1) A quartic oscillator is defined by the Hamiltonian \( \hat{H} = -\frac{\hbar^2}{8\pi^2 m} \frac{d^2}{dx^2} + \kappa x^4 \). Note this oscillator differs from the linear harmonic oscillator discussed earlier in that its potential energy goes like \( x^4 \).

a) Using a trial wave function of the form \( \psi_0(x) = \left( \frac{\alpha^2}{\pi} \right)^{1/4} e^{-\alpha x^2/2} \), where \( \alpha \) is a constant, calculate the energy \( \langle E \rangle = -\frac{\int_{-\infty}^{\infty} \psi_0^* \hat{H} \psi_0 dx}{\int_{-\infty}^{\infty} \psi_0^2 dx} \).

b) Using the Variational Principle, determine the value of \( \alpha \) that minimizes the energy in part a. That is, using your expression for \( \langle E \rangle \) from part a, set \( \frac{\partial \langle E \rangle}{\partial \alpha} = 0 \) and solve for \( \alpha \).

c) Using your result from part b, calculate the ground state energy of the quartic oscillator and the ground state wave function.

2) Suppose we apply a field in the x direction to a particle in a one dimensional box. The particle has a charge of Q and the dipole moment is therefore \( xQ \). Evaluate the transition moment integral for a transition from

a) \( n=1 \) to \( n=2 \).

b) \( n=1 \) to \( n=3 \).

c) \( n=1 \) to \( n=4 \).

d) \( n=1 \) to \( n=5 \).

e) Based on these trends, can you determine the selection rule for a transition for a particle in a box from the m to the n energy state? See text problem 18.22 for a hint.

3) The instantaneous dipole moment of hydrogen is \( \vec{\mu} = -e \vec{r} \). The vector \( \vec{r} \) is a three dimensional vector which we can decompose into its x, y, and z components:

\( \vec{r} = -er \left( \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \right) \) The transition moment integral for an electronic transition is then
\[ p_{nlm, n'lm'} = \frac{2\pi}{\hbar} \int_0^{2\pi} d\phi \int_0^\pi d\sin \theta \int_0^\infty dr r^2 \psi_{nlm} \bar{\mu} \psi_{n'lm'} \]

a) By evaluating the transition moment integrals, show that the \( \psi_{1s} \rightarrow \psi_{2s} \) transition is not allowed.

b) Determine if the \( \psi_{1s} \rightarrow \psi_{3p_x} \) or the \( \psi_{1s} \rightarrow \psi_{3p_z} \) transitions are allowed.

c) Based on these calculations, what are the selection rules for \( \Delta l \) and \( \Delta m \)?

I recommend you know how to work these problems in preparation for the exam:

4) Carbon monoxide absorbs radiation at a wave number \( \nu = 2144 \text{ cm}^{-1} \) because of its internal vibration.

a) Calculate the vibrational energy level spacing for CO. Assume the vibration of CO can be modeled as a harmonic oscillation.

b) Calculate \( k_B T \) for \( T=300 \text{ K} \). Based on this value and your answer in part a, will CO vibrations be thermally excited at \( T=300 \text{ K} \)? Explain.

c) At what temperature will thermal excitation of CO vibrations become important?

d) Calculate the vibrational heat capacity of 1 mole of CO at the temperature calculated in part c. Assume first the expression for the vibrational heat capacity based on the equipartition theorem. Compare this result to the quantum expression for the heat capacity

\[ C_V = \frac{N_A k_B \left( \frac{\hbar \nu}{k_B T} \right)^2 e^{-\hbar \nu/k_B T}}{(1 - e^{-\hbar \nu/k_B T})^2} \]

5) Particle-in-Box Problems:

a) Calculate the probability of finding a particle of mass \( m \) in a box of length \( L \) between \( x=L/4 \) and \( x=3L/4 \).

b) Assume a large number of helium atoms are in a one dimensional box. Calculate the average energy of the helium atoms, assuming the pressure is low enough that the helium behaves ideally. Assume \( T=1000 \text{ K} \).

c) Assume the translational energy of the helium atoms can be modeled using the particle in the box energy expression. Calculate the quantum number \( n \) that would have to be achieved such that the translational energy equals the average energy calculated in part b.

d) Calculate the energy level spacing for the \( n \) calculated in part c. Based on this answer, determine how important quantization is to translational motions of helium atoms in a one dimensional box of length \( L=1.00 \text{ cm} \) at \( T=1000 \text{ K} \).