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Lecture 2: The Characteristic Function and the Biased Random Flight
Text Reading: Chandrasekhar, Ch. 1.1

A. Biased Random Walk: The Characteristic Function

- The Fourier transform of the probability is also called a characteristic function in the theory of stochastic processes. Call the characteristic function ψ(s,t). Then the distribution function and characteristic functions form a Fourier pair:

$$\psi(s,t)=\int_{-\infty}^{+\infty} f(x,t)e^{isx}dx; \quad and \quad f(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(s,t)e^{-isx}ds \tag{2.1}$$

- The utility of the characteristic function lies in the fact that its logarithm can be expressed as a Taylor expansion around s=0, and the first few elements of this expansion are familiar features of a distribution. The expansion also converges rapidly so we need not evaluate usually more than the first few terms.
- Here we can use the characteristic function to obtain the distribution function for a biased one dimensional random walk.
- Assume x varies randomly in time and its distribution in time is described by the function f(x,t). We define the expectation value of x <x> and the variance of x: D(x):

$$\langle x\rangle=\int_{-\infty}^{+\infty} xf(x,t)dx; \quad and \quad D(x)=\int_{-\infty}^{+\infty}(x-\langle x\rangle)^2 f(x,t)dx \tag{2.2}$$

- Consider the Taylor expansion of lnψ(s) around s=0…

$$\ln\psi(s)=\ln\psi(0)+s\left.\frac{d\ln\psi}{ds}\right|_{s=0}+\frac{s^2}{2}\left.\frac{d^2\ln\psi}{ds^2}\right|_{s=0}+\ldots \tag{2.3}$$

- Recall $\psi(s,t)=\int_{-\infty}^{+\infty} f(x,t)e^{isx}dx$. Then

$$\begin{aligned}\frac{d\ln\psi}{ds}&=\frac{1}{\psi(s)}\frac{d\psi}{ds}=\frac{1}{\psi(s)}\frac{d}{ds}\int_{-\infty}^{+\infty} f(x,t)e^{isx}dx=\frac{i}{\psi(s)}\int_{-\infty}^{+\infty} xf(x,t)e^{isx}dx\\ \therefore \left.\frac{d\ln\psi}{ds}\right|_{s=0}&=\frac{i}{\psi(0)}\int_{-\infty}^{+\infty} xf(x,t)\,dx=i\langle x\rangle\end{aligned} \tag{2.4}$$

- Similarly:

$$\left.\frac{d^2\ln\psi}{ds^2}\right|_{s=0}=-D(x) \tag{2.5}$$

- Then

$$\ln\psi(s,t) = is\langle x\rangle - \frac{s^2}{2}D(x) + \ldots \tag{2.6}$$

- Define a variable ℓ that randomly varies between +1 or -1 such that the probability of ℓ at any time t being +1 or -1 is f(1)=p and f(-1)=q and p+q=1. Define the discrete FT;

$$\psi(s) = \sum_{\ell=\pm1} f(\ell)e^{i\ell s} = pe^{is} + qe^{-is} \tag{2.7}$$

- It can be shown that $\langle\ell\rangle = p-q$ and $D(\ell) = 4pq$ so that

$$\ln\psi(s) \approx is(p-q) - \frac{s^2}{2}(4pq) \tag{2.8}$$

- Assume N random processes occur with identical statistics as shown in equation 2.7. Note that the definition of ℓ changes for N jumps. ℓ is now the displacement or total number of jumps to the right or left of the starting point after N jumps. As such $|\ell|$ can be much greater than 1! Designate the characteristic function for N such processes

$$\psi_N(s) = \left(\sum_{\ell=\pm1} f(\ell)e^{i\ell s}\right)^N = \left(pe^{is} + qe^{-is}\right)^N \tag{2.9}$$

- It can be shown that

$$\ln\psi_N(s) \approx isN\langle x\rangle - \frac{s^2N}{2}D(x) = isN(p-q) - \frac{s^2N}{2}(4pq) \tag{2.10}$$

- The distribution function for N such systems jumping randomly as described in equation (2.1) is

$$f(\ell,N) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi_N(s)e^{-is\ell}ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left[isN\langle\ell\rangle - \frac{s^2N}{2}D(\ell)\right]e^{-is\ell}ds \tag{2.11}$$

- Making the substitution $u = i\left(\ell - \langle\ell\rangle\right)$ and evaluating the integral, equation (2.11) becomes:

$$f(\ell,N) = \frac{1}{2\pi}\sqrt{\frac{2}{D(\ell)}}e^{u^2/2D(\ell)} = \frac{1}{\sqrt{8\pi Npq}}\exp\left[-\frac{\left(\ell - N(p-q)\right)^2}{8Npq}\right] \tag{2.12}$$

- Equation (2.12) has f as a function of a discrete variable ℓ, which is the excess number of jumps to the right or left after N jumps. We can obtain an expression for f as a function of a continuous variable x. Designate the length of the unit jump δ. Then $x = \ell\delta$.
- Now $f(x,N)dx$ is the probability of the net displacement occurring between x and x+dx. Similarly $f(\ell,N)d\ell$ is the probability of a net displacement occurring between ℓ and $\ell + d\ell$. Both of these functions express the same thing using diffterent distance metrics. So they can be equated:

$$f(x,N)dx = f(\ell,N)d\ell = f(\ell,N)\frac{dx}{\delta} \tag{2.13}$$

$$\therefore f(x,N) = \frac{f(\ell,N)}{\delta} = \frac{1}{\delta\sqrt{8\pi Npq}} \exp\left[-\frac{\left(\ell - N(p-q)\right)^2}{8Npq}\right]$$

$$= \frac{1}{\sqrt{8\pi\delta^2 Npq}} \exp\left[-\frac{\left(x - \delta N(p-q)\right)^2}{8\delta^2 Npq}\right] \quad (2.14)$$

- Note for p=q=1/2 we obtain the unbiased random walk result:

$$f(x,N) = \frac{1}{\sqrt{2\pi\delta^2 N}} \exp\left[-\frac{x^2}{2\delta^2 N}\right] \quad (2.15)$$

for which $\langle x^2 \rangle = D(x) = \delta^2 N$. This result is the hallmark of an unbiased random flight process. We will apply this relationship in the next lecture.