t I_t$ newly infected people, where $\pi_3 > 0$.
- The total number of newly infected people in period t is thus given by:

$$T_t = \pi_1 (S_t C_t^S) \left(I_t C_t^I \right) + \pi_2 (S_t N_t^S) \left(I_t N_t^I \right) + \pi_3 S_t I_t. \quad (1)$$

- Kermack and McKendrick's (1927) SIR model is a special case of ERT's model in which the propagation of the disease is unrelated to economic activity ($\pi_1 = \pi_2 = 0$).

The SIR-Macro Model, Continued

- The number of susceptible people at time $t+1$ is equal to the number of susceptible people at time t minus the number of susceptible people that got infected at time t:

$$S_{t+1} = S_t - T_t. \tag{2}$$

- The number of infected people at time $t+1$ is equal to the number of infected people at time t plus the number of newly infected (T_t) minus the number of infected people that recovered ($\pi_R I_t$) and the number of infected people that died ($\pi_D I_t$):

$$I_{t+1} = I_t + T_t - (\pi_R + \pi_D) I_t. \tag{3}$$

  - Here, π_R is the rate at which infected people recover and π_D is the mortality rate, that is the probability that an infected person dies.

- The timing convention implicit in equation (3) is as follows.
  - Social interactions happen in the beginning of the period (infected and susceptible people meet).
  - Then, changes in health status unrelated to social interactions (recovery or death) occur.
  - At the end of the period, the consequences of social interactions materialize: T_t susceptible people become infected.

The SIR-Macro Model, Continued

- The number of recovered people at time $t+1$ is the number of recovered people at time t plus the number of infected people who just recovered ($\pi_R I_t$):

$$R_{t+1} = R_t + \pi_R I_t. \tag{4}$$

- Finally, the number of deceased people at time $t+1$ is the number of deceased people at time t plus the number of new deaths ($\pi_D I_t$):

$$D_{t+1} = D_t + \pi_D I_t. \tag{5}$$

- Total population (Pop) evolves according to:

$$Pop_{t+1} = Pop_t - \pi_D I_t.$$

The SIR-Macro Model, Continued

- ERT assume that at time $t = 0$, a fraction ε of susceptible people is infected by a virus through zoonotic exposure, that is the virus is directly transmitted from animals to humans.

- This implies:

$$I_0 = \varepsilon \quad \text{and} \quad S_0 = 1 - \varepsilon.$$

- Everybody is aware of the initial infection and understands the laws of motion governing population health dynamics.

- Critically, everybody takes aggregate variables like $I_t C_t^I$ and $I_t N_t^I$ as given.

The SIR-Macro Model, Continued

Individual Behavior in the Epidemic

- The variable U_t^j denotes the time-t lifetime utility of a type-j agent ($j = S, I, R$).
- The budget constraint of a type-j person is:

$$(1 + \tau_t)\, c_t^j = w_t \phi^j n_t^j + \Gamma_t. \tag{6}$$

- The parameter ϕ^j governs labor productivity of agents of type j:
  - It is equal to 1 for susceptible and recovered people ($\phi^S = \phi^R = 1$), but it and less than 1 for infected people ($\phi^I < 1$).

The SIR-Macro Model, Continued

- The budget constraint (6) embodies the assumption that there is no way for agents to pool risk associated with the infection.

- ERT model an economy in which the epidemics implies heterogeneity across agents in different groups.

- Going to the opposite extreme and assuming complete markets (asset markets that make it possible to insure all idiosyncratic risks across agents—an assumption that we made in all the models we studied so far—would be too unrealistic for the purposes of this paper.

The SIR-Macro Model, Continued

Susceptible People

- The lifetime utility of a susceptible person, U_t^S, can be written as:

$$U_t^S = \ln c_t^S - \frac{\varphi}{2}\left(n_t^S\right)^2 + \beta\left[\alpha_t U_{t+1}^I + (1-\alpha_t)\, U_{t+1}^S\right]. \tag{7}$$

- The variable α_t represents the probability that a susceptible person becomes infected during period t.

- It is given by:

$$\alpha_t = \pi_1 c_t^S\left(I_t C_t^I\right) + \pi_2 n_t^S\left(I_t N_t^I\right) + \pi_3 I_t. \tag{8}$$

  - You should recognize that this is an individual-agent version of equation (1).
  - Critically, susceptible people understand that consuming and working less reduces the probability of becoming infected.

The SIR-Macro Model, Continued

- The representative susceptible agent chooses consumption, labor effort, and the probability of becoming infected to maximize (7) subject to the constraints (6) and (8).
- FOCs for consumption and hours worked are, respectively:

$$\left(c_t^S\right)^{-1} - \lambda_{b,t}^S(1+\tau_t) + \lambda_{\alpha,t}^S \pi_1(I_t C_t^I) = 0,$$

$$\varphi n_t^S + \lambda_{b,t}^S w_t + \lambda_{\alpha,t}^S \pi_2(I_t N_t^I) = 0,$$

where $\lambda_{b,t}^S$ and $\lambda_{\alpha,t}^S$ are the Lagrange multipliers on the constraints (6) and (8), respectively.

- The FOC for the probability of becoming infected is:

$$\beta\left(U_{t+1}^I - U_{t+1}^S\right) + \lambda_{\alpha,t}^S = 0. \tag{9}$$

The SIR-Macro Model, Continued

Infected People

- The lifetime utility of an infected person, U_t^I, can be written as:

$$U_t^I = \ln c_t^I - \frac{\varphi}{2}\left(n_t^I\right)^2 + \beta\left[\pi_R U_{t+1}^R + (1-\pi_R-\pi_D)\, U_{t+1}^I\right]. \tag{10}$$

- This expression embodies a common assumption in macro and health economics that the cost of death is the foregone utility of life.

- The representative infected person chooses consumption and labor effort to maximize (10) subject to the constraint (6).

- The FOCs for consumption and hours worked are, respectively:

$$\left(c_t^I\right)^{-1} - \lambda_{b,t}^I(1+\tau_t) = 0,$$

$$\varphi n_t^I + \lambda_{b,t}^I \phi^I w_t = 0,$$

where $\lambda_{b,t}^I$ is the Lagrange multiplier associated with constraint (6).

The SIR-Macro Model, Continued

Recovered People

- The lifetime utility of a recovered person, U_t^R, can be written as:

$$U_t^R = \ln c_t^R - \frac{\varphi}{2}\left(n_t^R\right)^2 + \beta U_{t+1}^R. \tag{11}$$

- Note: This expression embodies the assumption that recovering confers immunity from the disease.
- The representative recovered agent chooses consumption and labor effort to maximize (11) subject to the constraint (6).
- The FOCs for consumption and hours worked are, respectively:

$$\left(c_t^R\right)^{-1} - \lambda_{b,t}^R(1+\tau_t) = 0,$$

$$\varphi n_t^R + \lambda_{b,t}^R w_t = 0,$$

where $\lambda_{b,t}^R$ is the Lagrange multiplier associated with constraint (6).

The SIR-Macro Model, Continued

Government Budget Constraint

- The government budget constraint is:

$$\tau_t \left(S_t c_t^S + I_t c_t^I + R_t c_t^R \right) = \Gamma_t \left(S_t + I_t + R_t \right).$$

Equilibrium

- In equilibrium, each person solves their maximization problem and the government budget constraint is satisfied.

- $c_t^S = C_t^S$, $c_t^I = C_t^I$, and $c_t^R = C_t^R$.

- And the goods and labor markets clear:

$$S_t C_t^S + I_t C_t^I + R_t C_t^R = A_t N_t$$

and

$$S_t N_t^S + I_t N_t^I + R_t N_t^R = N_t.$$

- ERT solve their model numerically for the dynamics from period 0 on.

Medical Preparedness, Treatments, and Vaccines

- In addition to their baseline scenario, ERT extend their SIR-macro model in three ways:

  a. They allow for the possibility that the mortality rate increases as the number of infections rises.

  b. They allow for the probabilistic development of a cure for the disease.

  c. They allow for the probabilistic development of a vaccine that inoculates susceptible people against the virus.

Medical Preparedness, Treatments, and Vaccines, Continued

The Medical Preparedness Model

- In their basic SIR-macro model, ERT abstracted from the possibility that the efficacy of the healthcare system will deteriorate if a substantial fraction of the population becomes infected.

- A simple way to model this scenario is to assume that the mortality rate depends on the number of infected people.

- ERT do this by assuming:

$$\pi_{D,t} = \pi_D + \kappa I_t^2.$$

- This functional form implies that the mortality rate is a convex function of the fraction of the population that becomes infected.

- The basic SIR-macro model corresponds to the special case $\kappa = 0$.

The Treatment Model

- The basic SIR-macro model abstracts from the possibility that an effective treatment against the virus will be developed.

- Suppose instead that an effective treatment that cures infected people is discovered with probability δ_C each period, where the subscript C stands for "cure."

- Once discovered, treatment is provided to all infected people in the period of discovery and all subsequent periods transforming them into recovered people.

- As a result, the number of new deaths from the disease goes to zero.

- The lifetime utility of an infected person before the treatment becomes available can now be written as:

$$U_t^I = \ln c_t^I - \frac{\varphi}{2}\left(n_t^I\right)^2 + \delta_C \beta U_{t+1}^R + (1-\delta_C)\,\beta\left[\pi_R U_{t+1}^R + (1-\pi_R-\pi_D)\,U_{t+1}^I\right]. \qquad (12)$$

- This expression reflects the fact that with probability δ_C an infected person receives treatment and becomes recovered, and with probability $1-\delta_C$ the infected person does not receive the treatment and may recover without treatment (with probability π_R), remain infected (with probability $1-\pi_R-\pi_D$), or die (with probability π_D.but $U_{t+1}^D = 0$).

Medical Preparedness, Treatments, and Vaccines, Continued

- How does an effective treatment impact population dynamics?
- Before the treatment is discovered, population dynamics evolve according to equations (1), (2), (3), (4), and (5).
- Suppose that the treatment is discovered at the beginning of time t^*.
- Then all infected people become recovered.
- The number of deceased stabilizes once the treatment arrives:

$$D_t = D_{t^*} \quad \text{for } t \geq t^*.$$

- Since the model assumes that anyone can be instantly cured, ERT normalize the number of susceptible and infected people to 0 for $t \geq t^*$.
- The number of recovered people is given by:

$$R_t = 1 - D_t.$$

The Vaccination Model

- ERT's basic SIR-macro model abstracts from the possibility that a vaccine against the virus will be developed.
- Suppose instead that a vaccine is discovered with probability δ_V each period, where the subscript V stands for "vaccine."
- Once discovered, the vaccine is provided to all susceptible people in the period of discovery and in all subsequent periods.
- The lifetime utility of a susceptible person in this scenario can be written as:

$$U_t^S = \ln c_t^S - \frac{\varphi}{2}\left(n_t^S\right)^2 + \delta_V \beta U_{t+1}^R + (1-\delta_V)\,\beta\left[\alpha_t U_{t+1}^I + (1-\alpha_t)\,U_{t+1}^S\right]. \tag{13}$$

- This expression reflects the fact that with probability δ_V a person is vaccinated and becomes immune to the virus (like a person who recovered), and with probability $1-\delta_V$ there is no vaccine, and the person remains susceptible to the virus in period $t+1$.
- The vaccine has no impact on people who were infected or have recovered.
  - The lifetime utilities of infected and recovered people person are given by (10) and (11), respectively.

Medical Preparedness, Treatments, and Vaccines, Continued

- How do vaccinations impact population dynamics?
- Before the vaccine is discovered, these dynamics evolve according to equations (1), (2), (3), (4), and (5).
- Suppose that the vaccine is discovered at the beginning of time $\tilde{t}$.
- Then, all susceptible people become recovered, and since no one is susceptible, there are no new infections.
- Denote the number of susceptible and recovered people right after a vaccine is introduced at time $\tilde{t}$ by $S'_{\tilde{t}}$ and $R'_{\tilde{t}}$.
- The values of these variables are:

$$S'_{\tilde{t}} = 0 \quad \text{and} \quad R'_{\tilde{t}} = R_{\tilde{t}} + S_{\tilde{t}}.$$

- For $t \geq \tilde{t}$, we have:

$$R_{t+1} = \begin{array}{cc} R'_t + \pi_R I_t & \text{for } t = \tilde{t} \\ R_t + \pi_R I_t & \text{for } t > \tilde{t}. \end{array}$$

- The laws of motion for I_t and D_t are given by (3) and (5).

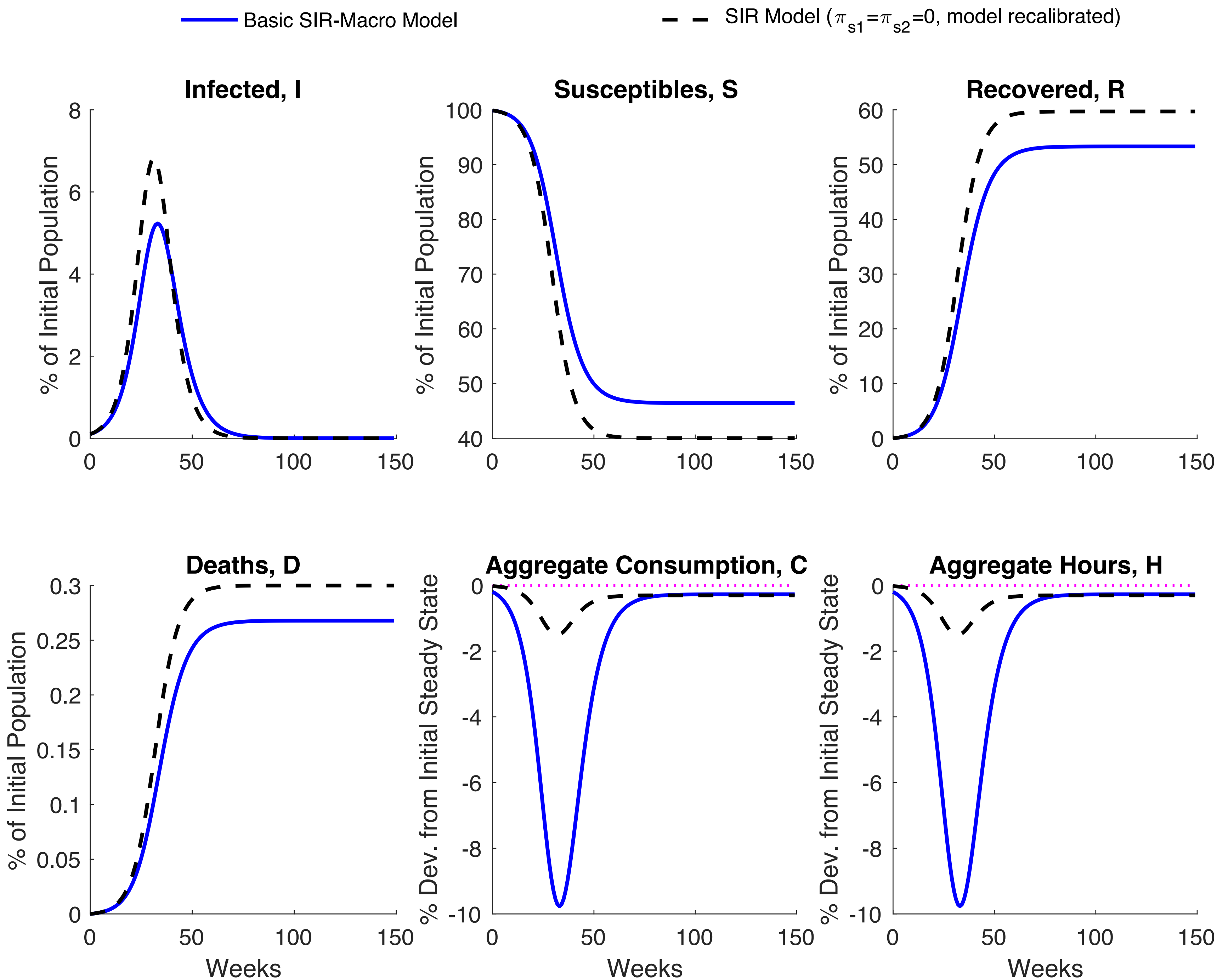
Next

- ERT solve their model numerically using an algorithm described in the Appendix of their paper after setting the values of parameters at conventional levels for the parameters of standard macroeconomic models or to match existing evidence on the coronavirus.
  - The calibration procedure is described in detail in the paper.
  - The model is calibrated so that periods represent weeks.
- The following slides present the results of model simulations in different scenarios.
  - The baseline SIR and SIR-Macro scenarios are computed under the assumption that no containment policy is in place ($\tau_t = 0$).

The SIR Model Results

- The black dashed lines in Figure 1 display the equilibrium population dynamics implied by the basic SIR model that abstracts from the effect of economic behavior on population dynamics (the special case of the SIR-Macro model in which $\pi_1 = \pi_2 = 0$).
- The share of the initial population that is infected peaks at 6.8 percent in week 31.
- Thereafter, this share falls because there are fewer susceptible people to infect.
- Eventually, 60 percent of the population becomes infected.
- Assuming a U.S. population of 330 million people, this scenario implies that roughly 200 million Americans eventually become infected.
- A mortality rate of 0.5 percent implies that the virus kills roughly one million people in the U.S.

Figure 1: Basic SIR-Macro Model vs. SIR Model



The SIR Model Results, Continued

- Even if there is no feedback from economic behavior to population dynamics in the basic SIR scenario, the epidemic and population dynamics do impact the economy.

- Figure 1 shows that the epidemic induces a recession: aggregate consumption falls by roughly 1.5 percent from peak to trough for two reasons:

  a. The virus causes infected people to be less productive at work ($\phi^I = 0.8$).

     · The associated negative income effect lowers consumption of the infected.

     · The dynamics of aggregate consumption mimic the share of infected agents in overall population.

  b. The death toll caused by the epidemic permanently reduces the size of the work force.

     · Since production is constant returns to scale, per capita income is the same in the post and pre-epidemic steady states.

     · In the post-epidemic steady state, population and real GDP are both 0.3 percent lower than in the initial steady state.

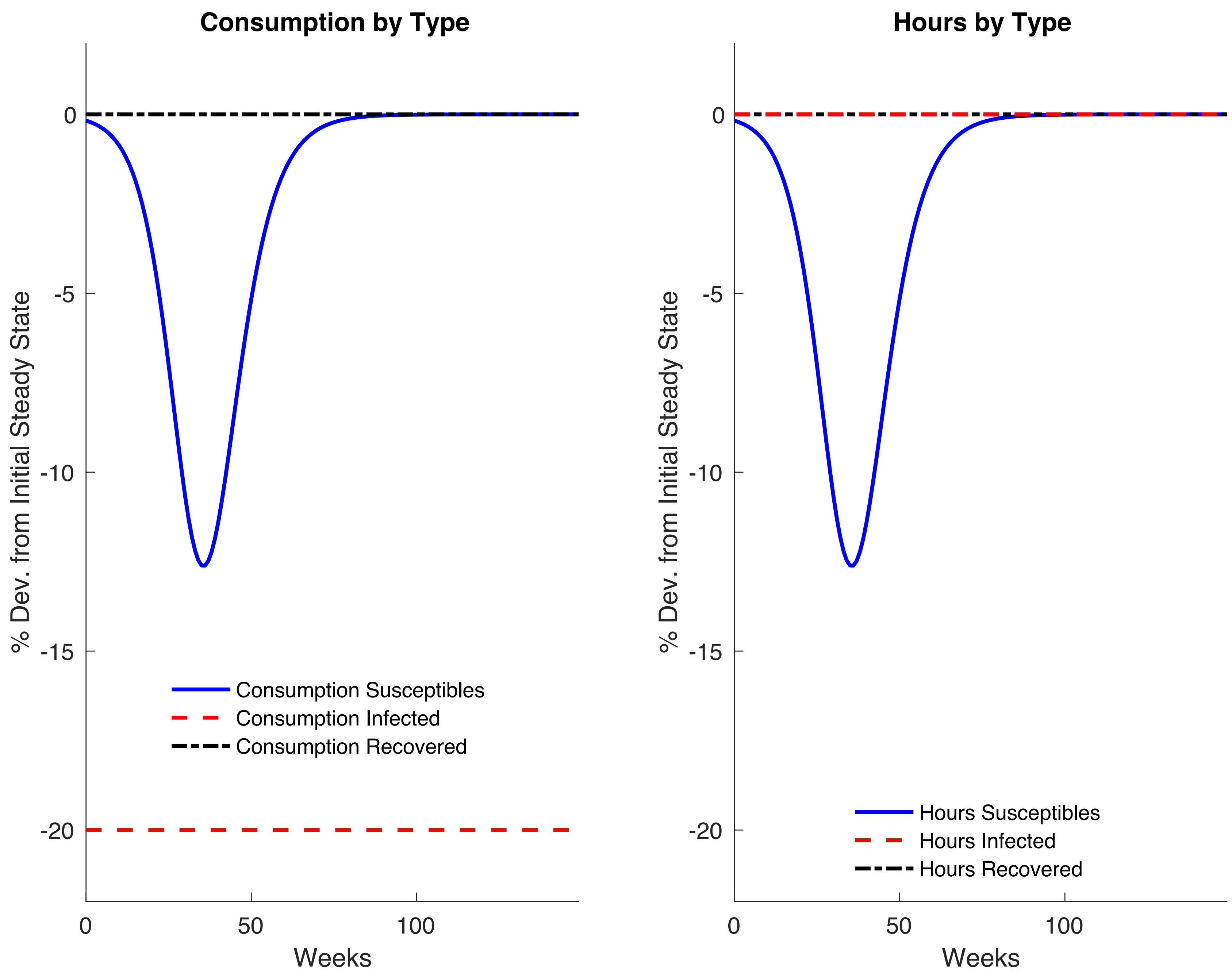
The SIR-Macro Model Results

- In the SIR model economic decisions about consumption and work do not influence the dynamics of the epidemic.
- In the SIR-Macro model, susceptible households can lower the probability of being infected by reducing their consumption and hours worked.
- The solid blue lines in Figure 1 show how the epidemic unfolds in this case.
- The share of the initial population that is infected peaks at 5.3 percent in week 33.
  - The peak is substantially smaller and occurs somewhat later than the corresponding peak in the SIR model.
- Eventually, 54 percent of the population becomes infected.
- So, for the U.S., roughly 180 million people eventually become infected and 890 thousand people die.

The SIR-Macro Model Results, Continued

- Figure 1 shows that the infection is less severe in the SIR-Macro model than in the SIR model.
- The reason is that in the SIR-Macro model susceptible people severely reduce their consumption and hours worked to lower the probability of being infected.
  - As Figure 2 shows, there are no offsetting effects arising from the behavior of recovered and infected people because they behave as in the SIR model.
- Consistent with these observations, the recession is much more severe in the SIR-Macro model:
  - Average aggregate consumption in the first year of the epidemic falls by 4.7 percent, a fall seven times larger than in the SIR model.

Figure 2: Consumption and Hours by Type in Basic SIR-Macro Model



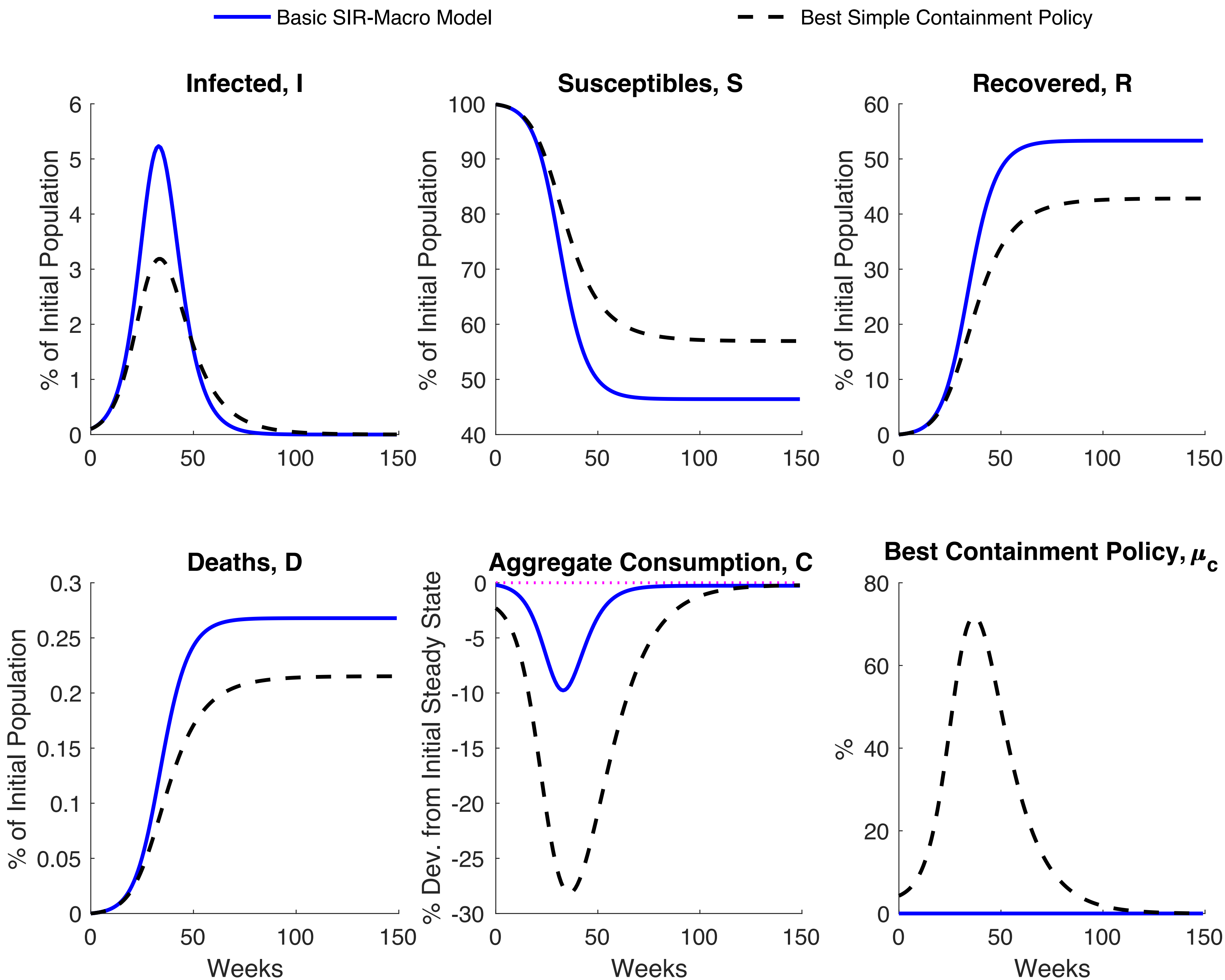
The SIR-Macro Model Results, Continued

- For similar reasons, the dynamics and magnitude of the drop in work hours is very different in the two models.
- In the SIR model, hours worked decline smoothly, falling by 0.3 percent in the post-epidemic steady state.
  - This decline entirely reflects the impact of the death toll on the workforce.
- In the SIR-macro model, hours worked follow a U-shaped pattern.
  - The peak decline of 9.8 percent occurs in week 33.
  - Thereafter, aggregate hours rise converging to a new steady state from below.
  - These dynamics are driven by the labor-supply decisions of susceptible agents.

The SIR-Macro Model Results, Continued

- The long-run decline in hours worked is lower in the SIR-Macro model (0.27 percent) than in the SIR model (0.30 percent).
- The reason is that fewer people die in the epidemic so the population falls by less in the SIR-Macro model than in the SIR model.
- Figure 3 compares the dynamics of the SIR-Macro model without policy intervention ($\tau_t = 0$) to those under the optimal containment policy.
- Optimal containment (explained below) results in a considerably smaller death toll at the cost of a much larger recession.

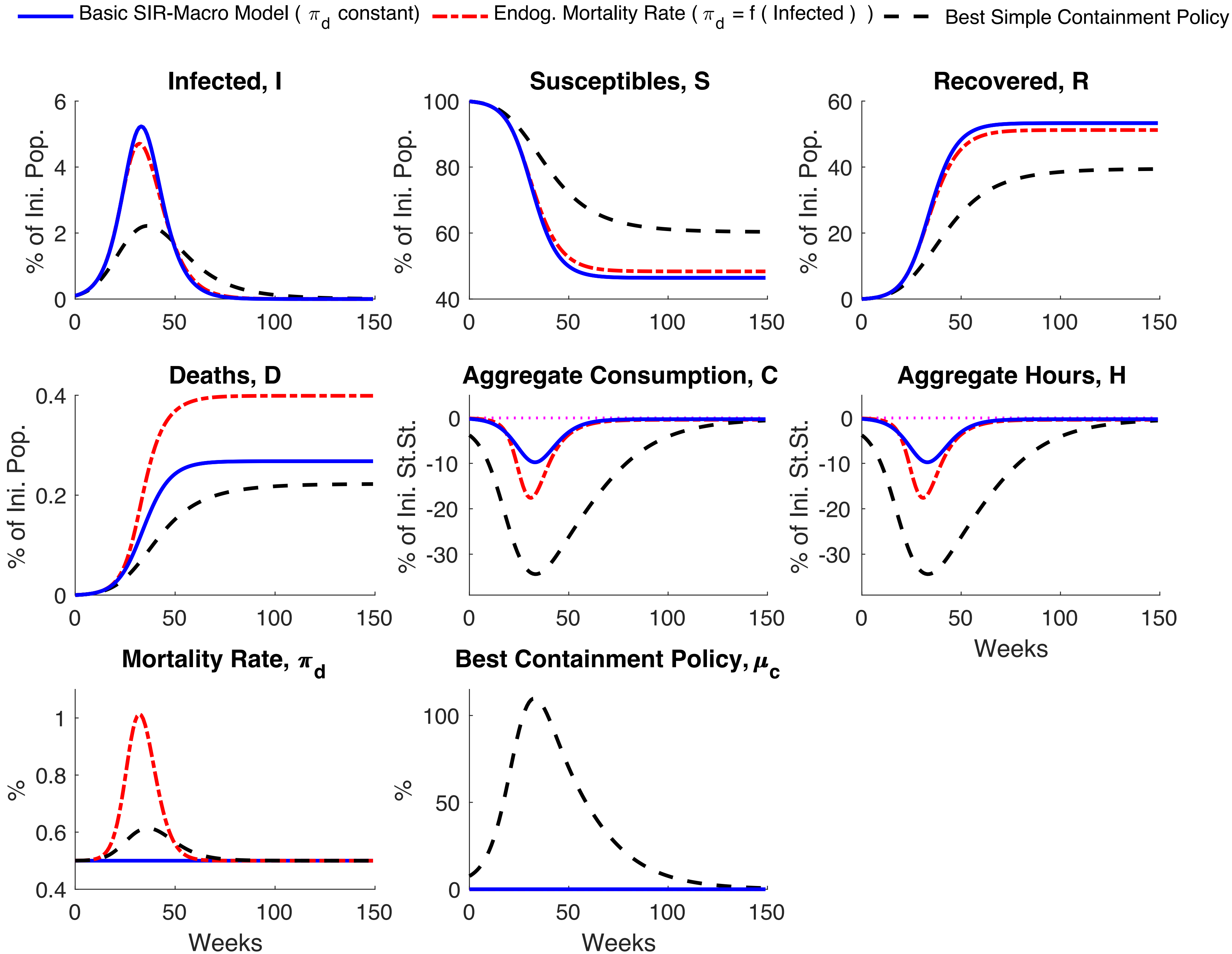
Figure 3: Basic SIR-Macro Model With and Without Containment



The Medical Preparedness Model Results

- The red dashed-dotted lines in Figure 4 show that the model without containment policy but with an endogenous mortality rate involves a much larger recession than in the basic SIR-Macro model (blue solid lines).
- The reason is that people internalize the higher mortality rates associated with an healthcare system that is overburdened with infected people.
- Since the costs of becoming infected are much higher, people cut back on consumption and work to reduce the probability of becoming infected.
- The net result is that fewer people are infected but more people die.

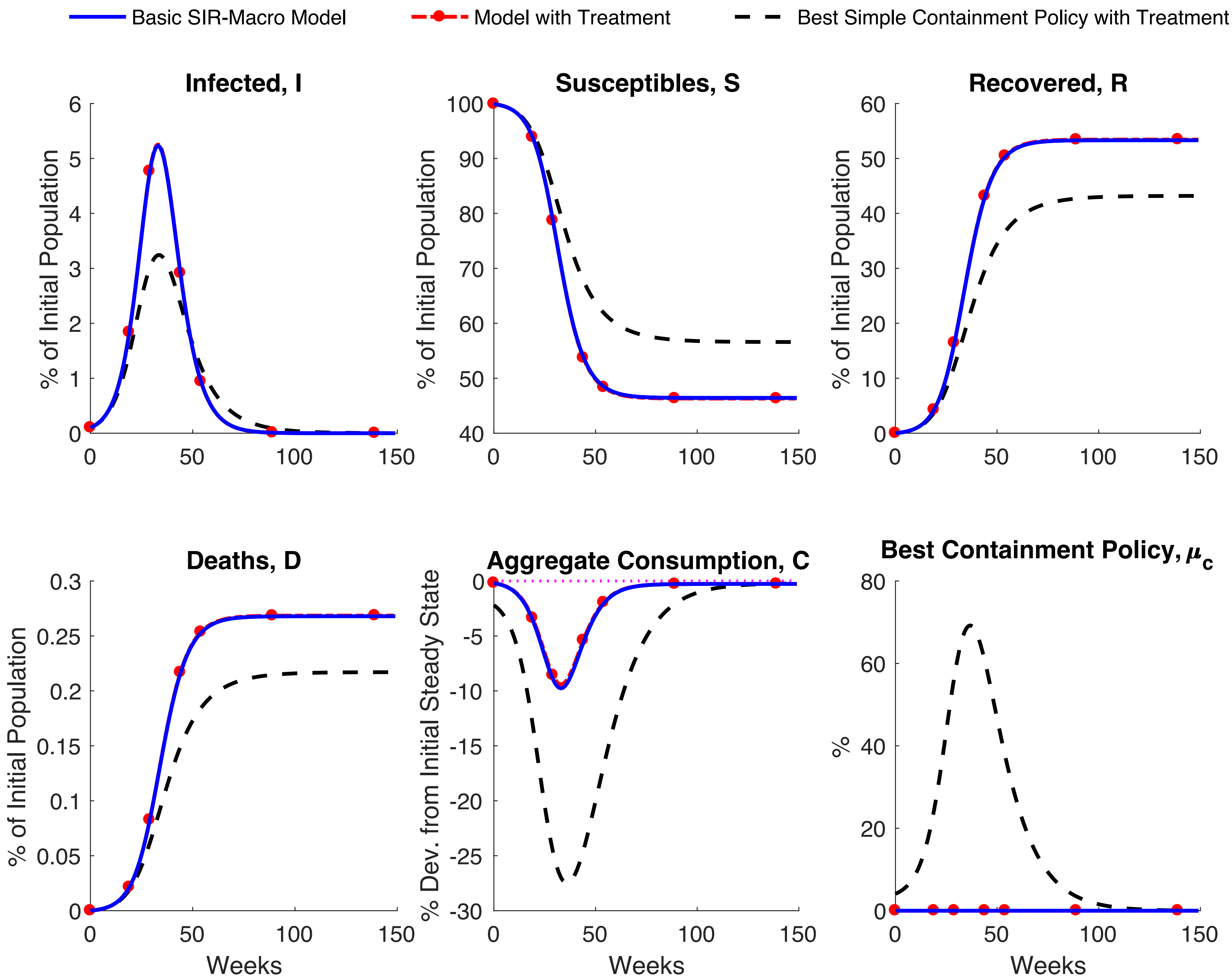
Figure 4: Medical Preparedness



The Treatment and Vaccines Models Results

- The possibility of treatment and vaccination have similar qualitative effects in the absence of containment policy.
- Compared to the basic SIR-Macro model people become more willing to engage in market activities.
- The reason is that the expected costs associated with being infected are smaller.
- Because of this change in behavior, the recession is less severe.
- In Figures 5 and 6 the blue-solid and red-dashed-dotted lines virtually coincide.
- So, in practice the quantitative effect of the possibility of treatments or vaccinations is quite small.

Figure 5: SIR-Macro Model With Treatments



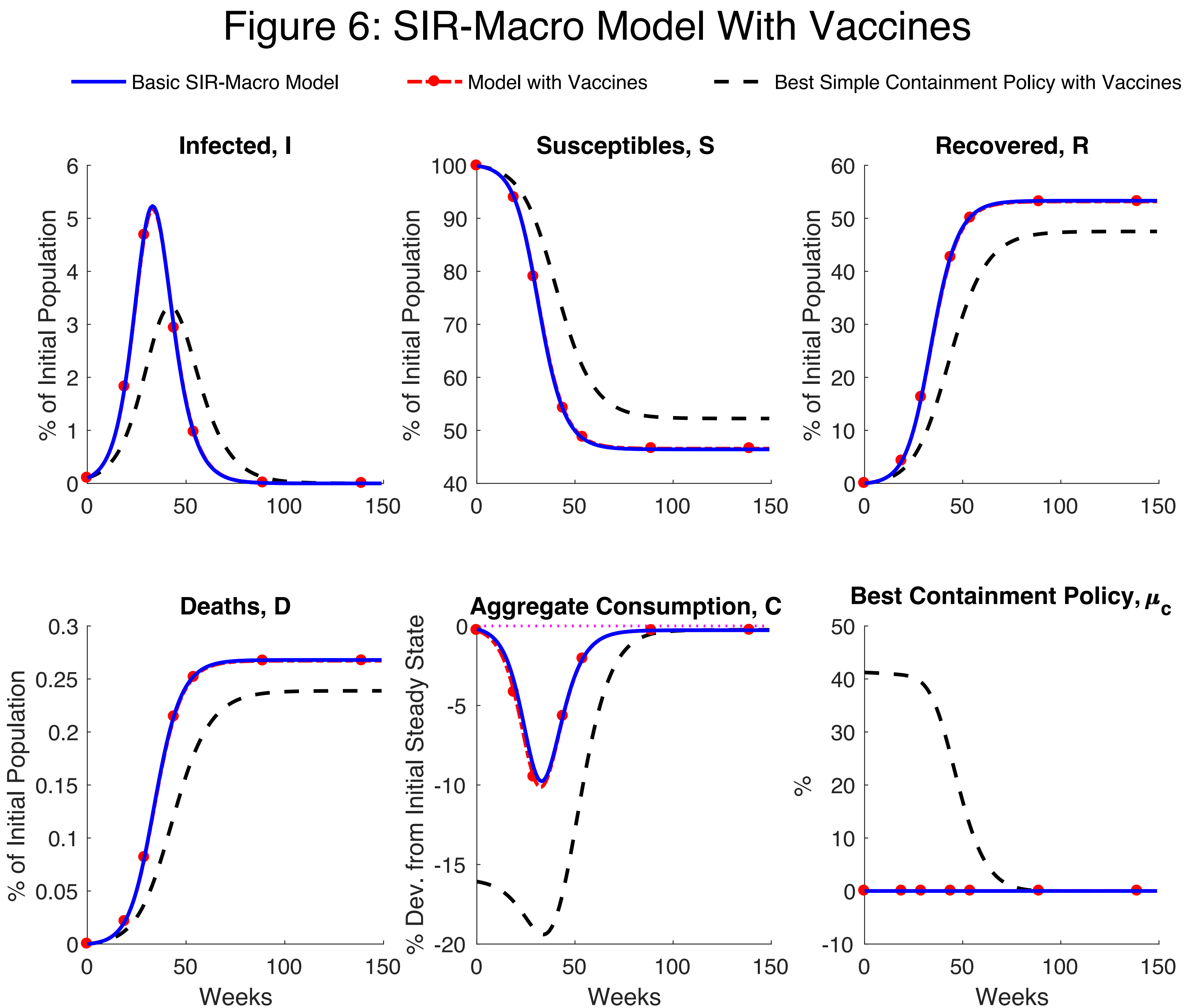


Figure 6: SIR-Macro Model With Vaccines
Basic SIR-Macro Model
Model with Vaccines
Best Simple Containment Policy with Vaccines
Infected, I
Susceptibles, S
Recovered, R
Deaths, D
Aggregate Consumption, C
Best Containment Policy, μ_c
% of Initial Population
% Dev. from Initial Steady State
%
Weeks

Robustness

- ERT report the results of a series of robustness exercises where they vary key parameters of the basic SIR-Macro model.
- Overall, both the qualitative and the quantitative conclusions of the benchmark calibration are robust to the perturbations they consider.

Containment Policy

- The competitive equilibrium of the model economy—the equilibrium in absence of policy intervention—is not Pareto optimal.
  - The outcome is not efficient.
- There is a classic externality associated with the behavior of infected agents.
- Because agents are atomistic, they do not take into account the impact of their actions on the infection and death rates of other agents.
- As we noted, ERT model containment measures as a tax on consumption, the proceeds of which are rebated lump sum to all agents.
- They compute the optimal sequence of containment rates that maximize social welfare, U_0, defined as a weighted average of the lifetime utility of the different agents.
- Since at time 0 $R_0 = D_0 = 0$, the value of U_0 is

$$U_0 = S_0 U_0^S + I_0 U_0^I.$$

- Figure 3 displays the results.

Containment Policy, Continued

- First, it is optimal to escalate containment measures gradually over time.
- The optimal containment rate rises from 4.5 percent in period 0 to a peak value of 72 percent in period 37.
- The rise in containment rates roughly parallels the dynamics of the infection rate itself.
- The basic intuition is that containment measures internalize the externality caused by the behavior of infected people.
- So, as the number of infected people rises, it is optimal to intensify containment measures.
  - For example, at time 0 very few people are infected, so the externality is relatively unimportant.
  - A high containment rate at time 0 would have a high social cost relative to the benefit.
  - As the infection rises, the externality becomes important and the optimal containment rate rises.

Containment Policy, Continued

- The optimal containment policy greatly reduces the peak level of infections from 5.3 to 3.2 percent, reducing the death toll from 0.27 to 0.21 percent of the initial population.
- For a country like the U.S., this reduction represents roughly two hundred thousand lives saved.
- This beneficial outcome is associated with a much more severe recession.
- The fall in average aggregate consumption in the first year of the epidemic more than triples, going from about 4.7 percent without containment measures to about 17 percent with containment measures.
- Higher containment rates make consumption more costly, so people cut back on the amount they consume and work.

Containment Policy, Continued

- Why not choose initial containment rates that are sufficiently high to induce an immediate, persistent decline in the number of infected?
- Absent vaccines, the only way to prevent a recurrence of the epidemic is for enough of the population to acquire immunity by becoming infected and recovering.
- The optimal way to reach this critical level of immunity is to gradually increase containment measures as infections rise and slowly relax them as new infections wane.

Containment Policy in the Medical Preparedness Model

- Comparing Figures 3 and 4, we see that the optimal containment policy is more aggressive in the medical preparedness model than in the basic SIR-Macro model.
  - The peak containment rate is higher (110 versus 72 percent) and occurs earlier (at week 33 versus week 37).
  - In addition, the containment rate comes down much more slowly in the medical preparedness model.
- These differences reflect that, other things equal, the social cost of the externality is much larger.
- Not only do agents not internalize the cost of consumption and work on infection rates, they also do not internalize the aggregate increase in mortality rates.
- The optimal containment policy greatly reduces the peak level of infections from 4.7 without containment to 2.2 percent with containment.
- The death toll falls from 0.40 to 0.22 percent of the initial population.
  - For a country like the U.S., this reduction represents roughly 600 thousand lives saved.

Containment Policy in the Treatment and Vaccines Models

- Comparing Figures 3 and 5 we see that the optimal containment policy in the treatment and basic SIR-Macro models are very similar.
- In the treatment model, along a path were no treatment is discovered, the optimal containment policy reduces the peak level of infections from 5.3 to 3.2 percent, reducing the death toll from 0.27 to 0.21 percent of the initial population.
  - This reduction corresponds to roughly 200 thousand lives saved in the U.S.
  - The latter figure pertains to a worst-case scenario in which a treatment is never discovered.

Containment Policy in the Treatment and Vaccines Models, Continued

- The black-dashed lines in Figure 6 show that optimal policy is very different in the basic SIR-Macro model and the vaccination model.
- With vaccines as a possibility, it is optimal to immediately introduce severe containment measures to minimize the number of deaths.
- Those containment measures cause a very large, persistent recession:
  - average consumption in the first year of the epidemic falls by about 17 percent.
- But this recession is worth incurring in the hope that the vaccination arrives before many people get infected.
- It is optimal to reduce and delay the peak of the infections in anticipation of a vaccine being discovered.

Containment Policy in the Treatment and Vaccines Models, Continued

- Figure 6 displays the behavior of the vaccines model under optimal containment policy on a path where a vaccine does not arrive.
- Compared to the outcome without containment policy (red-dashed-dotted lines), the peak of the infection rate drops from 5.3 to 3.3 percent of the initial population.
- Moreover, the infection peak occurs in period 42 rather than in period 33.
- Absent a vaccine being discovered, the optimal containment policy reduces the death toll as a percent of the initial population from 0.27 percent to 0.24 percent.
  - For the U.S., this reduction amounts to about a one hundred thousand lives.
  - Remember that this reduction pertains to a worst-case scenario in which vaccines never arrive.

Containment Policy in the Treatment and Vaccines Models, Continued

- Why is optimal policy so different in the vaccination model?
- The basic reason is that unlike treatment, a vaccine does not cure infected people.
- The expected arrival of a vaccine also reduces the importance of building up the fraction of the population that is immune to a level that prevents the recurrence of an epidemic.

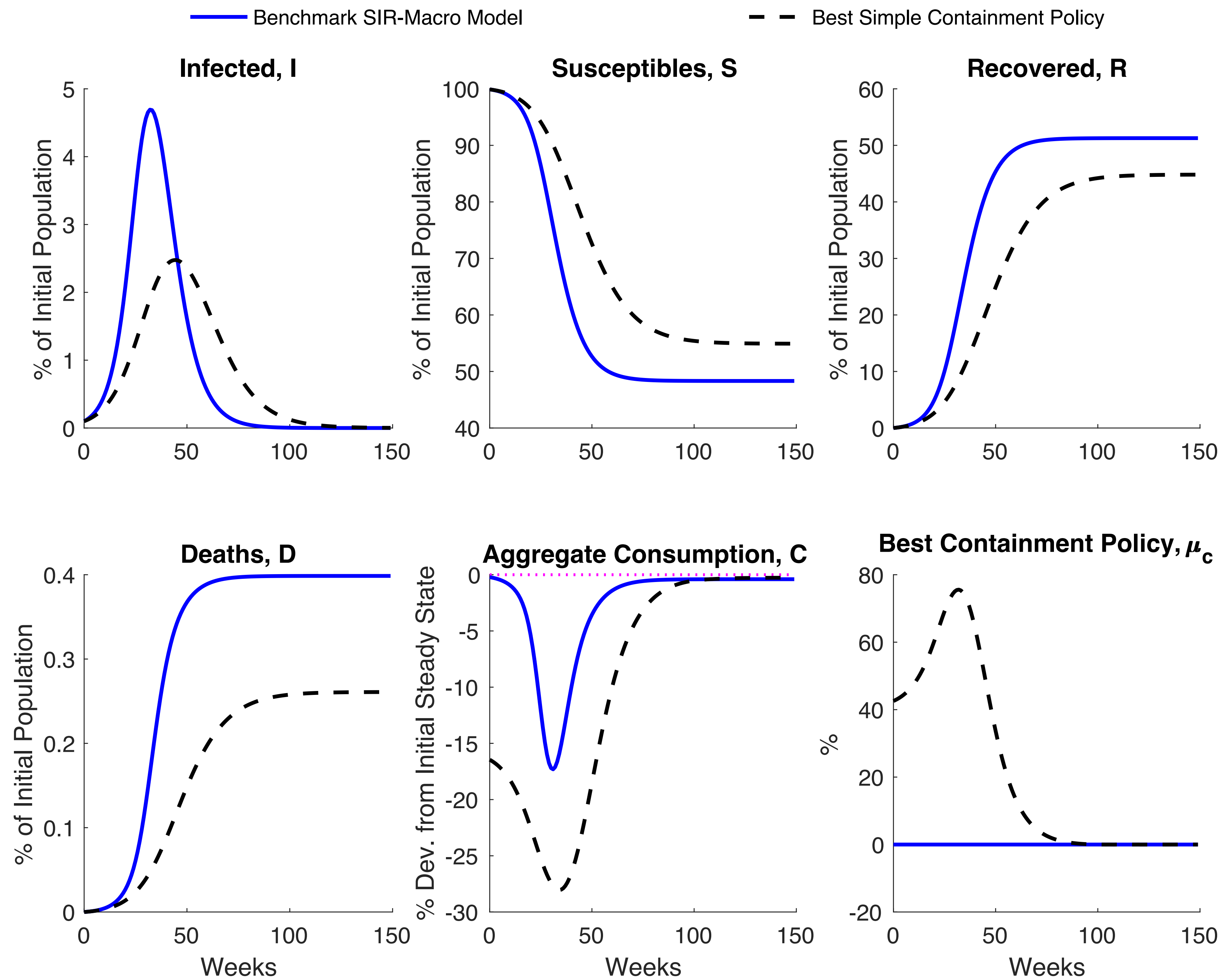
The Complete Model

- The exercises above are useful for understanding the mechanisms at work in ERT's model.
- But the most meaningful version of the model allows for both the possibility of vaccines and medical treatment, as well as the impact of a large number of infections on the efficacy of the healthcare system.

Optimal Policy in the Complete Model

- The solid blue and black dashed lines in Figure 7 correspond to the evolution of the economy without containment policy and under optimal containment policy, respectively.
- Consistent with previous figures, the figure displays a path along which vaccines and treatments are not discovered.
- From a qualitative point of view, the complete model inherits key features of its underlying components.
- Consistent with the vaccination model, it is optimal to immediately introduce severe containment (43 percent).
- Consistent with the treatment and medical preparedness models, it is optimal to ramp containment up as the number of infections rise.
  - The maximal containment rate reaches 76 percent in period 32.

Figure 7: Benchmark SIR-Macro Model (Vaccines, Treatment, Med. Preparedness)



Optimal Policy in the Complete Model, Continued

- The optimal containment measures substantially increase the severity of the recession.
  - Without containment, average consumption in the first year of the epidemic falls by about 7 percent.
  - With containment, this fall is 22 percent.
- Notably, the size of the recession is smaller than in the medical preparedness model.
- The reason is that the prospect of vaccinations and treatments reduce the magnitude of the externality associated with the medical preparedness problem.

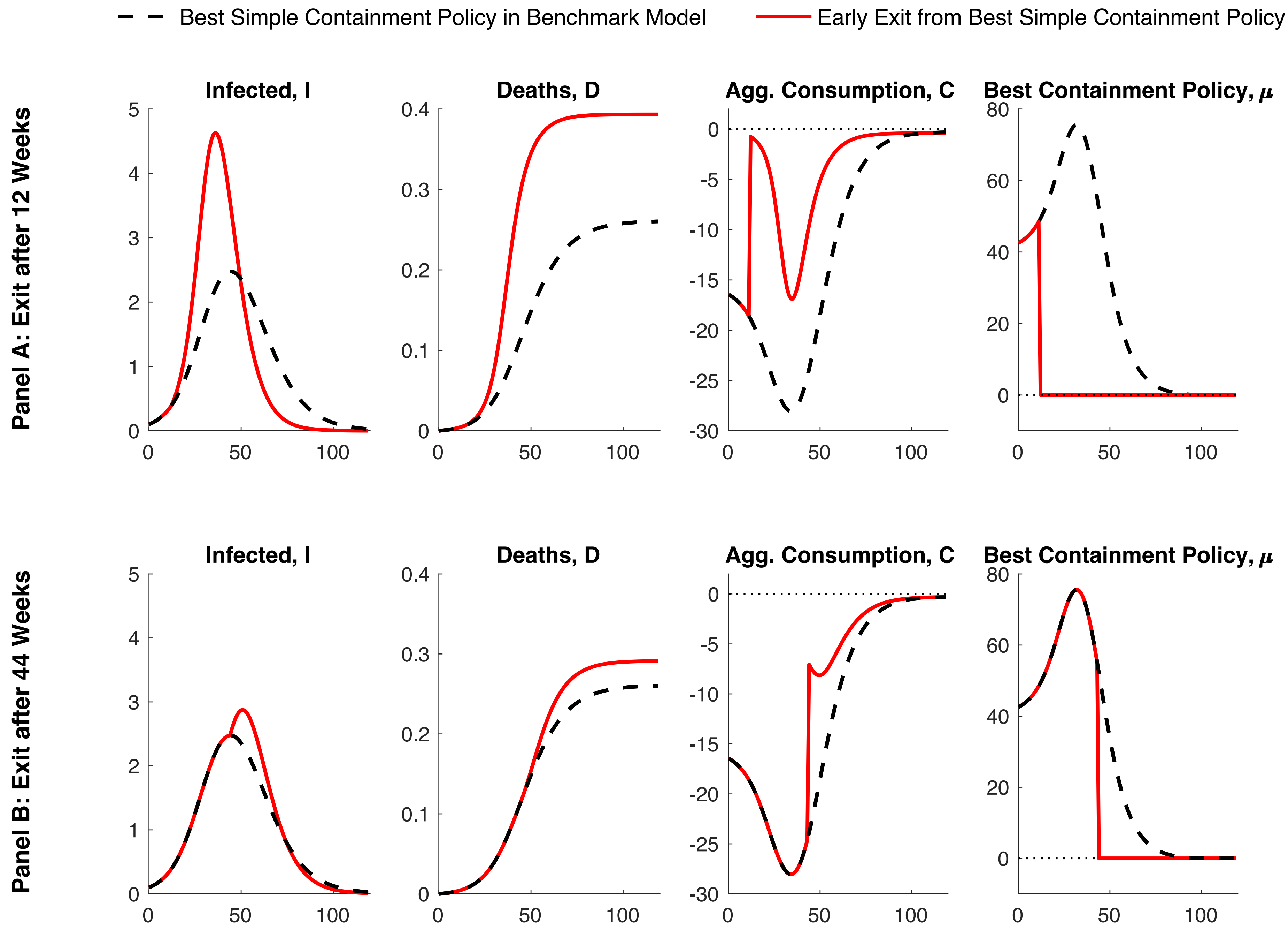
Optimal Policy in the Complete Model, Continued

- The benefit of the large recession associated with optimal containment in the combined model is a less severe epidemic.
- Compared to the outcome without containment, the peak infection rate drops from 4.7 to 2.5 percent of the initial population.
- The optimal policy reduces the death toll as a percent of the initial population from 0.4 percent to 0.26 percent.
  - For the U.S., this reduction amounts to about half-a-million lives.
- Remember that the latter reduction pertains to a worst-case scenario in which vaccines and treatments never arrive.
  - If they did arrive, many more lives would saved—by medicine rather than by containment policies.

The Costs of Ending Containment Too Early

- Policymakers could face intense pressure to prematurely end containment measures because of their impact on economic activity.
- What are the costs of doing so?
- The solid red lines in Panels A and B of Figure 8 display the response of the economy to an unanticipated end of optimal containment policy after weeks 12 and 44, respectively.
  - Week 44 is when infections peak under optimal containment.
- The black dashed lines pertain to the behavior of the economy when optimal policy is fully implemented.
- From Panel A, we see that abandoning containment initially generates a large recovery of the economy with consumption surging by roughly 17 percent.

Figure 8: Benchmark SIR-Macro Model (Vaccines, Treatment, Med. Preparedness)



Notes: x-axis in weeks; infected and deaths in % of ini. population; consumption in % dev. from ini. steady state; best containment policy in %.

The Costs of Ending Containment Too Early, Continued

- Unfortunately, this surge results in a large rise in infection rates.
- The latter rise plunges the economy into a second, persistent recession.
- So, prematurely abandoning containment brings about a temporary rise in consumption but no long-lasting economic benefits.
- Tragically, abandonment leads to a substantial rise in the total number of deaths caused by the epidemic.

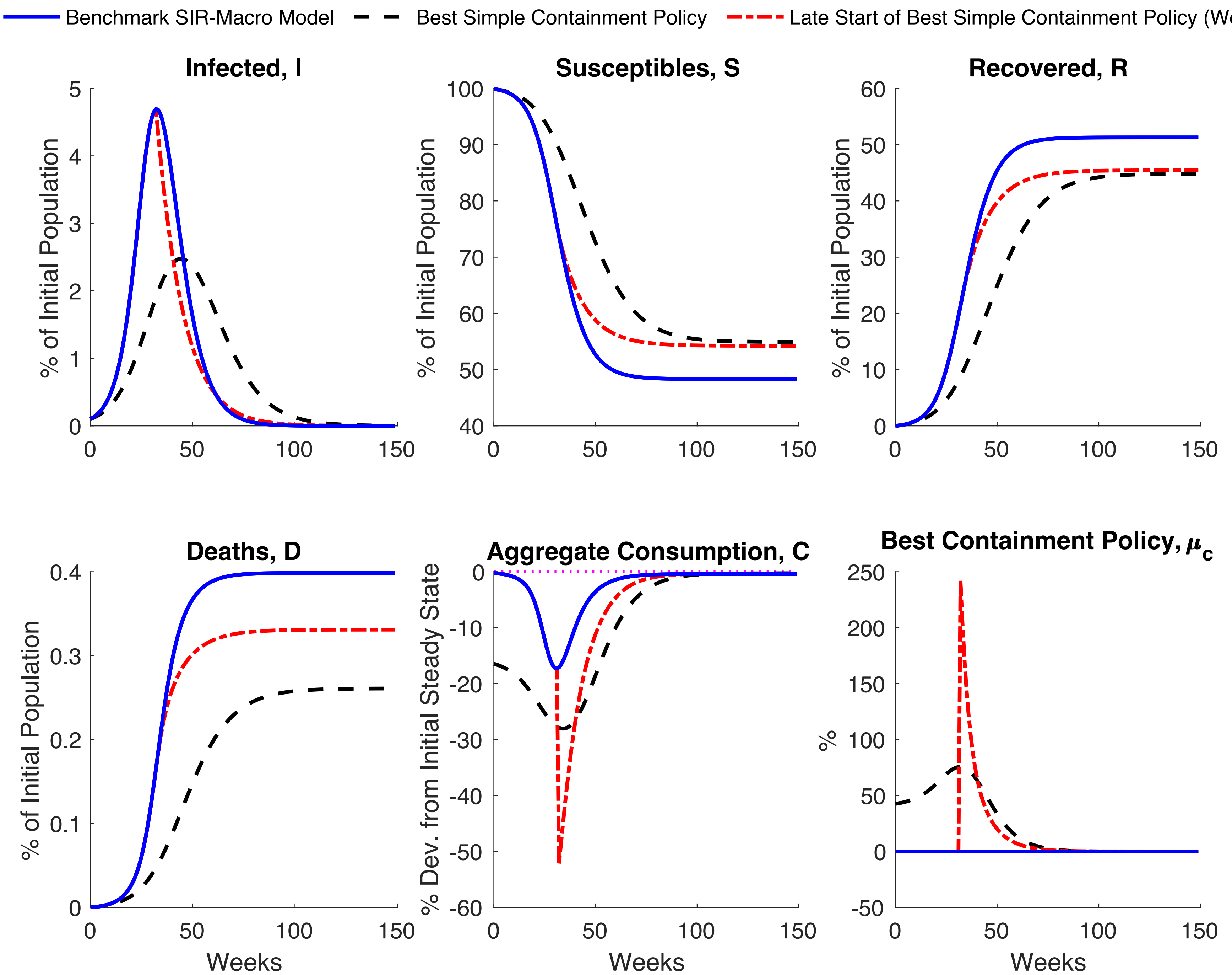
The Costs of Ending Containment Too Early, Continued

- Panel B shows that the longer policy makers pursue optimal containment policy, the better.
- Both the temporary gains and the losses of abandoning optimal policy in Panel B are smaller than those in Panel A.
- The implications of ERT's model about the cost of ending containment too early are consistent with the evidence for the 1918 Spanish.
- The conclusion is that it is important for policymakers to resist the temptation to pursue transient economic gains associated with abandoning containment measures.

The Costs of Starting Containment Too Late

- Policymakers can also face pressures to delay implementing optimal containment measures.
- The red dashed-dotted lines in Figure 9 display the impact of only beginning containment in week 33, the period in which infections peak.
- The assumption is that optimal policy is calculated and implemented from that point on.
- The black dashed lines pertain to the behavior of the economy when the optimal containment policy is implemented from week zero on.
- The solid blue line corresponds to the outcome with no containment measures.

Figure 9: Benchmark SIR-Macro Model (Vaccines, Treatment, Med. Preparedness)



The Costs of Starting Containment Too Late, Continued

- The optimal policy that begins in week 33 involves draconian containment measures that lead to an enormous drop in economic activity.
- The reason is simple: with infections raging, the economic externalities associated with economic activity are very large.
- Despite the draconian measures, the total number of deaths associated with the epidemic is much larger than if the optimal containment policy is implemented without delay.
- Still, as far as the death toll of the epidemic is concerned, late containment (red dashed-dotted lines) is better than no containment at all (blue solid lines).
- The implications of ERT's model about the cost of starting containment too late are consistent with the evidence for the 1918 Spanish flu
- The conclusion is that it is important for policymakers to resist the temptation to delay optimal containment measures for the sake of initially higher short-run levels of economic activity.

Ideal Containment Policy

- The exercises above considered simple containment policies corresponding to a problem in which the government chooses the same consumption containment rate for all agents.

- But ideal containment policy would correspond to a social planning problem where the planner directly chooses consumption and hours worked of susceptible, infected, and recovered people.

- The planner would maximizes social welfare, U_0, but choosing $C_t^S, C_t^I, C_t^R, N_t^S, N_t^I$, and N_t^R for all periods subject to the expressions for the lifetime utility of the different agents, the transmission function (1), and the laws of motion for the population, (2), (3), (4), and (5).

- The lifetime utilities of infected and recovered people are given by (10) and (11), respectively.

- The lifetime utility of susceptible people that is relevant for the planner would be:

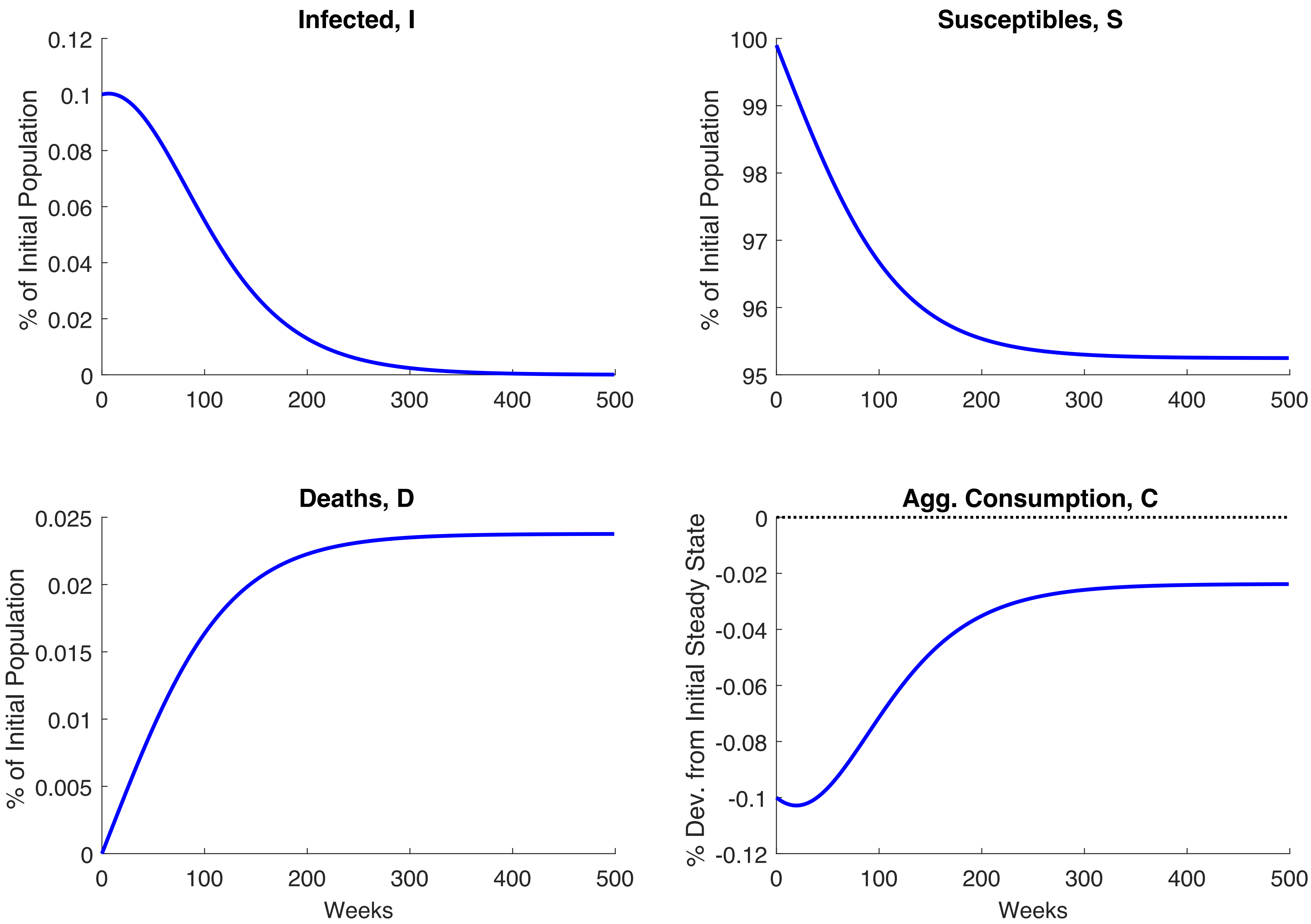
$$U_t^S = \ln C_t^S - \frac{\varphi}{2}(N_t^S)^2 + \beta\left[T_t U_{t+1}^I + (1 - T_t)\, U_{t+1}^S\right]$$

  because the planner would internalize the infection externalities, implying that lifetime utility of the susceptible would be computed using the aggregate transition probabilities.

Ideal Containment Policy, Continued

- Figure 10 shows the dynamics that the planner would generate.
- Note that infected people do not work unless they recover.
- As a result, all susceptible people can work without fear of becoming infected.
- The planner sets the consumption of infected people to a minimum.
- Because infected people are completely isolated, the initial infection quickly dies out without causing a recession.

Figure 10: Smart Containment in the Benchmark SIR-Macro Model



Ideal Containment Policy, Continued

- The previous analysis assumes that infected people have to be in contact with other people to get consumption goods.
- This assumption explains the draconian implication that consumption of infected people should be kept at a minimum.
- Suppose instead that the planner can directly deliver consumption goods to the infected so they do not need to go shopping.
- The solution to this modified problem continues to have the property that infected people do not work.
- But they consume the same as other people.
- Since there is such a small number of infected people at time 0, aggregate consumption and hours worked are essentially the same as in the pre-epidemic steady state.

Ideal Containment Policy, Continued

- ERT's analysis of ideal containment assumes that policymakers know the health status of different individuals.
- In reality, this knowledge would require antigen and antibody tests for immunity and infection that are sufficiently accurate to act upon.
- ERT's results suggest that there are enormous social returns to having these tests and the policy instruments to implement smart (if possible, ideal) containment.
- This conclusion is consistent with the emphasis placed by epidemiologists on early detection and early response.

Conclusion

- ERT's paper highlights how economic incentives and infection dynamics interact to shape the outcome of pandemics.
- It highlights the importance of policy responses, their timing, and their nature.
- It does so by using some of the basic tools you have become familiar with in this class, combined with a workhorse epidemiology model.
- The SIR-Macro model abstracts from features of the economy that we know to be central to its dynamics:
  - There is no asset accumulation (no physical capital or financial assets), no search-and-matching friction in labor markets, no financial accelerator mechanism, no role for monetary policy, no fiscal policy other than the containment tax on consumption, and none of other extremely important features of real economies.

Conclusion, Continued

- But the simplicity of the SIR-Macro model is what allows ERT to obtain clear, transparent results and intuitions that can guide the understanding of more complicated models and policy exercises.
- A large number of macroeconomists are working to develop results and evidence on the effects of the COVID-19 crisis on the economy, the channels through which they happen, and the role of policy responses.
- And a large number of macroeconomists are of course continuing to work on many other interesting questions not directly related to the COVID-19 crisis.
- I hope this class made you curious about this work, its results, and how it can help policymakers.