

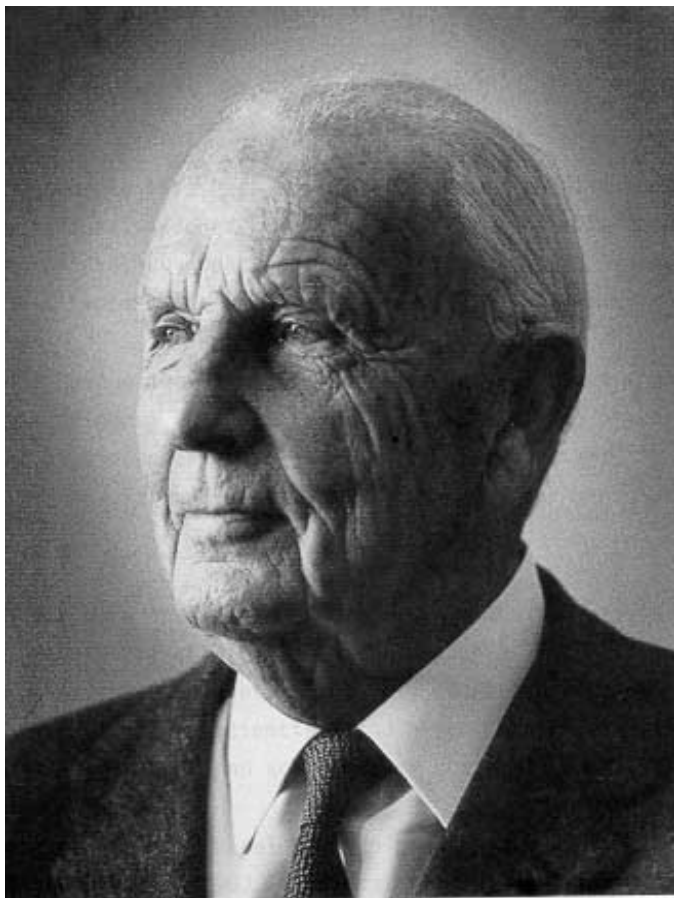


Weibull Reliability Analysis

⇒ <http://www.rt.cs.boeing.com/MEA/stat/scholz/>
<http://www.rt.cs.boeing.com/MEA/stat/reliability.html>
<http://www.rt.cs.boeing.com/MEA/stat/scholz/weibull.html>

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Mathematics & Computing Technology
Applied Statistics

Wallodi Weibull



W. Weibull 1887-1979
 photo: Sam C. Saunders, Pullman WA, USA

Societies and worked to the last day of his remarkable life. He died on October 12, 1979, in Annecy, France. * 6. 18. 1887

Perhaps the greatest tribute to anyone's work is the extent to which it is used and cited by subsequent researchers. The table and graph below, supplied by Mr. Göran Weibull, show the rapidly increasing use of the Weibull distribution and the wide range of fields in which it has been applied. Although the tabulation indicates the variety of applications, it underestimates the total number of references. Another 600 have appeared since the tabulations were prepared.

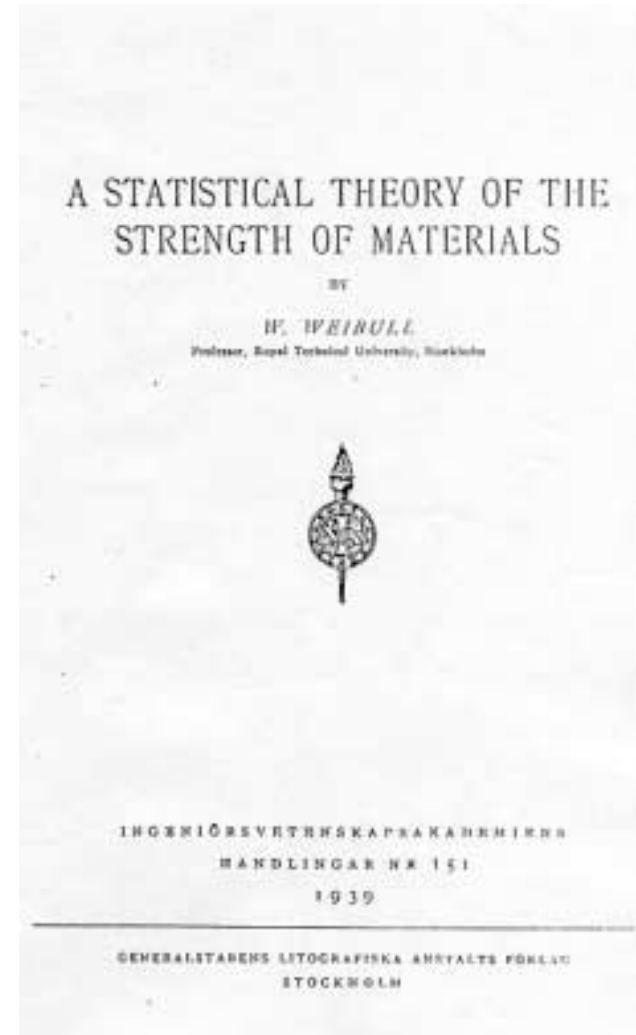
As some of you know, I am of Hungarian origin. In the Hungarian language, "waloddi", spelled "Walodi", means "the Real Thing". Waloddi Weibull was the real thing.

1984

Year	Number of publications
1940	0
1945	0
1950	0
1955	10
1960	30
1965	80
1970	200
1975	650

Publications concerning the Weibull distribution, theoretical properties and practical applications

Seminal Paper



The Weibull Distribution



- **Weibull distribution**, useful **uncertainty model** for
 - **wearout failure time T**
when governed by wearout of weakest subpart
 - **material strength T**
when governed by embedded flaws or weaknesses,
- It has often been found useful based on empirical data (e.g. Y2K)
- It is also theoretically founded on the **weakest link principle**

$$T = \min (X_1, \dots, X_n) ,$$

with X_1, \dots, X_n statistically independent random strengths or failure times of the n “links” comprising the whole.

The X_i must have a natural finite lower endpoint,
e.g., link strength ≥ 0 or subpart time to failure ≥ 0 . \hookrightarrow

Theoretical Basis



- Under weak conditions **Extreme Value Theory** shows¹ that for large n

$$P(T \leq t) \approx 1 - \exp\left(-\left[\frac{t - \tau}{\alpha}\right]^\beta\right) \quad \text{for } t \geq \tau, \alpha > 0, \beta > 0$$

- The above approximation has very much the same spirit as the **Central Limit Theorem** which under some weak conditions on the X_i asserts that the distribution of $T = X_1 + \dots + X_n$ is approximately bell-shaped normal or Gaussian for large n
- Assuming a Weibull model for T , material strength or cycle time to failure, amounts to treating the above approximation as an equality

$$F(t) = P(T \leq t) = 1 - \exp\left(-\left[\frac{t - \tau}{\alpha}\right]^\beta\right) \quad \text{for } t \geq \tau, \alpha > 0, \beta > 0$$

¹see: E. Castillo, *Extreme Value Theory in Engineering*, Academic Press, 1988

S. Coles, *An Introduction to Statistical Modeling of Extreme Values*, Springer Verlag, 2001

Weibull Reproductive Property



If X_1, \dots, X_n are statistically independent

with $X_i \sim \text{Weibull}(\alpha_i, \beta)$ then

$$\begin{aligned} P(\min(X_1, \dots, X_n) > t) &= P(X_1 > t) \times \dots \times P(X_n > t) \\ &= \prod_{i=1}^n \exp\left(-\left[\frac{t}{\alpha_i}\right]^\beta\right) = \exp\left(-\left[\frac{t}{\alpha}\right]^\beta\right) \end{aligned}$$

$$\text{with } \alpha = \left(\sum_{i=1}^n \alpha_i^{-\beta}\right)^{-1/\beta}$$

Hence $T = \min(X_1, \dots, X_n) \sim \text{Weibull}(\alpha, \beta)$

This is similar to the normal reproductive property

$$X_1, \dots, X_n \text{ independent normal} \implies \sum_{i=1}^n X_i \text{ is normal}$$

Weibull Parameters



The Weibull distribution may be controlled by 2 or 3 parameters:

- the **threshold parameter** τ

$T \geq \tau$ with probability 1

$\tau = 0 \implies$ 2-parameter Weibull model.

- the **characteristic life or scale parameter** $\alpha > 0$

$$P(T \leq \tau + \alpha) = 1 - \exp\left(-\left[\frac{\alpha}{\alpha}\right]^\beta\right) = 1 - \exp(-1) = .632$$

regardless of the value $\beta > 0$

- the **shape parameter** $\beta > 0$, usually $\beta \geq 1$

2-Parameter Weibull Model



- We focus on analysis using the **2-parameter Weibull model**
- Methods and software tools much better developed
- Estimation of τ in the 3-parameter Weibull model leads to complications
- When a 3-parameter Weibull model is assumed, it will be stated explicitly

Relation of α & β to Statistical Parameters ($\tau = 0$)



- The **expectation** or **mean value** of T

$$\mu = E(T) = \int_0^{\infty} t f(t) dt = \alpha \Gamma(1 + 1/\beta)$$

$$\text{with } \Gamma(t) = \int_0^{\infty} \exp(-x) x^{t-1} dx$$

- The **variance** of T

$$\sigma^2 = E(T - \mu)^2 = \int_0^{\infty} (t - \mu)^2 f(t) dt = \alpha^2 [\Gamma(1 + 2/\beta) - \Gamma^2(1 + 1/\beta)]$$

- **p -quantile** t_p of T , i.e., by definition $P(T \leq t_p) = p$

$$t_p = \alpha [-\log(1 - p)]^{1/\beta}, \quad \text{for } p = 1 - \exp(-1) = .632 \quad \implies \quad t_p = \alpha$$

Weibull Density



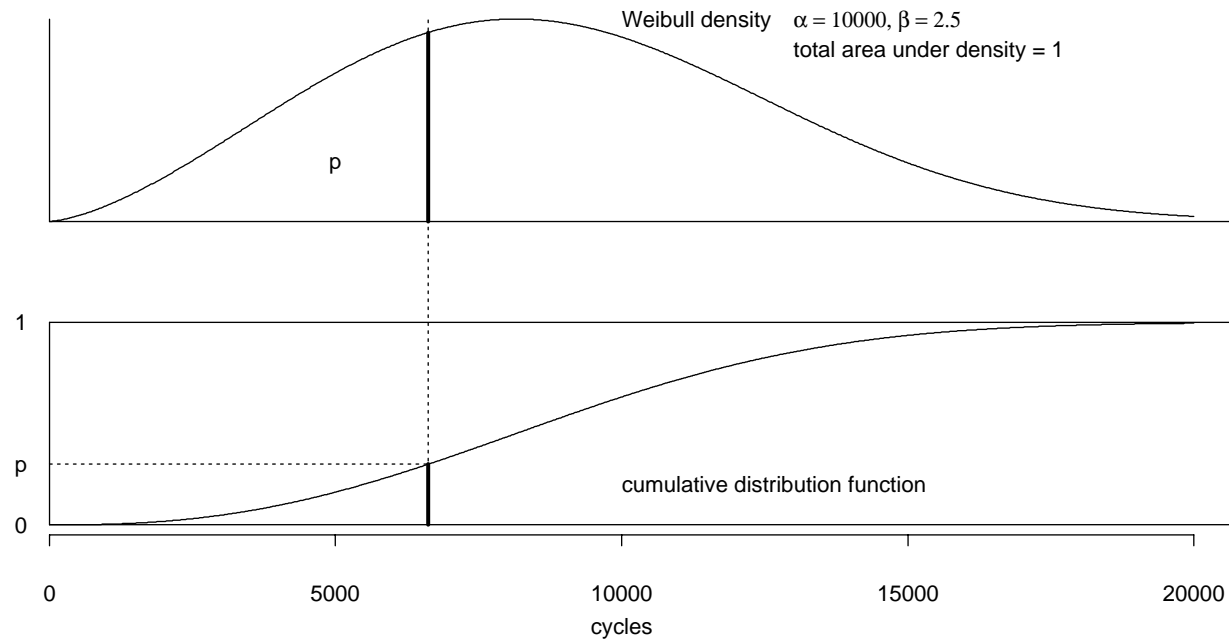
- The **cumulative distribution function** $F(t) = P(T \leq t)$ is just one way to describe the distribution of the random quantity T
- The **density function** $f(t)$ is another representation ($\tau = 0$)

$$f(t) = F'(t) = \frac{dF(t)}{dt} = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left(-\left[\frac{t}{\alpha}\right]^\beta\right) \quad t \geq 0$$

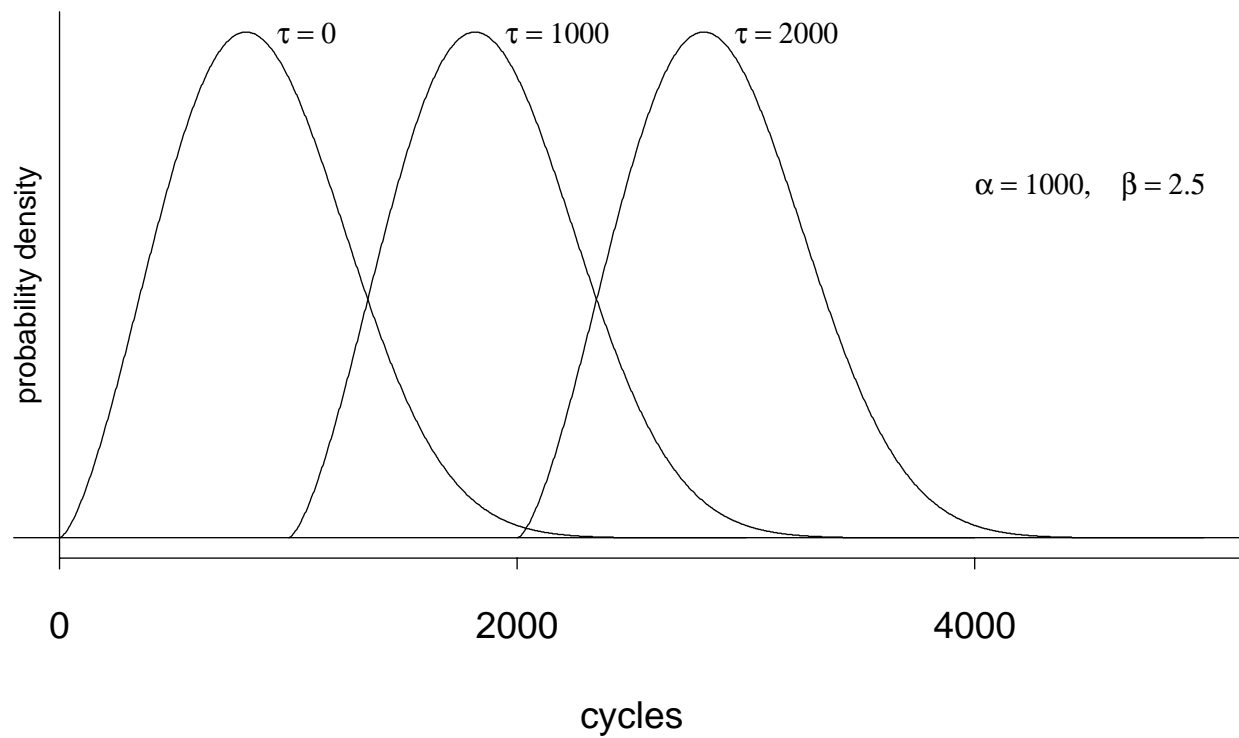
$$P(t \leq T \leq t + dt) \approx f(t) dt$$

$$F(t) = \int_0^t f(x) dx$$

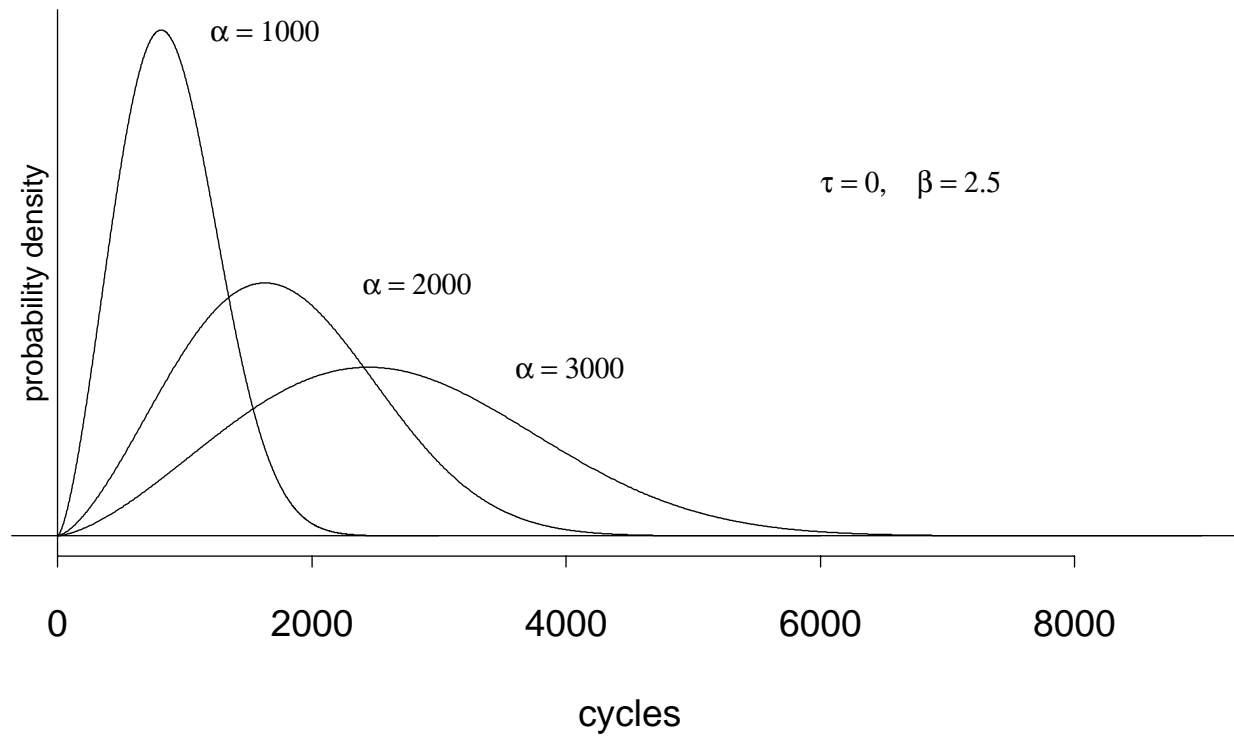
Weibull Density & Distribution Function



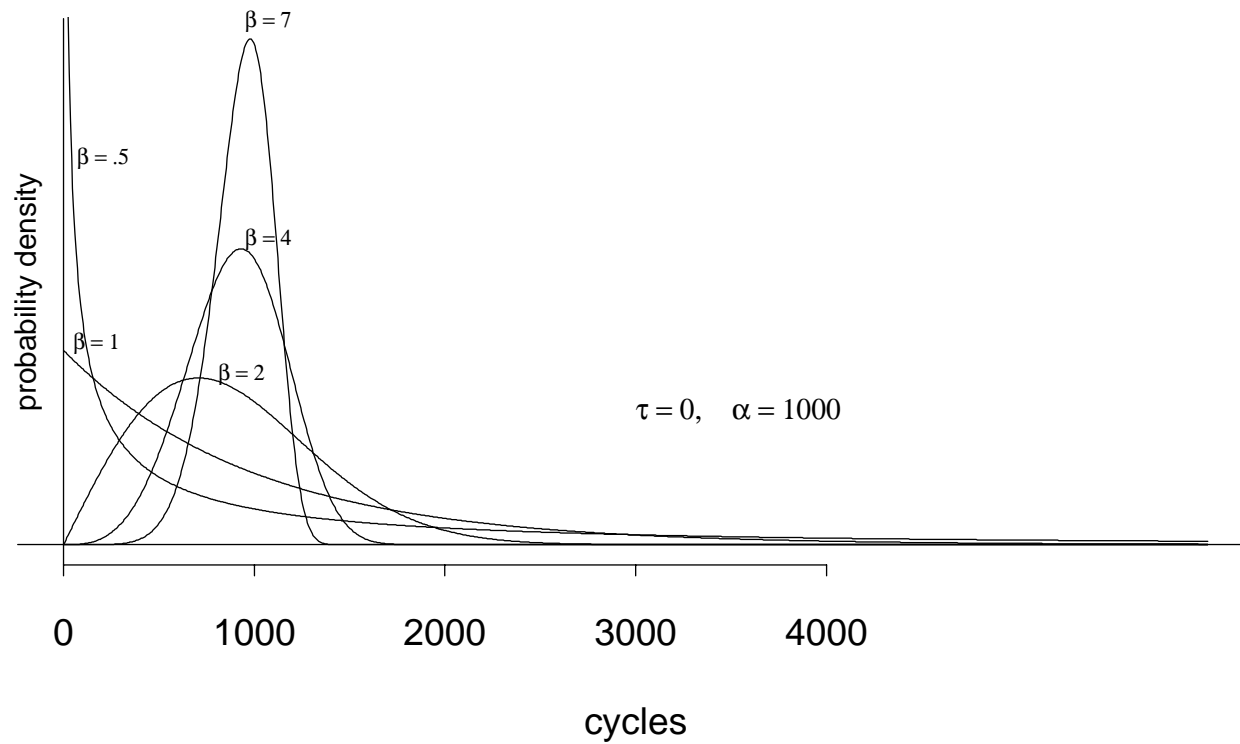
Weibull Densities: Effect of τ



Weibull Densities: Effect of α



Weibull Densities: Effect of β



Failure Rate or Hazard Function



- A third representation of the Weibull distribution is through the **hazard or failure rate function**

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} = \frac{d}{dt} [-\log(1 - F(t))]$$

- $\lambda(t)$ is increasing t for $\beta > 1$ (wearout)
- $\lambda(t)$ is decreasing t for $\beta < 1$
- $\lambda(t)$ is constant for $\beta = 1$ (exponential distribution)

$$P(t \leq T \leq t + dt | T \geq t) = \frac{P(t \leq T \leq t + dt)}{P(T \geq t)} \approx \frac{f(t) dt}{1 - F(t)} = \lambda(t) dt$$

$$F(t) = 1 - \exp\left(-\int_0^t \lambda(x) dx\right) \quad \text{and} \quad f(t) = \lambda(t) \exp\left(-\int_0^t \lambda(x) dx\right)$$

Exponential Distribution



- The **exponential distribution** is a **special case**: $\beta = 1$ & $\tau = 0$

$$F(t) = P(T \leq t) = 1 - \exp\left(-\frac{t}{\alpha}\right) \quad \text{for } t \geq 0$$

- This distribution is useful when parts fail due to **random external influences** and **not due to wear out**
- Characterized by the **memoryless property**,
a part that has not failed by time t is as good as new,
past stresses without failure are water under the bridge
- Good for describing lifetimes of electronic components,
failures due to external voltage spikes or overloads

Unknown Parameters

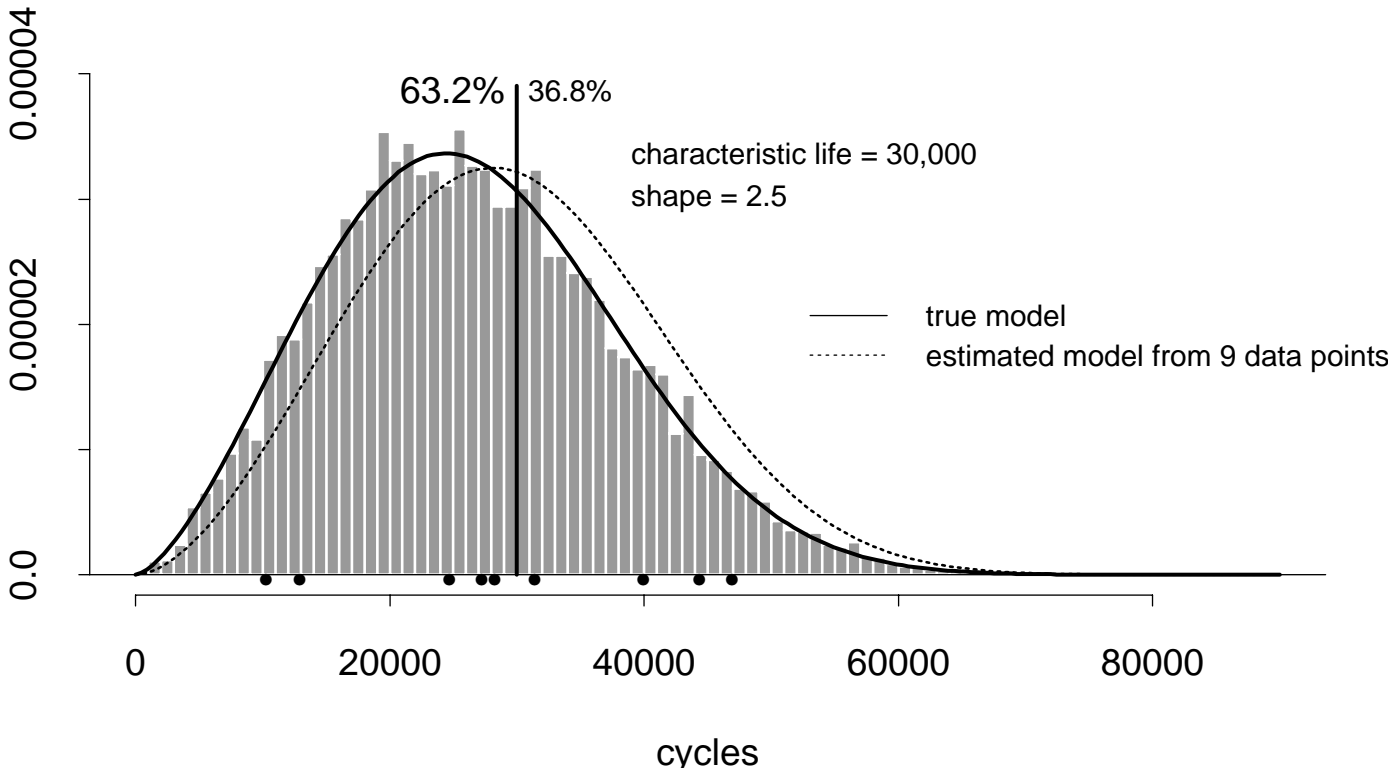


- Typically will not know the Weibull distribution: α , β unknown
- Will only have sample data \implies estimates $\hat{\alpha}$, $\hat{\beta}$
get **estimated** Weibull model for failure time distribution
 \implies **double uncertainty**
uncertainty of failure time & uncertainty of estimated model
- Samples of failure times are sometimes very small,
only 7 fuse pins or 8 ball bearings tested until failure,
long lifetimes make destructive testing difficult
- Variability issues are often not sufficiently appreciated
how do **small sample sizes** affect our confidence in
estimates and predictions concerning future failure experiences?

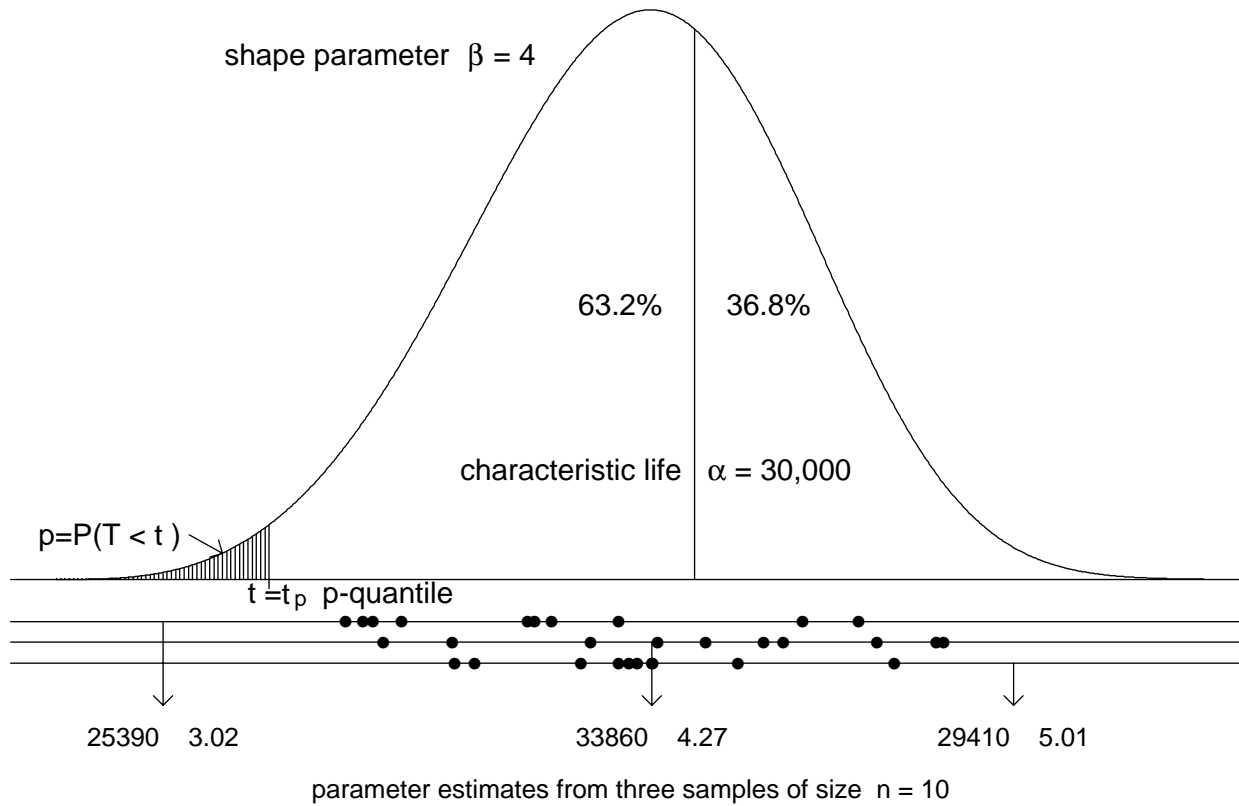
Estimation Uncertainty



A Weibull Population: Histogram for N = 10,000 & Density



Weibull Parameters & Sample Estimates



Generation of Weibull Samples



- Using the quantile relationship $t_p = \alpha [-\log(1 - p)]^{1/\beta}$
one can generate a Weibull random sample of size n by
 - generating a random sample U_1, \dots, U_n
from a uniform $[0, 1]$ distribution
 - and computing $T_i = \alpha [-\log(1 - U_i)]^{1/\beta}$, $i = 1, \dots, n$.
 - Then T_1, \dots, T_n can be viewed as a random sample of size n
from a Weibull population or Weibull distribution
with parameters α & β .
- Simulations are useful in gaining insight on estimation procedures

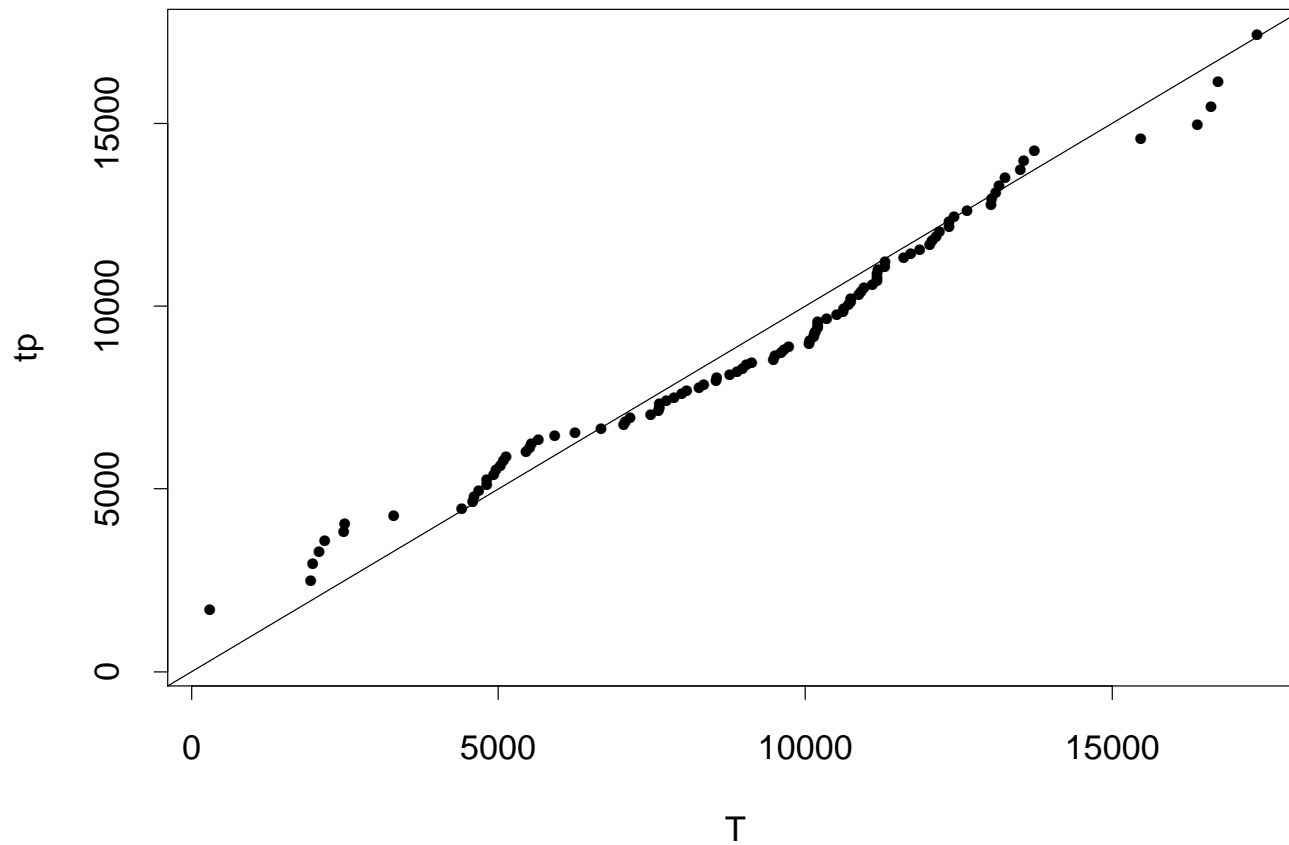
Graphical Methods



- Suppose we have a complete Weibull sample of size n : T_1, \dots, T_n
- Sort these values from lowest to highest: $T_{(1)} \leq T_{(2)} \leq \dots \leq T_{(n)}$
- Recall that the p -quantile is $t_p = \alpha [-\log(1 - p)]^{1/\beta}$
- Compute $t_{p_1} < \dots < t_{p_n}$ for $p_i = (i - .5)/n$, $i = 1, \dots, n$
- Plot the points $(T_{(i)}, t_{p_i})$, $i = 1, \dots, n$

We expect these points to cluster around main diagonal

Weibull Quantile-Quantile Plot: Known Parameters



Weibull QQ-Plot: Unknown Parameters



- Previous plot requires knowledge of the unknown parameters α & β
- Note that

$$\log(t_p) = \log(\alpha) + w_p/\beta, \quad \text{where } w_p = \log[-\log(1 - p)]$$

- Expect points $(\log[T_{(i)}], w_{p_i}), i = 1, \dots, n$, to cluster around line with slope $1/\beta$ and intercept $\log(\alpha)$
- This suggests estimating α & β from a fitted line, by eye or least squares

Maximum Likelihood Estimation



- If t_1, \dots, t_n are the observed sample values one can contemplate the probability of obtaining such a sample or of values nearby, i.e.,

$$\begin{aligned} P(T_1 \in [t_1 - dt/2, t_1 + dt/2], \dots, T_n \in [t_n - dt/2, t_n + dt/2]) \\ = P(T_1 \in [t_1 - dt/2, t_1 + dt/2]) \times \dots \times P(T_n \in [t_n - dt/2, t_n + dt/2]) \\ \approx f_{\alpha, \beta}(t_1)dt \times \dots \times f_{\alpha, \beta}(t_n)dt \end{aligned}$$

where $f(t) = f_{\alpha, \beta}(t)$ is the Weibull density with parameters (α, β)

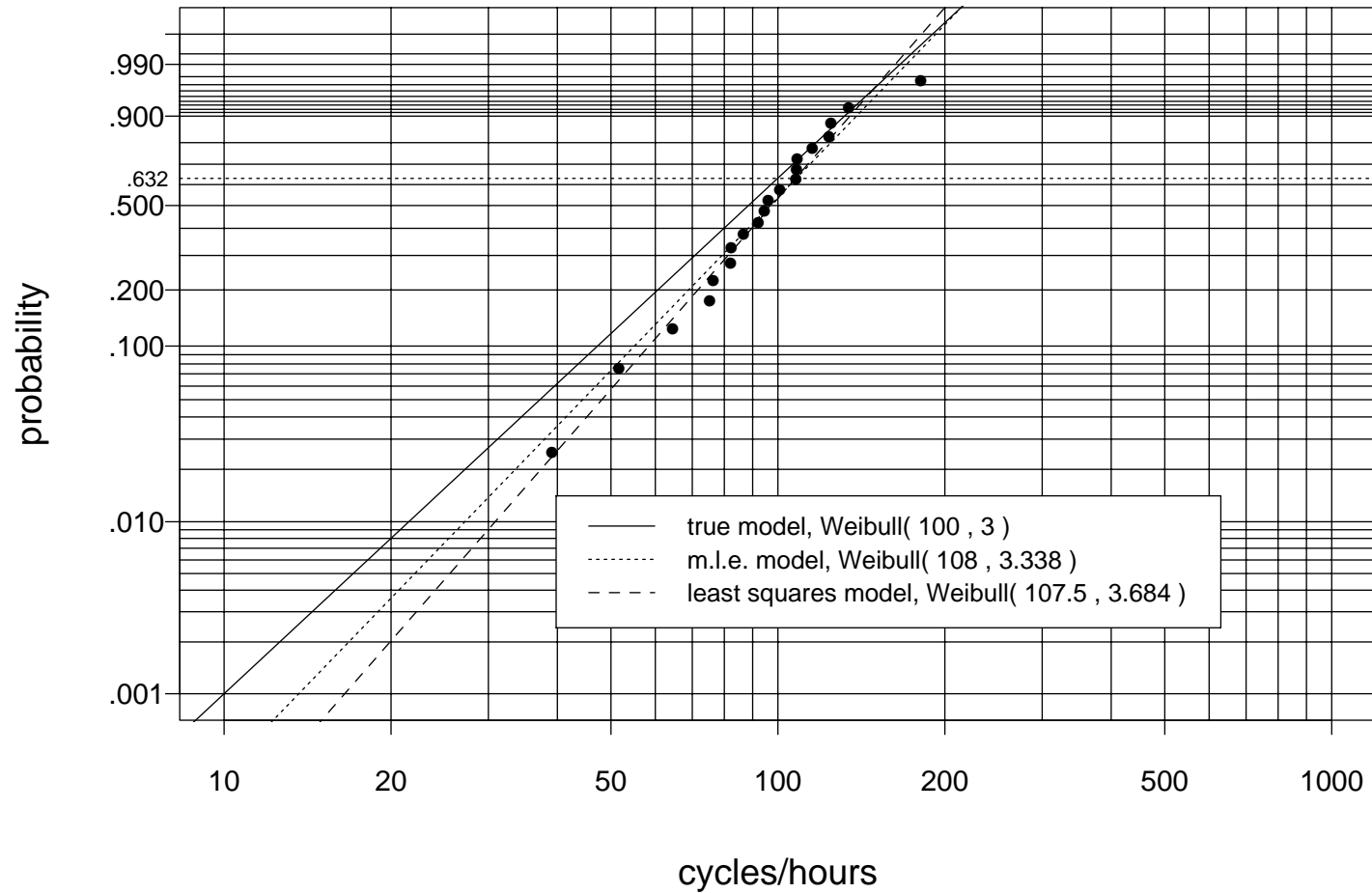
- Maximum likelihood estimation maximizes this probability over α & $\beta \implies$ maximum likelihood estimates (m.l.e.s) $\hat{\alpha}$ and $\hat{\beta} \implies$ the most likely Weibull model “explaining” the data

General Remarks on Estimation

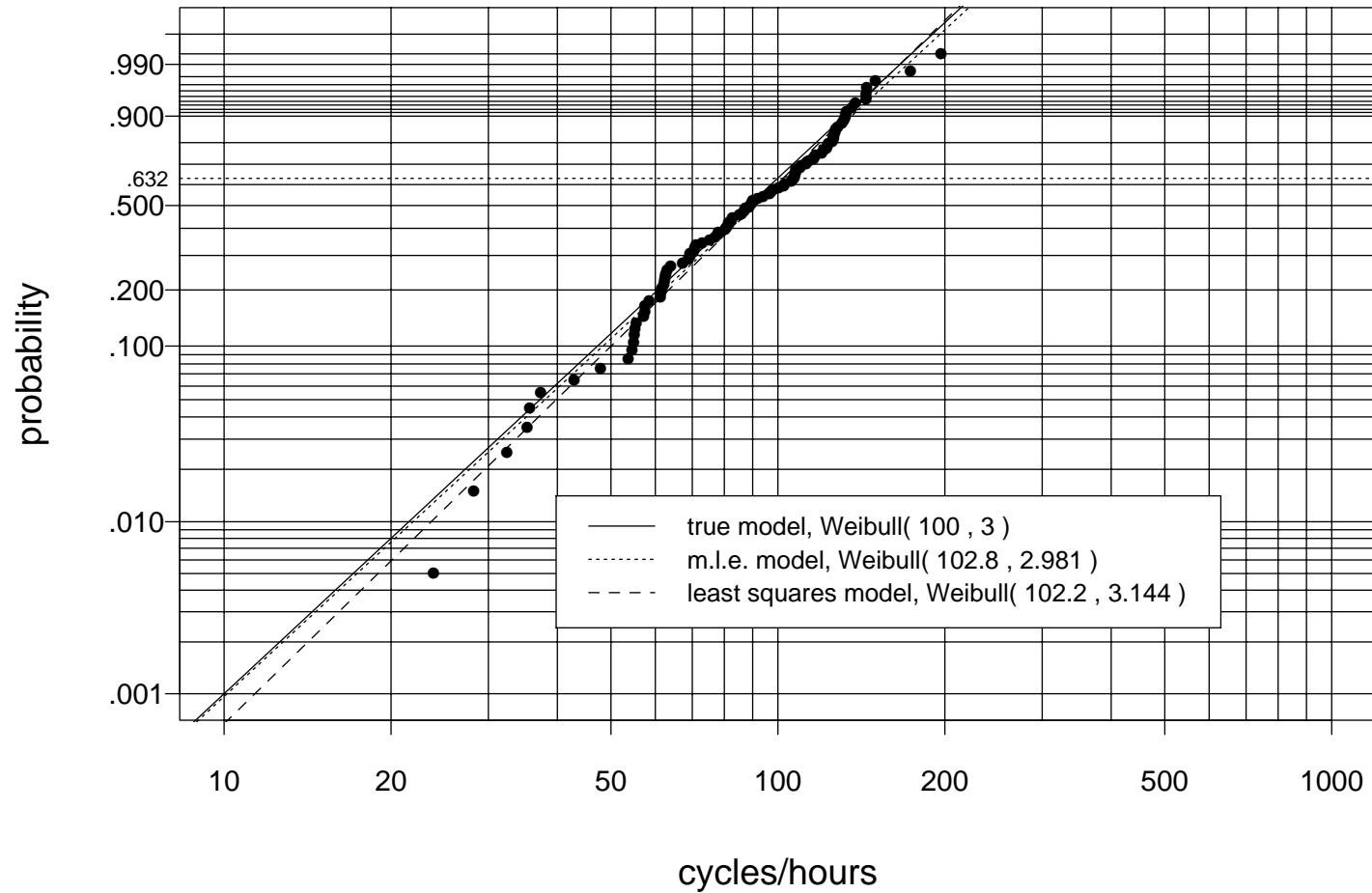


- MLEs tend to be optimal in large samples (lots of theory)
- Method is very versatile in extending to many other data scenarios
censoring and covariates
- Least squares method applied to QQ-plot is not entirely appropriate
tends to be unduly affected by stray observations
not as versatile to extend to other situations

Weibull Plot: $n = 20$



Weibull Plot: $n = 100$



Tests of Fit (Graphical)



- The Weibull plots provide an informal diagnostic for checking the Weibull model assumption
- The anticipated linearity is based on the Weibull model properties
- Strong nonlinearity indicates that the model is not Weibull
- Sorting out nonlinearity from normal statistical point scatter takes a lot of practice and a good sense for the effect of sample size on the variation in point scatter (simulation helps)
- Formal tests of fit are available for complete samples² and also for some other censored data scenarios

²R.B. D'Agostino and M.A. Stephens, *Goodness-of-Fit Techniques*, Marcel Dekker 1986

Formal Goodness-of-Fit Tests



- Let $F_{\hat{\alpha}, \hat{\beta}}(t)$ be the fitted Weibull distribution function
- Let $\widehat{F}_n(t) = \frac{\#\{T_i \leq t; i=1, \dots, n\}}{n}$ be the empirical distribution function
- Compute a discrepancy metric D between $F_{\hat{\alpha}, \hat{\beta}}$ and \widehat{F}_n ,

$$D_{KS}(F_{\hat{\alpha}, \hat{\beta}}, \widehat{F}_n) = \sup_t |F_{\hat{\alpha}, \hat{\beta}}(t) - \widehat{F}_n(t)| \quad \text{Kolmogorov-Smirnov}$$

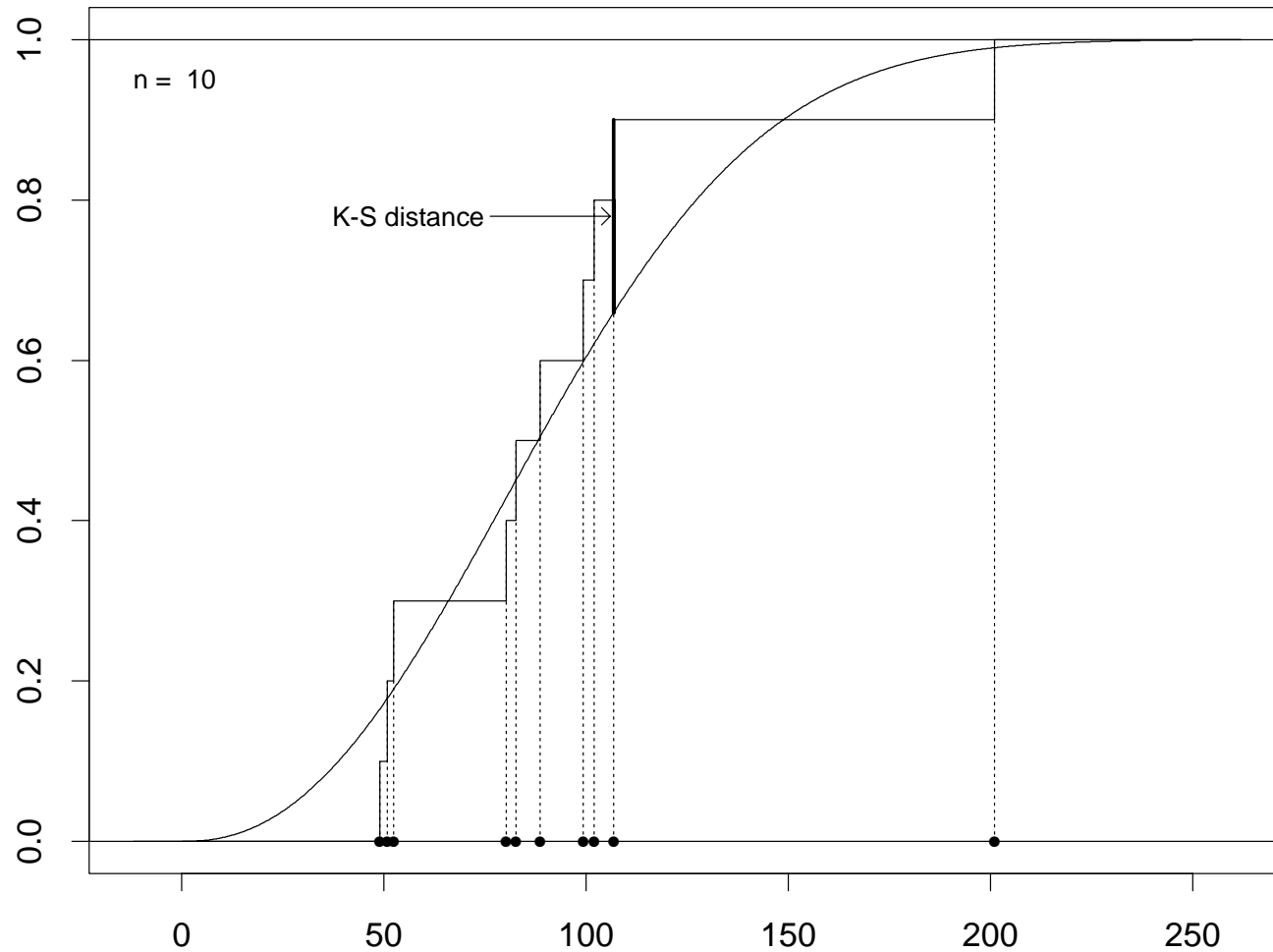
$$D_{CvM}(F_{\hat{\alpha}, \hat{\beta}}, \widehat{F}_n) = \int_0^\infty (F_{\hat{\alpha}, \hat{\beta}}(t) - \widehat{F}_n(t))^2 f_{\hat{\alpha}, \hat{\beta}}(t) dt \quad \text{Cramer-von Mises}$$

$$D_{AD}(F_{\hat{\alpha}, \hat{\beta}}, \widehat{F}_n) = \int_0^\infty \frac{(F_{\hat{\alpha}, \hat{\beta}}(t) - \widehat{F}_n(t))^2}{F_{\hat{\alpha}, \hat{\beta}}(t)(1 - F_{\hat{\alpha}, \hat{\beta}}(t))} f_{\hat{\alpha}, \hat{\beta}}(t) dt \quad \text{Anderson-Darling}$$

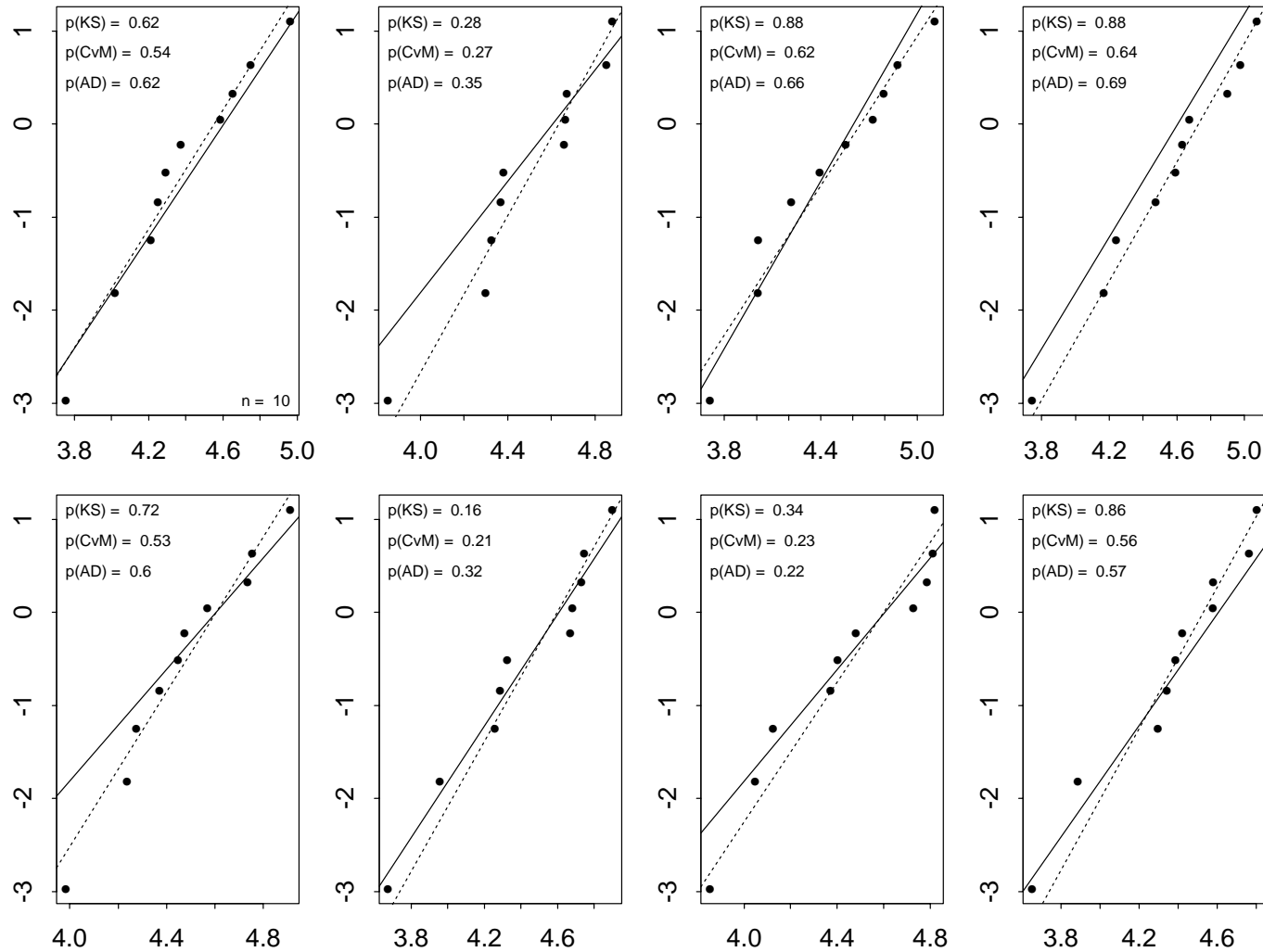
- The distributions of D , when sampling from a Weibull population, are known and p -values of observed values d of D can be calculated

$$p = P(D \geq d) \implies \text{BCSLIB: HSPFIT}$$

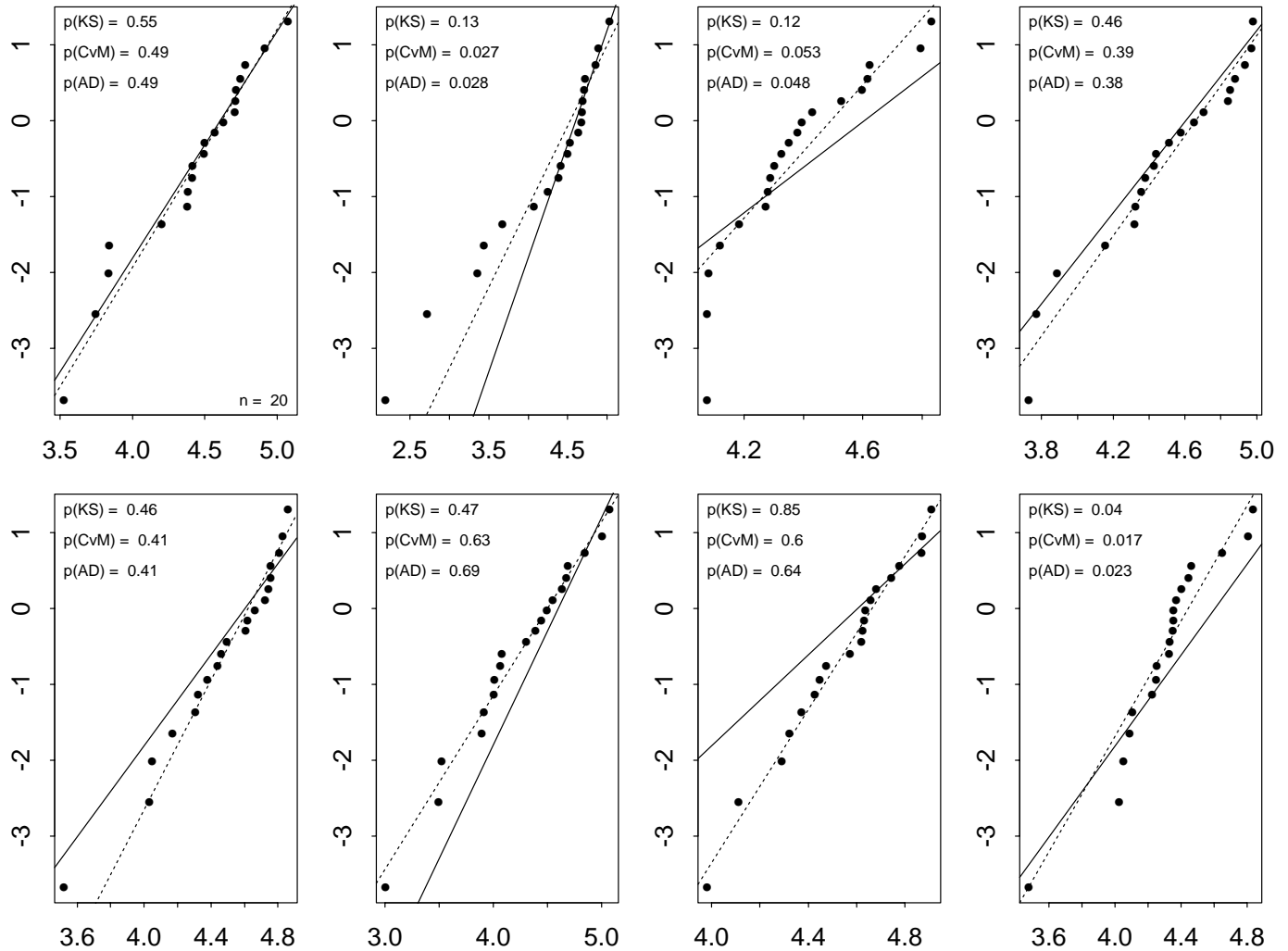
Kolmogorov-Smirnov Distance



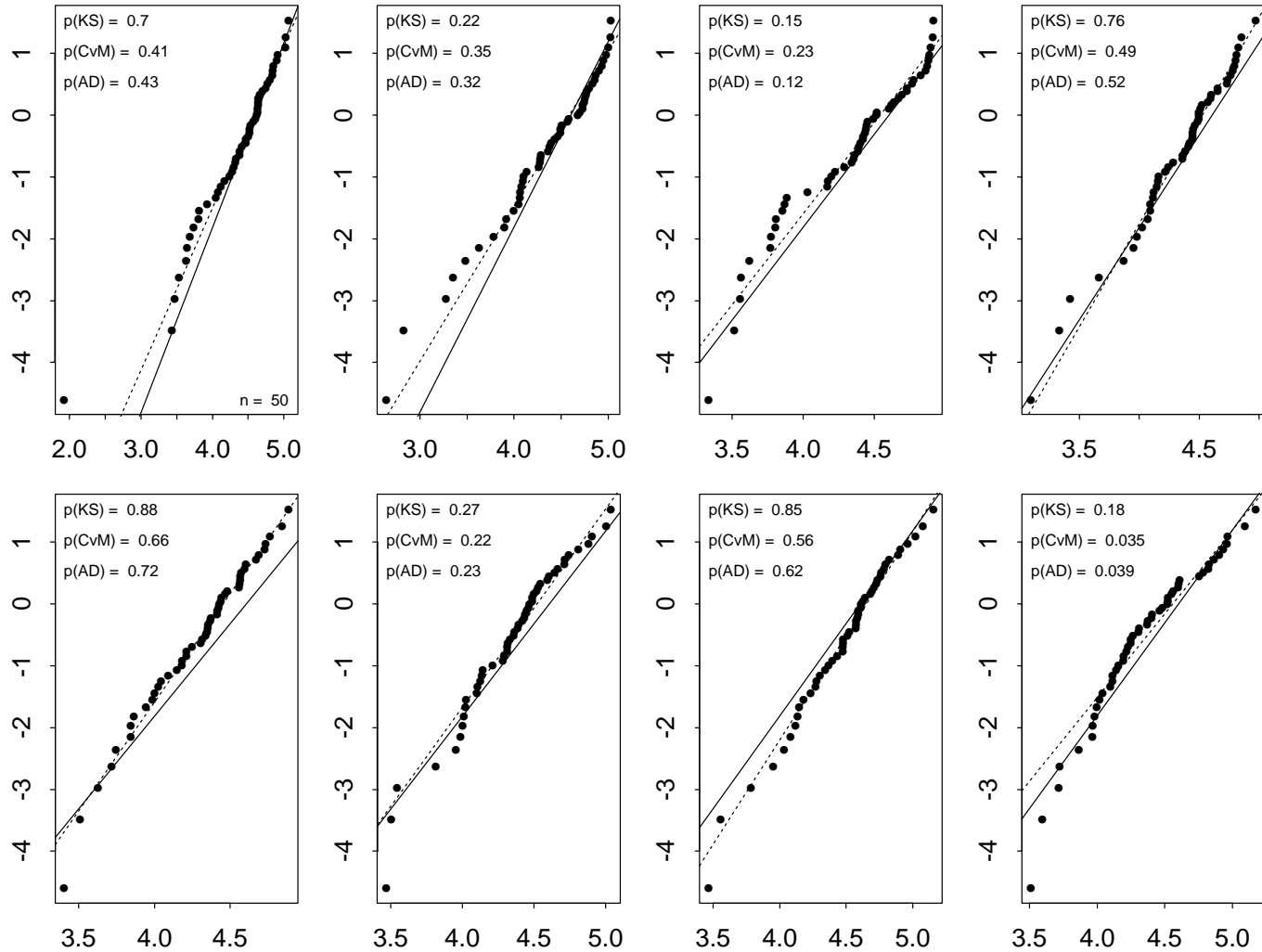
Weibull Plots: $n = 10$ (solid line is true model)



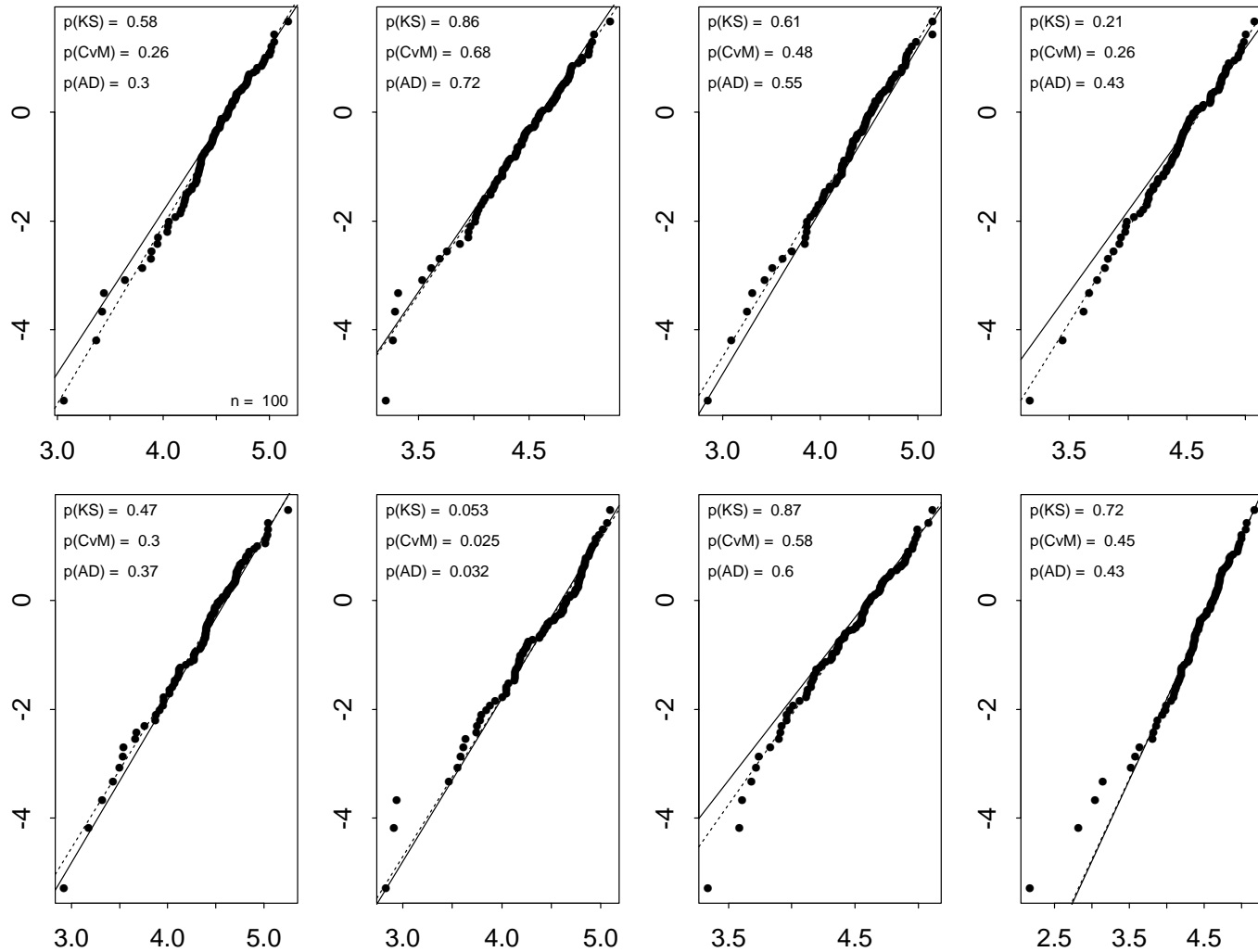
Weibull Plots: $n = 20$



Weibull Plots: $n = 50$



Weibull Plots: $n = 100$



Estimates and Confidence Bounds/Intervals



- For a target θ

$$\theta = \alpha, \quad \theta = \beta, \quad \theta = t_p = \alpha[-\log(1 - p)]^{1/\beta}, \quad \text{or} \quad \theta = P_{\alpha,\beta}(T \leq t)$$

one gets corresponding m.l.e $\hat{\theta}$ by replacing (α, β) by $(\hat{\alpha}, \hat{\beta})$

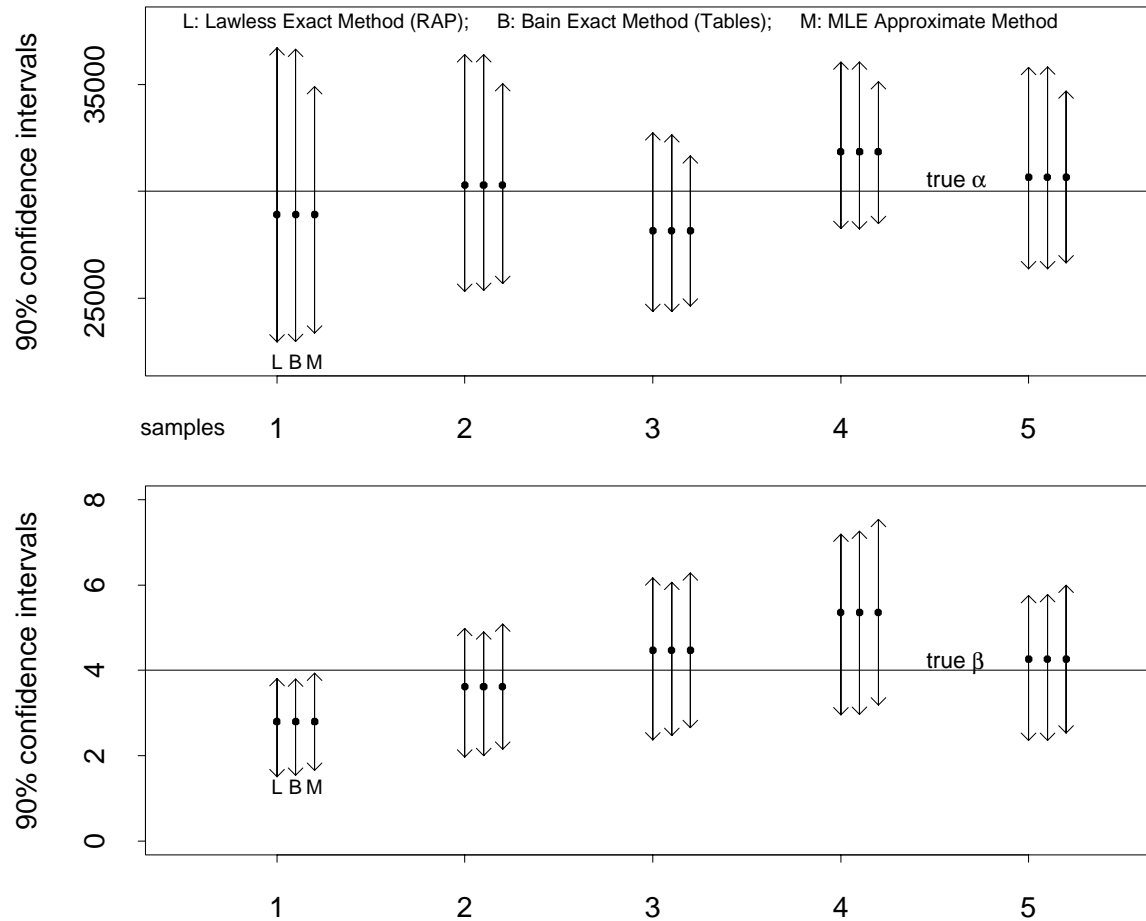
- Such estimates $\hat{\theta}$ vary around the target θ due to sampling variation
- Capture the estimation uncertainty via confidence bounds
- For $0 < \gamma < 1$ get $100\gamma\%$ lower/upper confidence bounds $\hat{\theta}_{L,\gamma}$ & $\hat{\theta}_{U,\gamma}$

$$P(\hat{\theta}_{L,\gamma} \leq \theta) = \gamma \quad \text{or} \quad P(\theta \leq \hat{\theta}_{U,\gamma}) = \gamma$$

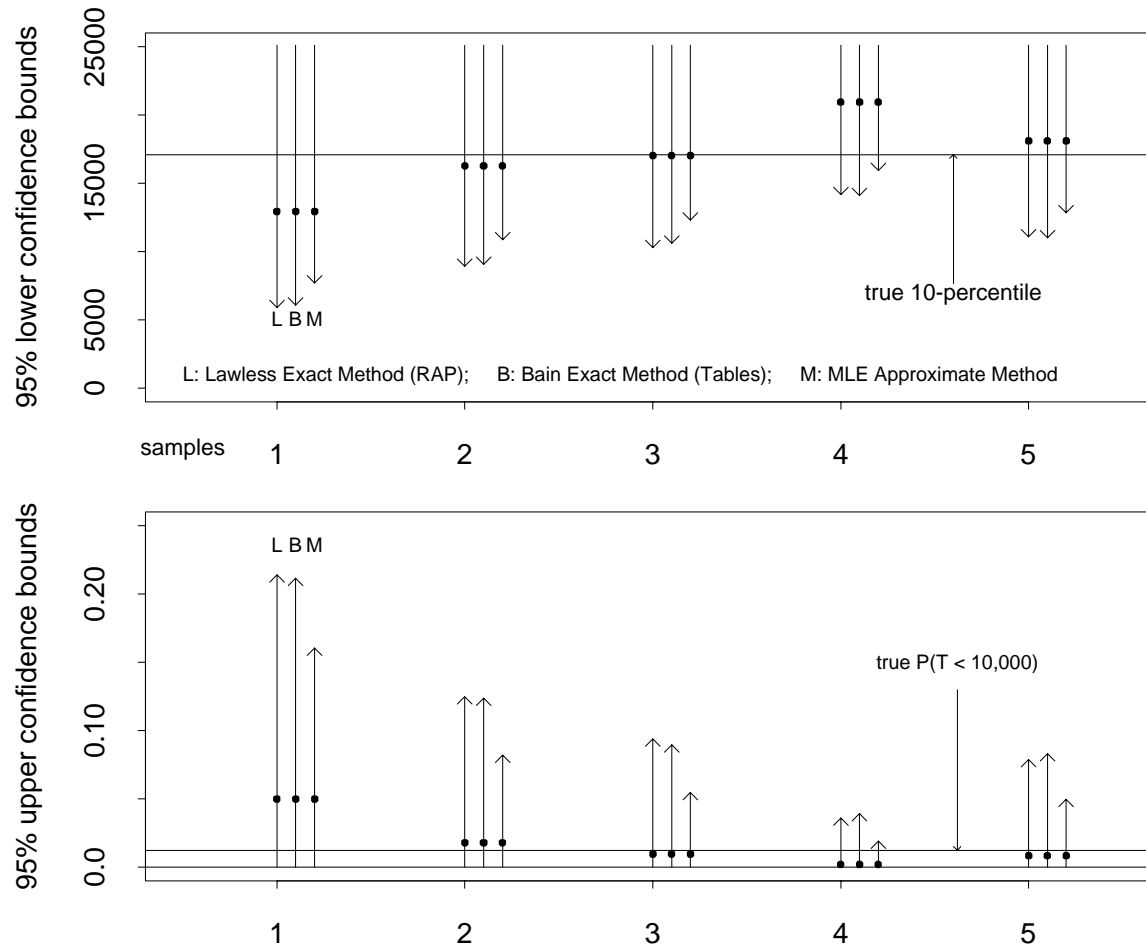
- For $\gamma > .5$ get a $100(2\gamma - 1)\%$ confidence interval by $[\hat{\theta}_{L,\gamma}, \hat{\theta}_{U,\gamma}]$

$$P(\hat{\theta}_{L,\gamma} \leq \theta \leq \hat{\theta}_{U,\gamma}) = 2\gamma - 1 = \gamma^*$$

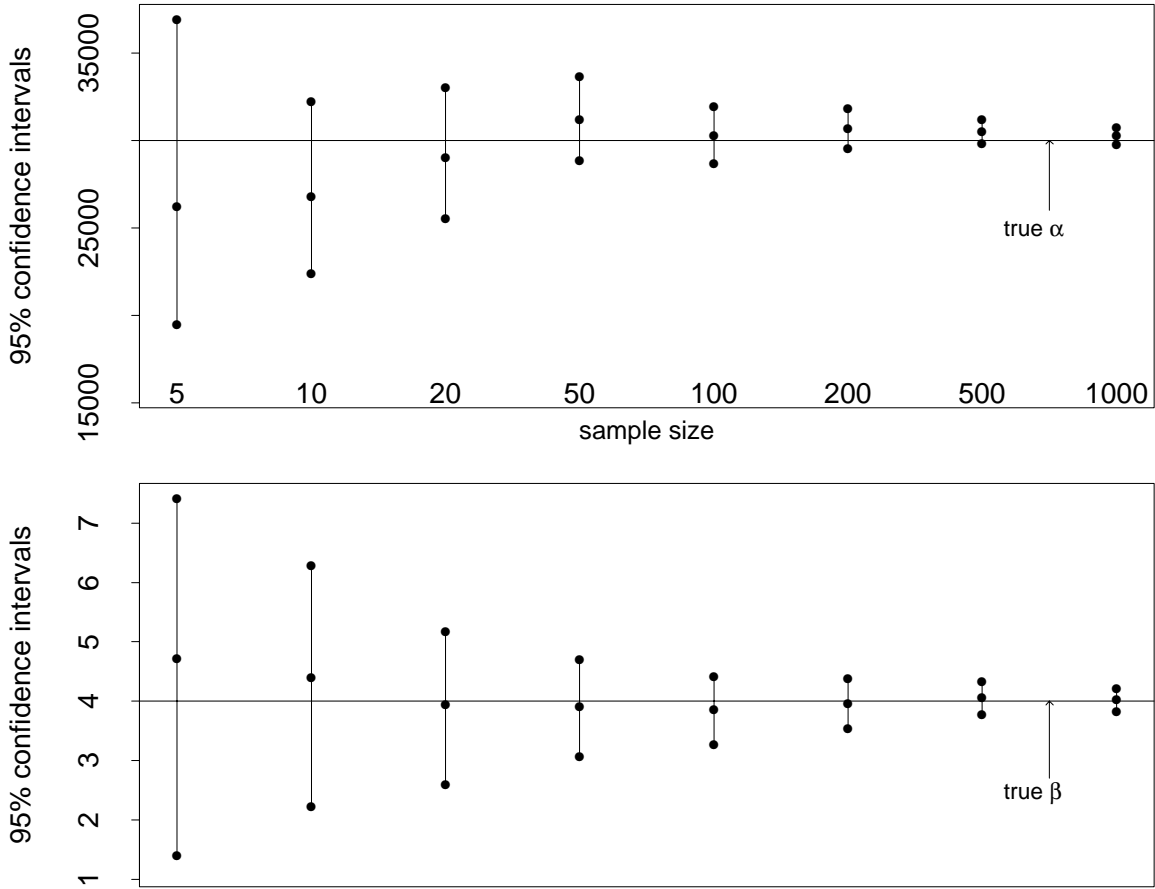
Confidence Bounds & Sampling Variation, $n = 10$



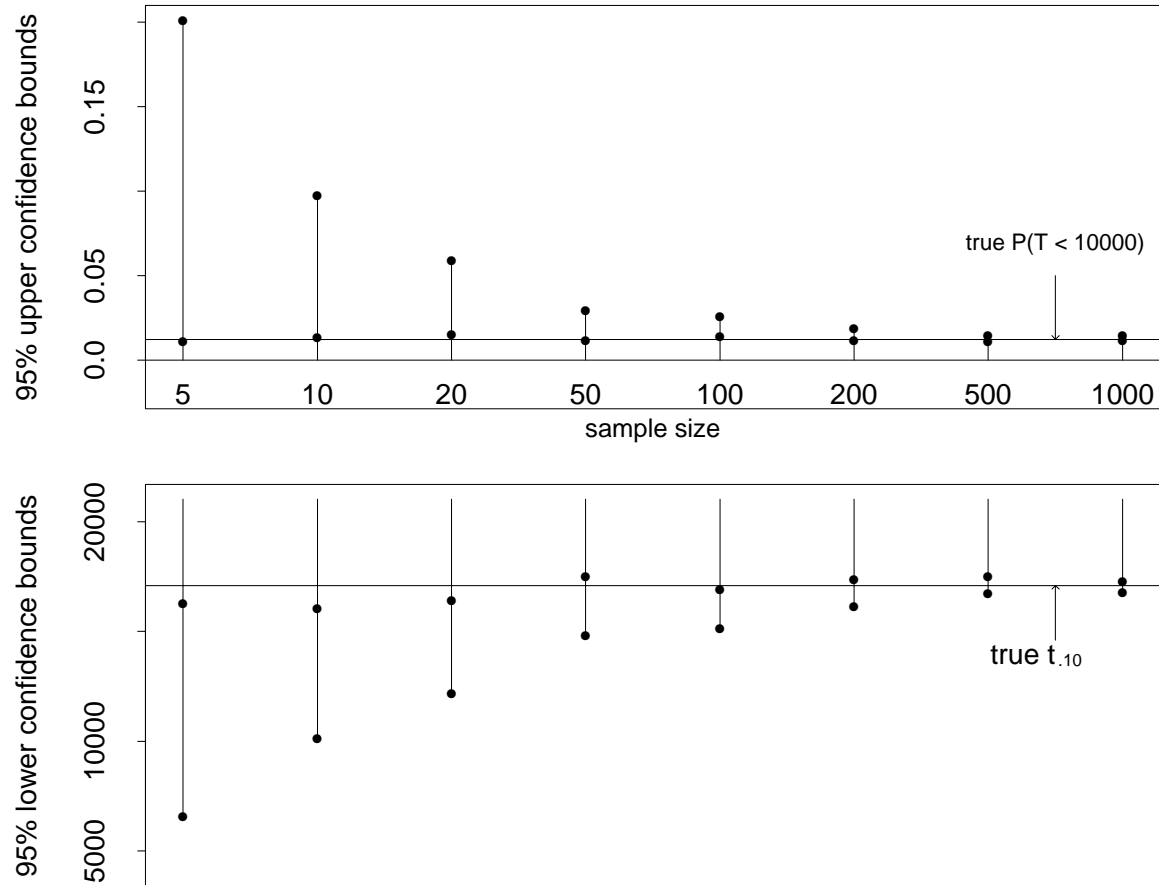
Confidence Bounds & Sampling Variation, $n = 10$



Confidence Bounds & Effect of n



Confidence Bounds & Effect of n



Incomplete Data or Censored Failure Times



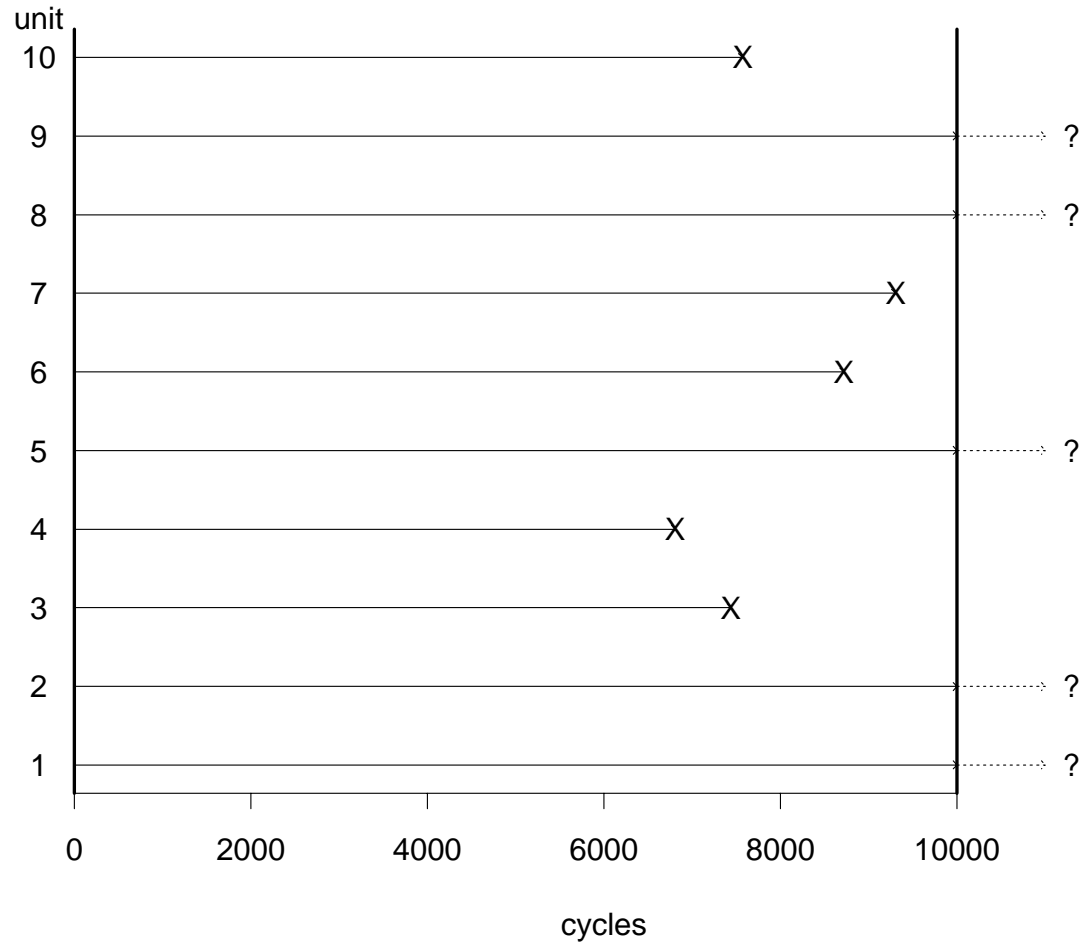
- **Type I censoring or time censoring:**
units are tested until failure or until a prespecified time has elapsed
- **Type II censoring or failure censoring:**
only the r lowest values of the total sample of size n become known
this shows up when we put n units on test at the same time
and terminate the test after the first r units have failed
- **Interval censoring, inspection data, grouped data:**
 i^{th} unit is only known to have failed between two known
inspection time points, i.e., failure time T_i falls in $(s_i, e_i]$
only bracketing intervals (s_i, e_i) , $i = 1, \dots, n$, become known

Incomplete Data or Censored Failure Times (continued)

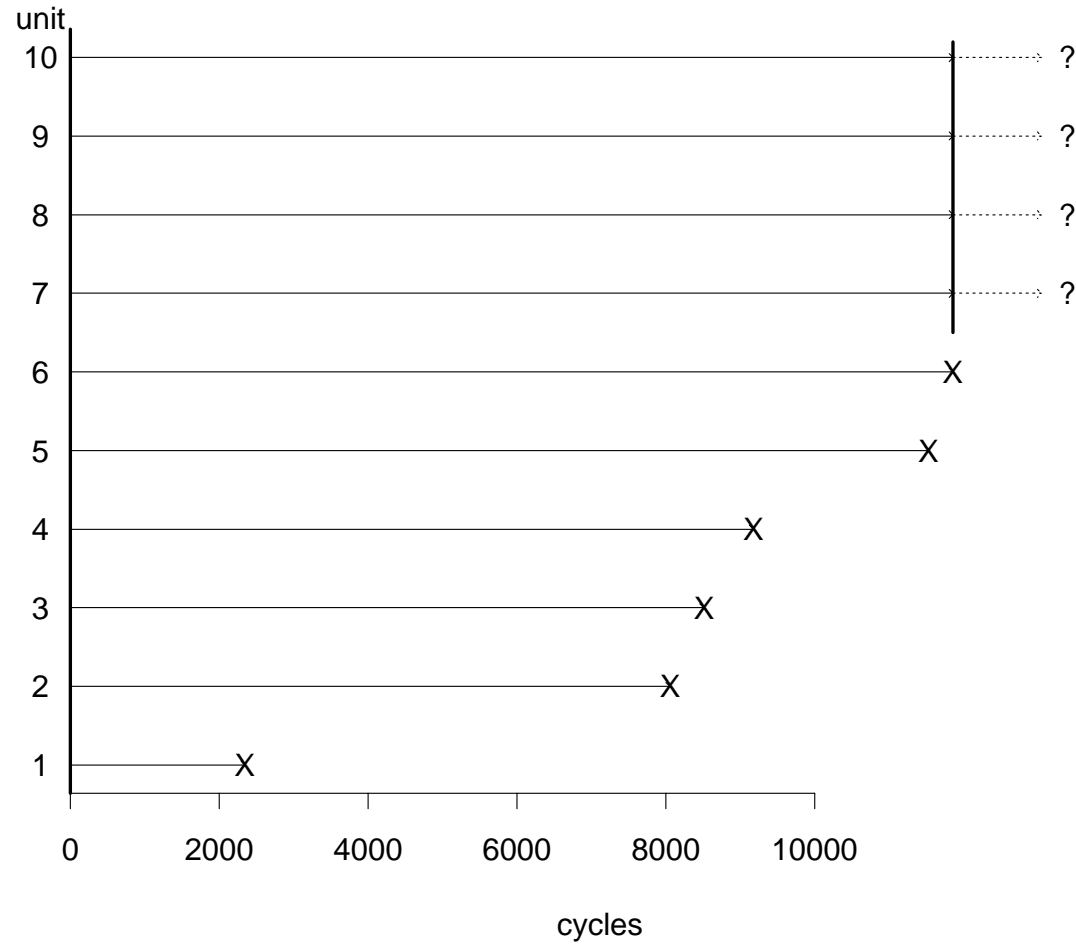


-
- **Random right censoring:**
units are observed until failed or removed from observation due to other causes (different failure modes, competing risks)
 - **Multiple right censoring:**
units are put into service at different times and times to failure or censoring are observed
 - Data can also combine several of the above censoring phenomena
 - It is important that the censoring mechanism should **not** correlate with the (potential) failure times, i.e., no censoring of anticipated failures
 - **All data (censored & uncensored) should enter analysis** otherwise bias results

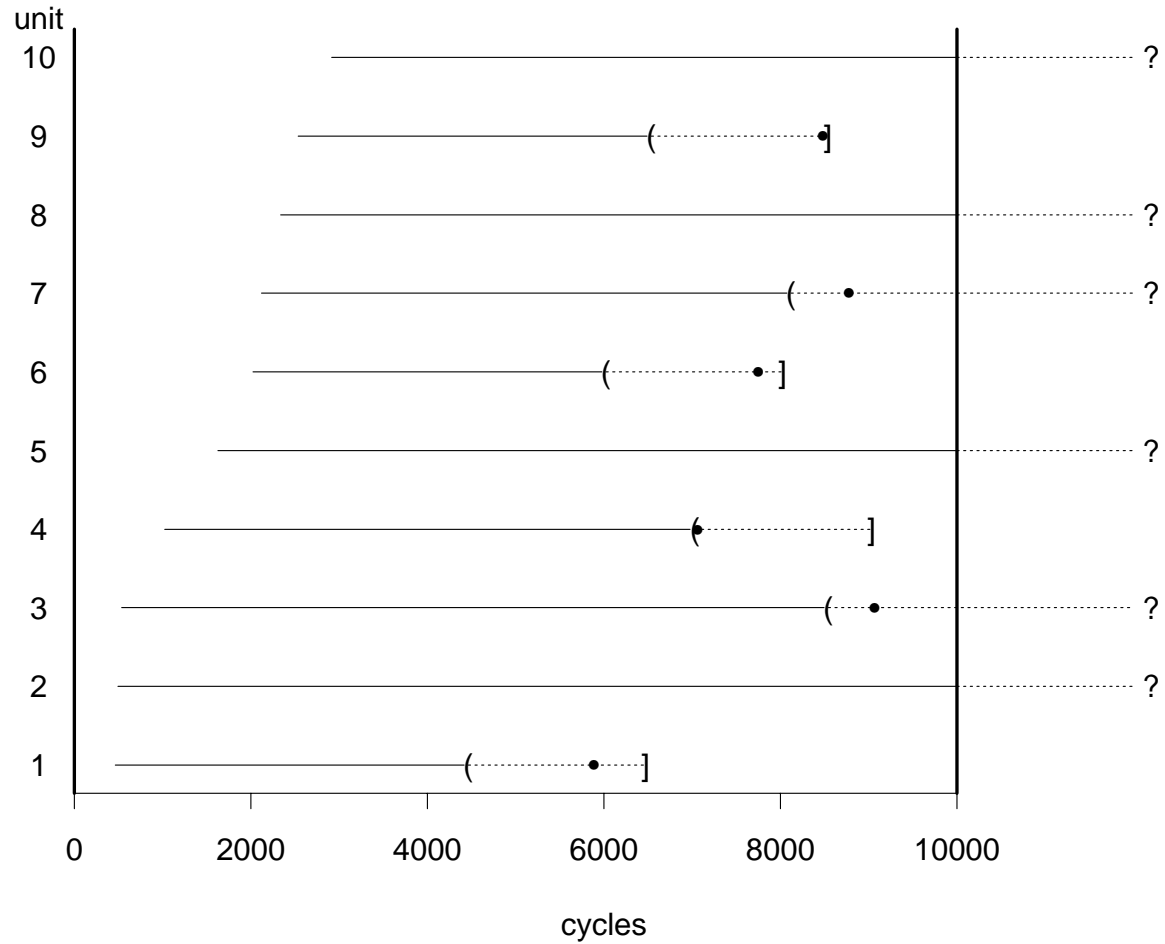
Type I or Time Censored Data: $n = 10$



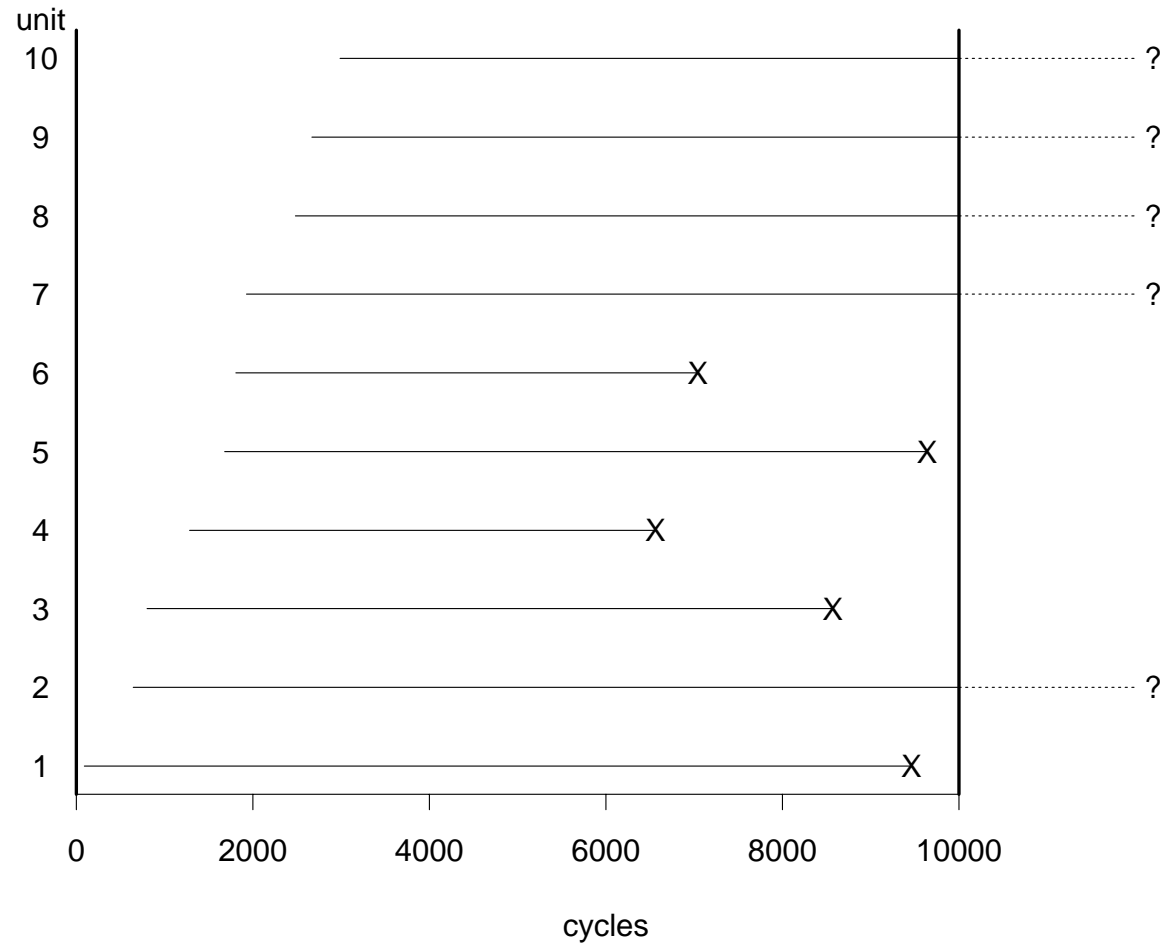
Type II or Failure Censored Data: $n = 10$



Interval Censored Data: $n = 10$



Multiply Right Censored Data: $n = 10$



Nonparametric MLEs of CDF



- For complete, type I & II censored data the nonparametric MLE is

$$\widehat{F}(t) = \frac{\#\{T_i \leq t; i = 1, \dots, n\}}{n},$$

– $-\infty < t < \infty$ for complete sample,

– $-\infty < t \leq t_0$ for type I censored (at t_0) sample,

– $-\infty < t \leq T_{(r)}$ for type II censored (at $T_{(r)}$, r^{th} smallest failure time)

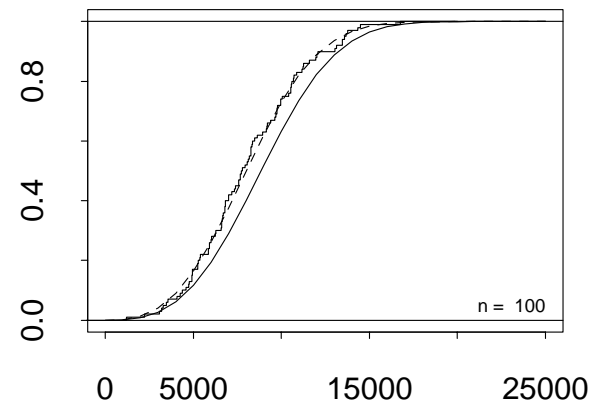
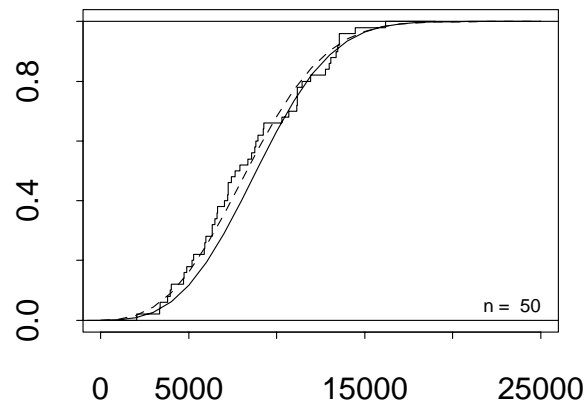
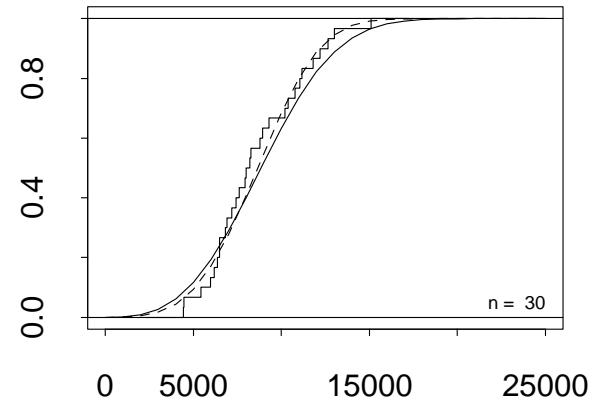
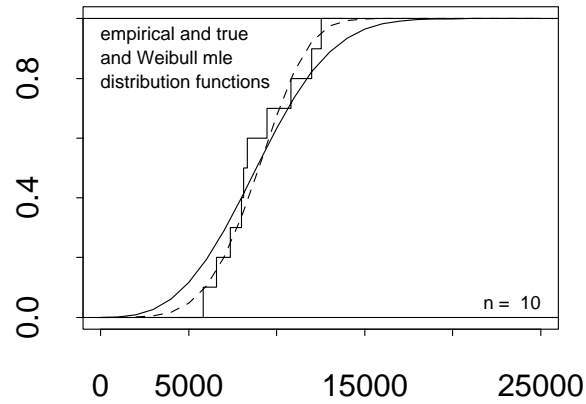
- For multiply right censored data with failures at $t_1^* < \dots < t_k^*$ the nonparametric MLE (Kaplan-Meier or Product Limit Estimator) is

$$\widehat{F}(t) = 1 - \left[(1 - \hat{p}_1)^{\delta_1(t)} \dots (1 - \hat{p}_k)^{\delta_k(t)} \right], \quad \delta_i(t) = \begin{cases} 1 & \text{for } t_i^* \leq t \\ 0 & \text{for } t_i^* > t \end{cases}$$

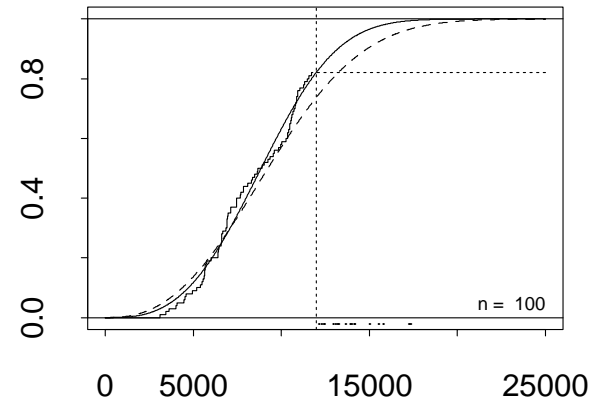
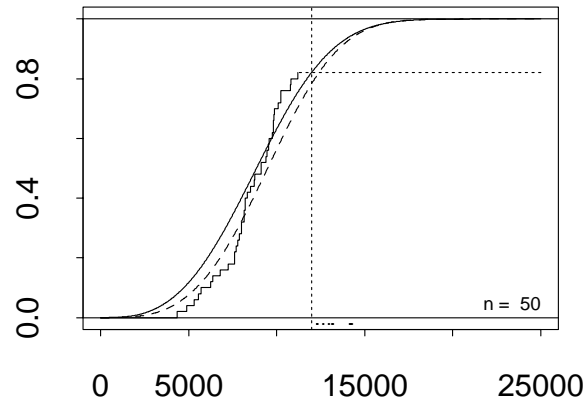
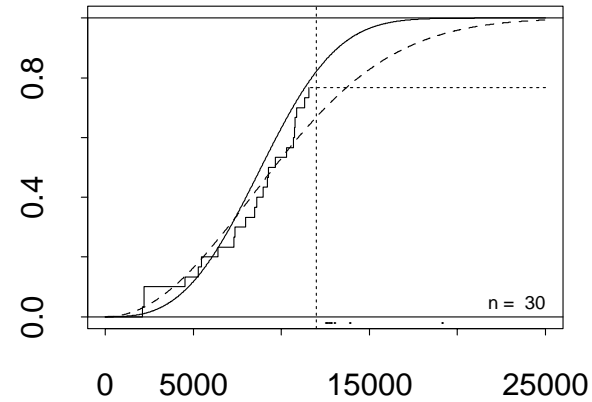
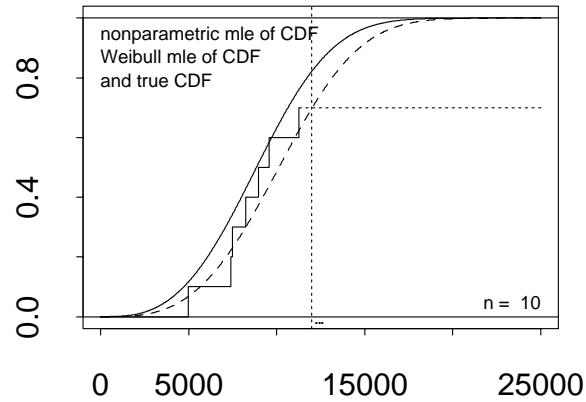
where $\hat{p}_i = d_i/n_i$, $n_i = \#$ units known to be at risk just prior to t_i^* and $d_i = \#$ units that failed at t_i^*

- The nonparametric MLE for interval censored data is complicated

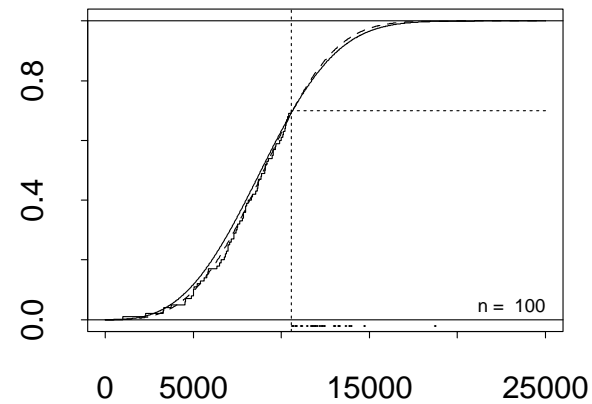
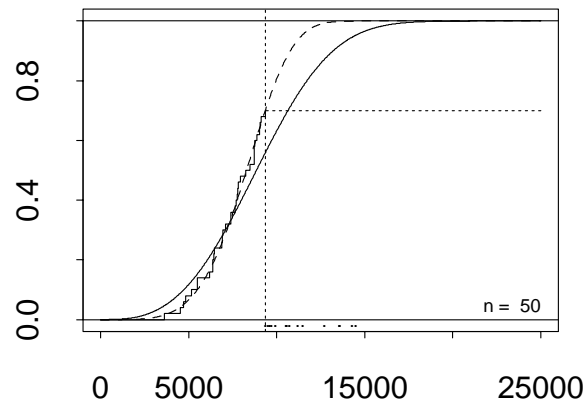
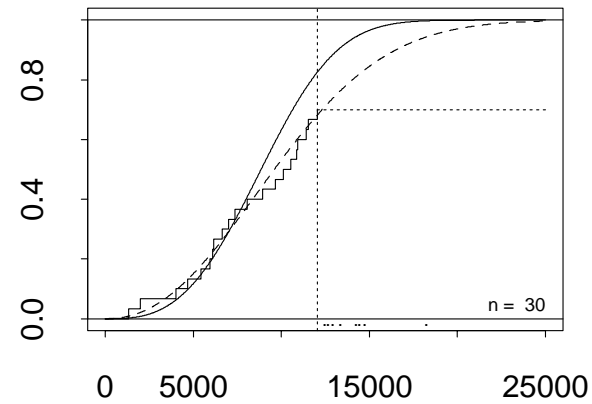
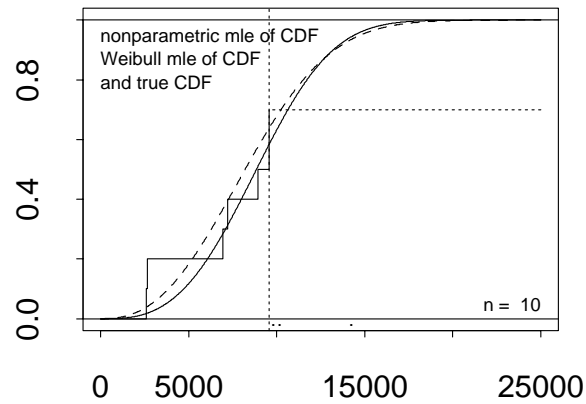
Empirical CDF (complete samples)



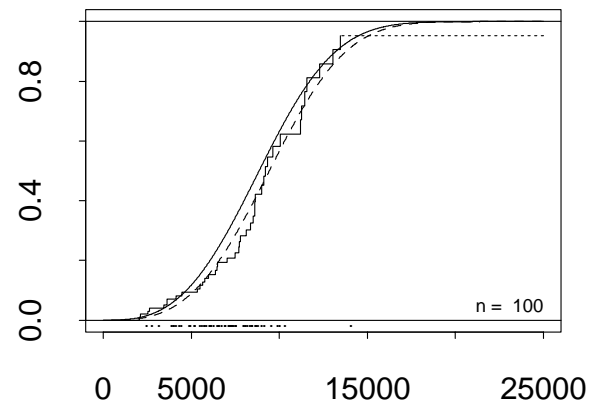
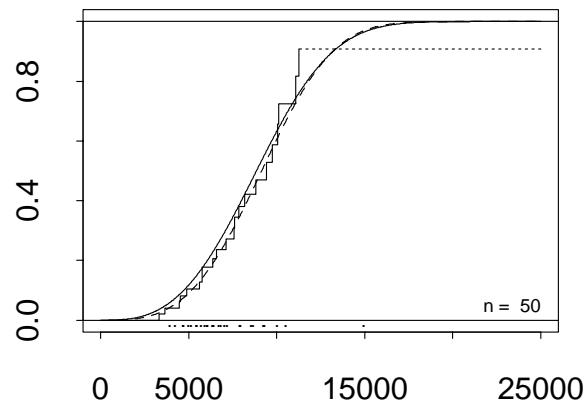
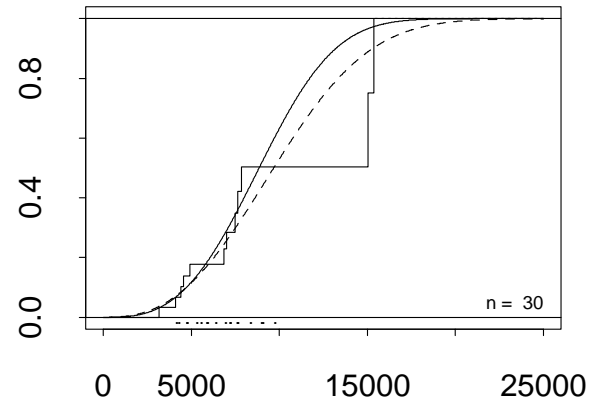
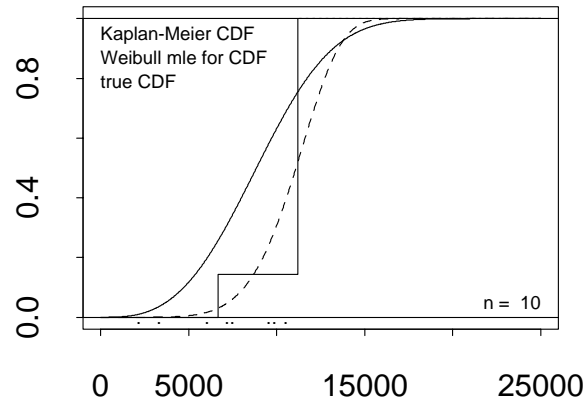
Nonparametric MLE for Type I Censored Data



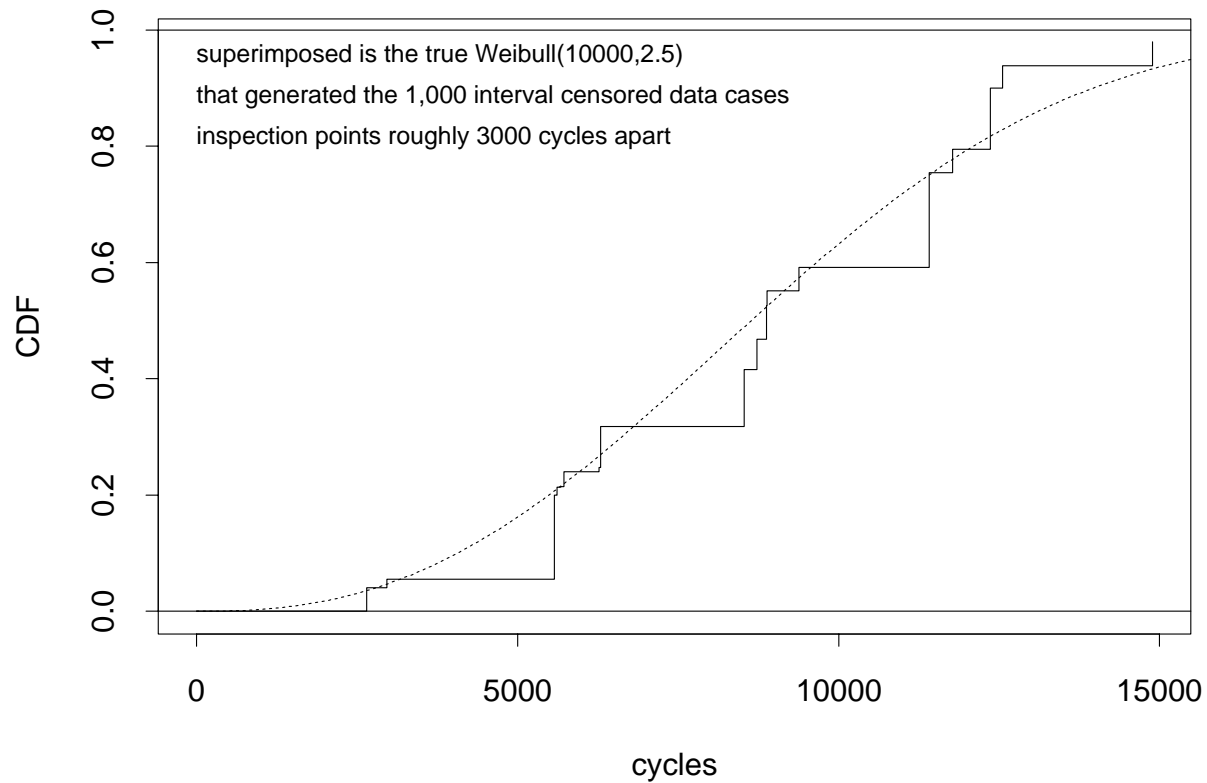
Nonparametric MLE for Type II Censored Data



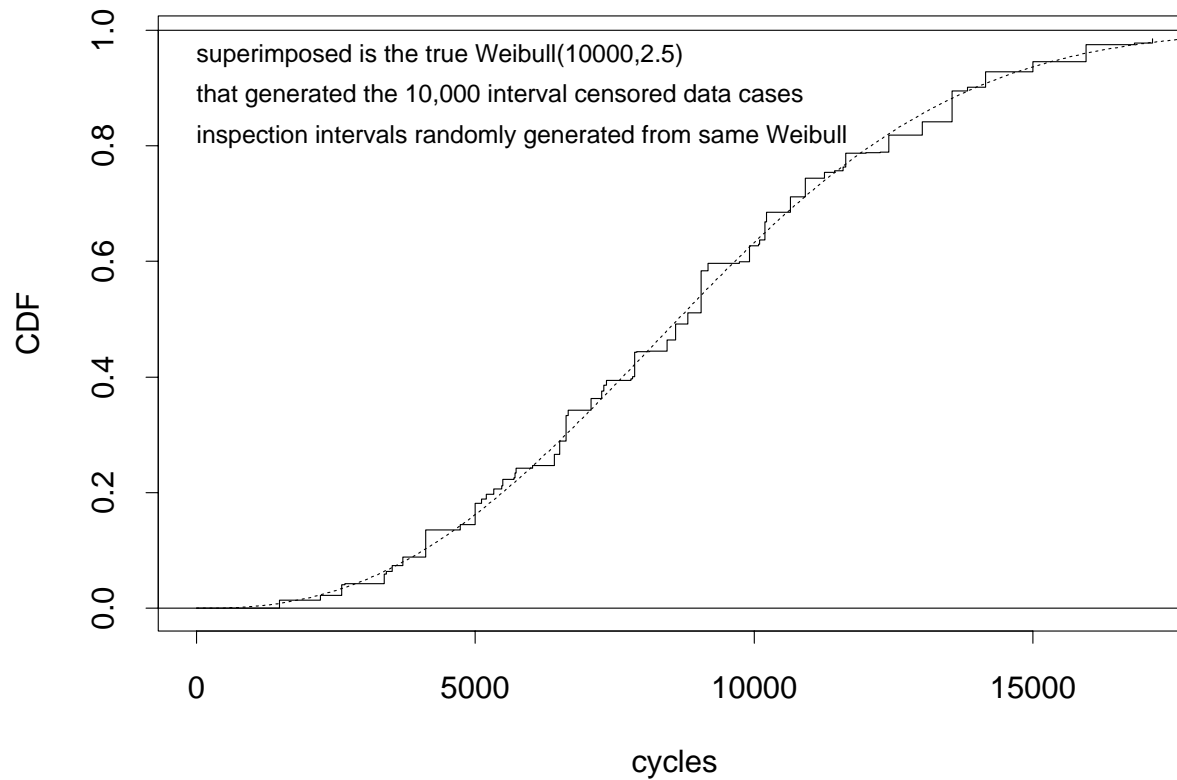
Kaplan-Meier Estimates (multiply right censored samples)



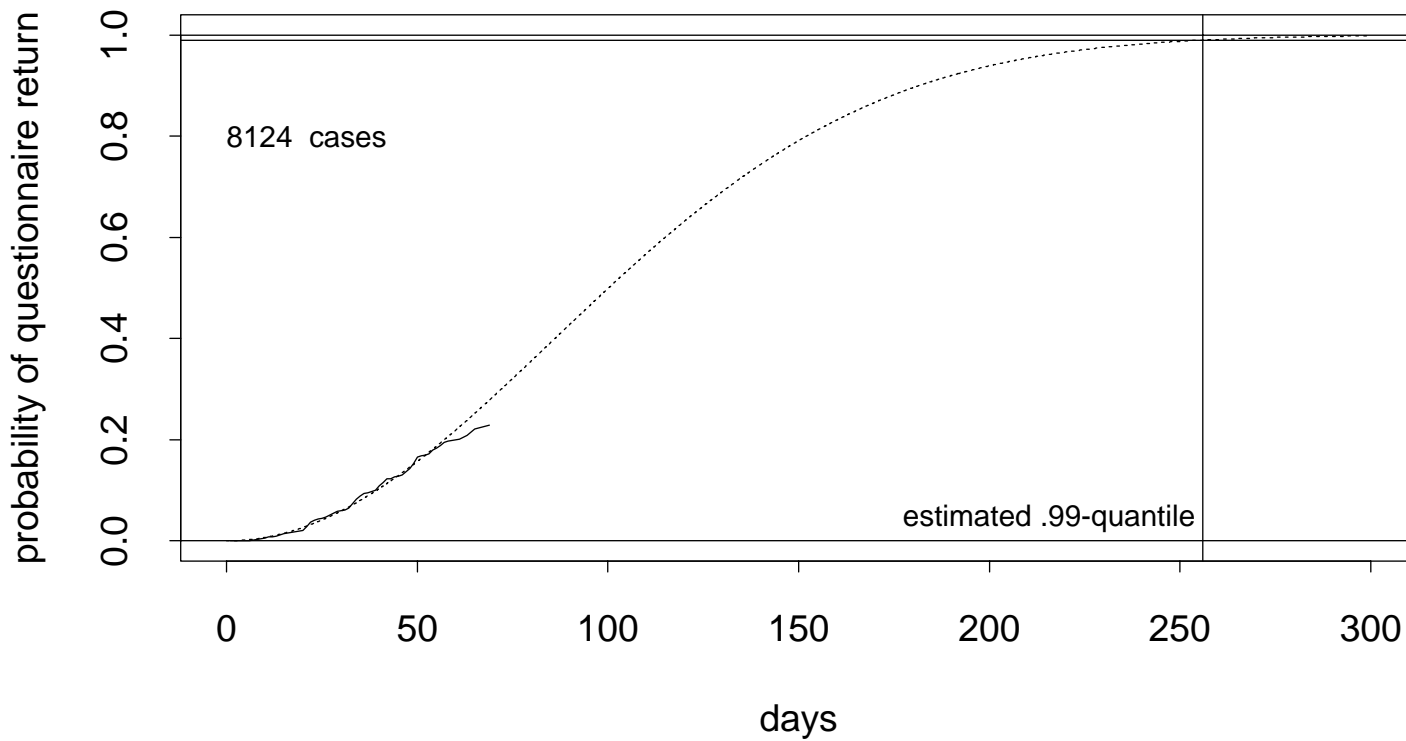
Nonparametric MLE for Interval Censored Data



Nonparametric MLE for Interval Censored Data



Application to Y2K Questionnaire Return Data



Plotting Positions for Weibull Plots (Censored Case)



- For complete samples we plotted $(\log[T_{(i)}], \log[-\log(1 - p_i)])$ with

$$p_i = \frac{i - .5}{n} = \frac{1}{2} [\widehat{F}(T_{(i)}) + \widehat{F}(T_{(i-1)})] = \frac{1}{2} \left[\frac{i}{n} + \frac{i-1}{n} \right]$$

- For type I & II & multiply right censored samples

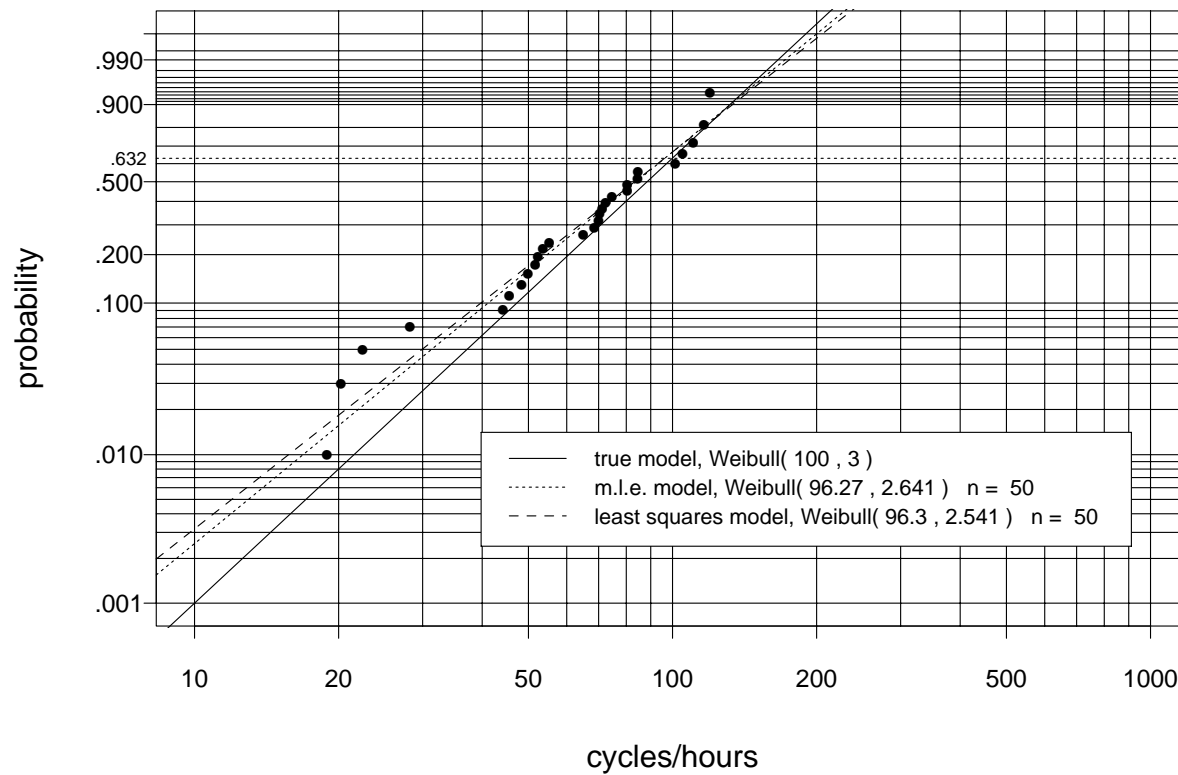
we plot in analogy $(\log[T_{(i)}^*], \log[-\log(1 - p_i)])$, where

$T_{(1)}^* < \dots < T_{(k)}^*$ are the k distinct observed failure times and

$$p_i = \frac{1}{2} [\widehat{F}(T_{(i)}^*) + \widehat{F}(T_{(i-1)}^*)]$$

and $\widehat{F}(t)$ is the nonparametric MLE of $F(t)$ for the censored sample

Weibull Plot, Multiply Right Censored Sample $n = 50$



Software: MLE's & Confidence Bounds for Censored Data



- For complete & type II censored data RAP computes exact coverage confidence bounds for α , β , t_p , and $P(T \leq t)$.
- WEIBREG and commercial software (Weibull++, WeibullSMITH, common statistical packages) compute approximate confidence bounds for above targets, based on large sample m.l.e. theory
- RAP & WEIBREG are Boeing code, runs within DOS mode of Windows95 (interface clumsy, but job gets done), contact me
- Boeing has a site license for Weibull++ Version 4 in the Puget Sound area (with upgrade pressure) contact David Twigg (425) 266-7919, david.twigg@pss.boeing.com
- There is a Weibull++ Version 6 out

Other Software



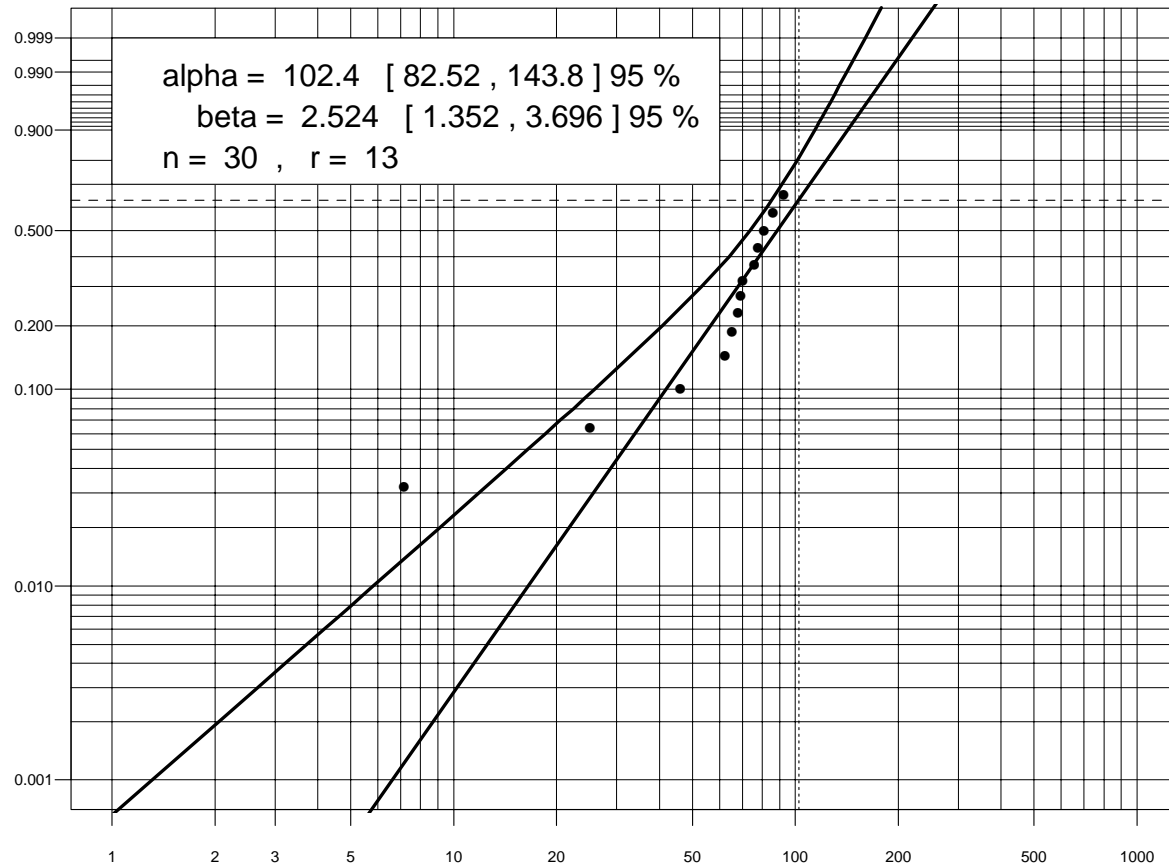
- S-Plus Statistical data analysis environment (\approx \$2000 for PC licence)
- R is a GNU version of S-plus, freely available
either from: <http://cran.r-project.org/> for lots of operating systems or inside Boeing for Windows
http://starfly.ca.boeing.com/KIRTSweb/oss_toolkit/toolkit/optional.html#L048
- Both S-Plus and R have Terry Therneau's survival analysis package built in or as add-on. Using it takes some sophistication.
- My web-based Weibull analysis tool is based on R
<http://www.rt.cs.boeing.com/MEA/stat/scholz/weibull.html>
- Also freely available is the very user friendly package SPLIDA by Bill Meeker which can be added to S-Plus 4.5 or S-Plus 2000
<http://www.public.iastate.edu/~splida/>

Software in Progress (Needs Pull!)

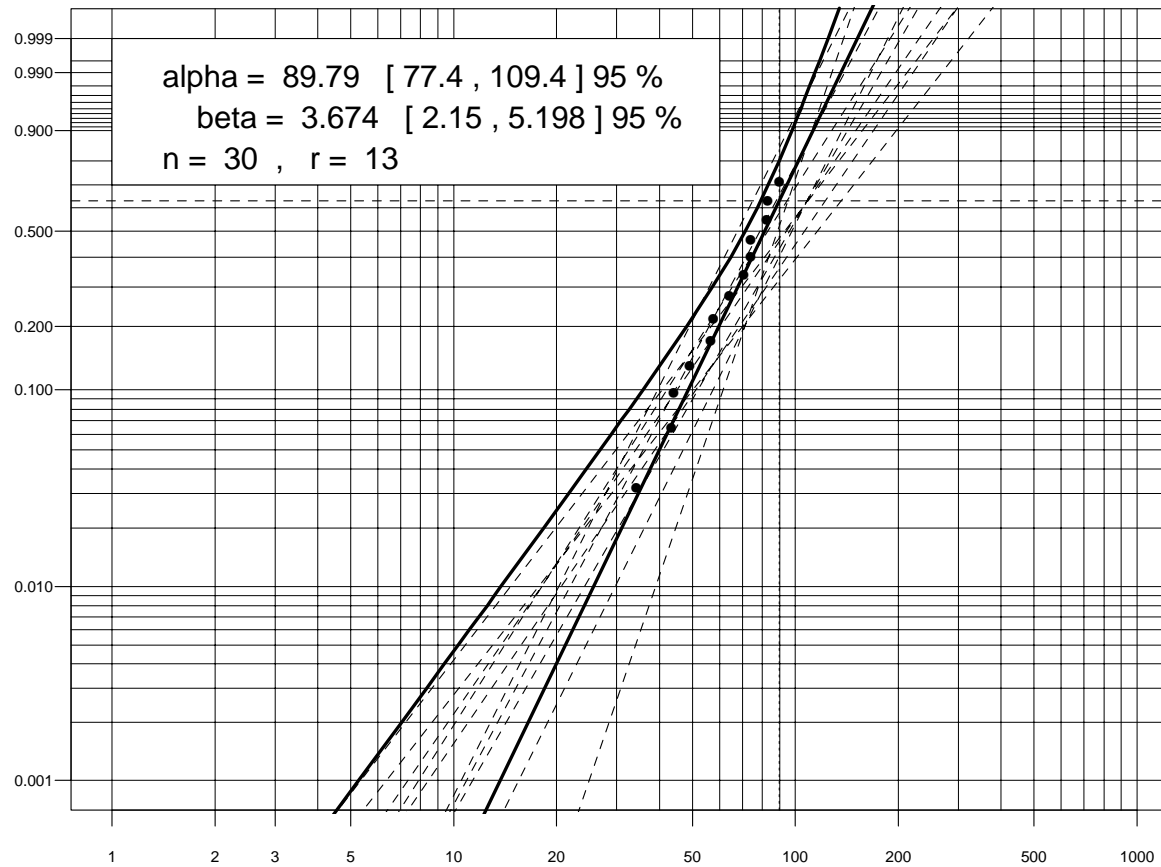


- The current version of web-based Weibull analysis tool and also most other versions based on large sample approximations for the MLE's have some defect for extreme sample with very few failures: Confidence bounds for quantiles t_p are not always monotone increasing in p , which does not make sense. Also quantile and probability bounds are not inverses to each other.
- I have an updated version of WEIBREG based on a new method that avoids this problem.
It also includes a bootstrap version of the new monotone method. However, it is not yet released.
Quantile and probability bounds are true inverses of each other.
- The plan is to update the web based tool and to also create web based versions of RAP and ABVAL to bypass the legacy issue.

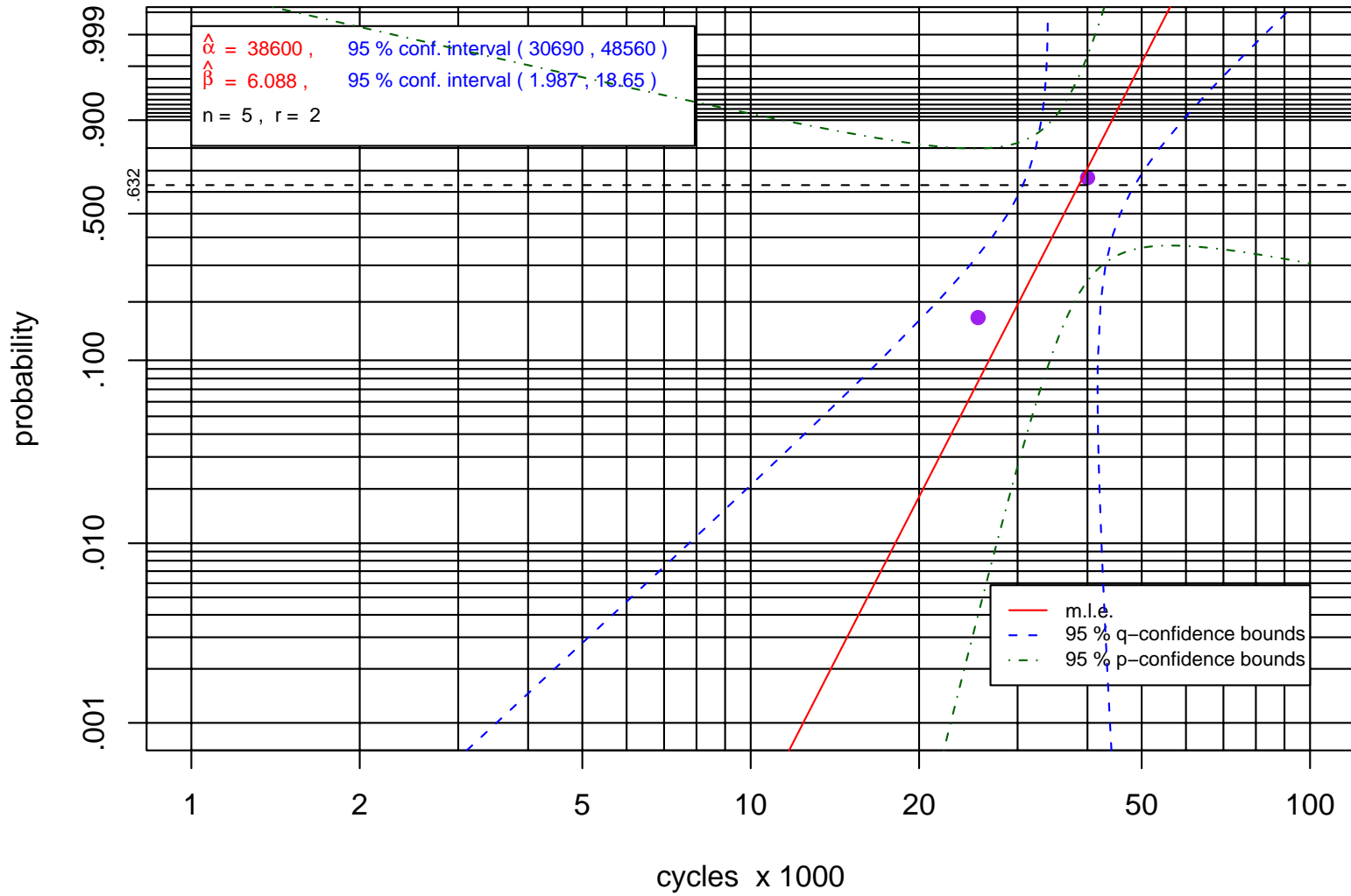
Weibull Plot With Confidence Bound



Weibull Plot With Confidence Bound (Variation)



Problem Weibull Plot



Data: 15900⁺, 20000⁺, 25500, 36000⁺, 40010

Weibull Regression Model & Analysis



- Recall $t_p = \alpha[-\log(1 - p)]^{1/\beta}$ or

$$\log(t_p) = \log(\alpha) + w_p/\beta \quad \text{with} \quad w_p = \log[-\log(1 - p)]$$

- Often we deal with failure data collected under different conditions
 - different part types
 - different environmental conditions
 - different part users
- Linear regression model for $\log(\alpha)$ (multiplicative on α)

$$\log(\alpha_i) = b_1 Z_{i1} + \dots + b_p Z_{ip} \quad \text{typically } Z_{i1} \equiv 1$$

where Z_{i1}, \dots, Z_{ip} are known covariates for i^{th} unit

- Now we have $p + 1$ unknown parameters β, b_1, \dots, b_p
- parameter estimates & confidence bounds using MLEs

Accelerated Life Testing: Inverse Power Law



- Units last too long under normal usage conditions
- Increase “stress” to accelerate failure time
- Increase voltage and accelerate life via inverse power law

$$T(\text{Volt}) = T(\text{Volt}_U) \left(\frac{\text{Volt}}{\text{Volt}_U} \right)^c, \quad \text{where usually } c < 0$$

this means

$$\alpha(\text{Volt}) = \alpha(\text{Volt}_U) \left(\frac{\text{Volt}}{\text{Volt}_U} \right)^c$$

$$\begin{aligned} \text{or } \log [\alpha(\text{Volt})] &= \log [\alpha(\text{Volt}_U)] + c \log (\text{Volt}) - c \log (\text{Volt}_U) \\ &= b_1 + b_2 Z \end{aligned}$$

with

$$b_1 = \log [\alpha(\text{Volt}_U)] - c \log (\text{Volt}_U), \quad b_2 = c, \quad \text{and } Z = \log (\text{Volt})$$

Accelerated Life Testing: Arrhenius Model



- Another way of accelerating failure in processes involving chemical reaction rates is to increase the temperature
- Arrhenius proposed the following acceleration model with temperature measured in Kelvin ($\text{temp}K = \text{temp}^{\circ}C + 273.15$)

$$T(\text{temp}) = T(\text{temp}_U) \exp \left[\frac{\kappa}{\text{temp}} - \frac{\kappa}{\text{temp}_U} \right]$$

or

$$\begin{aligned} \log [\alpha(\text{temp})] &= \log [\alpha(\text{temp}_U)] + \frac{\kappa}{\text{temp}} - \frac{\kappa}{\text{temp}_U} \\ &= b_1 + b_2 Z \end{aligned}$$

with

$$b_1 = \log [\alpha(\text{temp}_U)] - \frac{\kappa}{\text{temp}_U}, \quad b_2 = \kappa, \quad \text{and} \quad Z = \text{temp}^{-1}$$

Pooling Data With Different α 's



- Suppose we have three groups of failure data

$$T_1, \dots, T_{n_1} \sim \mathcal{W}(\tilde{\alpha}_1, \beta), \quad T_{n_1+1}, \dots, T_{n_1+n_2} \sim \mathcal{W}(\tilde{\alpha}_2, \beta),$$

$$T_{n_1+n_2+1}, \dots, T_{n_1+n_2+n_3} \sim \mathcal{W}(\tilde{\alpha}_3, \beta)$$

- We can analyze the whole data set of $N = n_1 + n_2 + n_3$ values jointly using the following model for α_j and dummy covariates $Z_{1,j}$, $Z_{2,j}$ & $Z_{3,j}$

$$\log(\alpha_j) = b_1 Z_{1,j} + b_2 Z_{2,j} + b_3 Z_{3,j}, \quad j = 1, 2, \dots, N$$

where $b_1 = \log(\tilde{\alpha}_1)$, $b_2 = \log(\tilde{\alpha}_2) - \log(\tilde{\alpha}_1)$ and $b_3 = \log(\tilde{\alpha}_3) - \log(\tilde{\alpha}_1)$
 $Z_{1,j} = 1$ for $j = 1, \dots, n_1$, $Z_{2,j} = 1$ for $j = n_1 + 1, \dots, n_1 + n_2$, $Z_{2,j} = 0$ else
and $Z_{3,j} = 1$ for $j = n_1 + n_2 + 1, \dots, N$ & $Z_{3,j} = 0$ else.

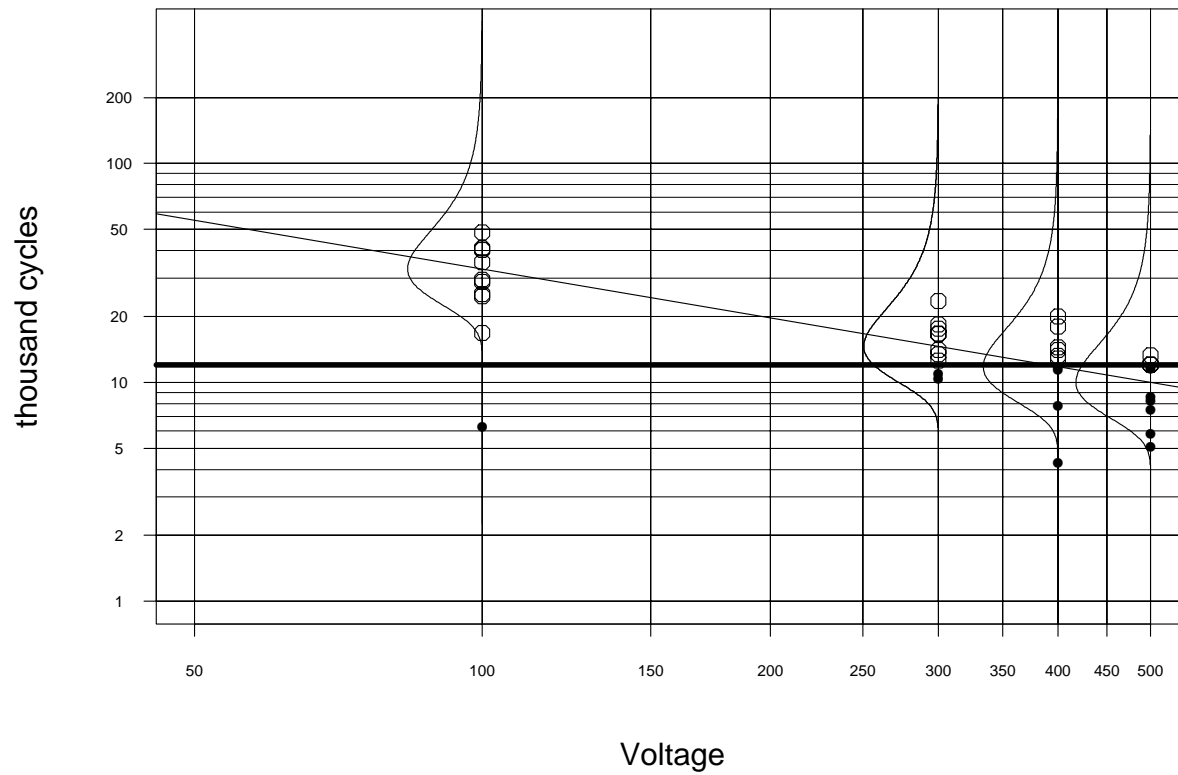
- Advantage: Smaller estimation error
all data are used in estimating the (assumed) common β

Pooling Data (continued)

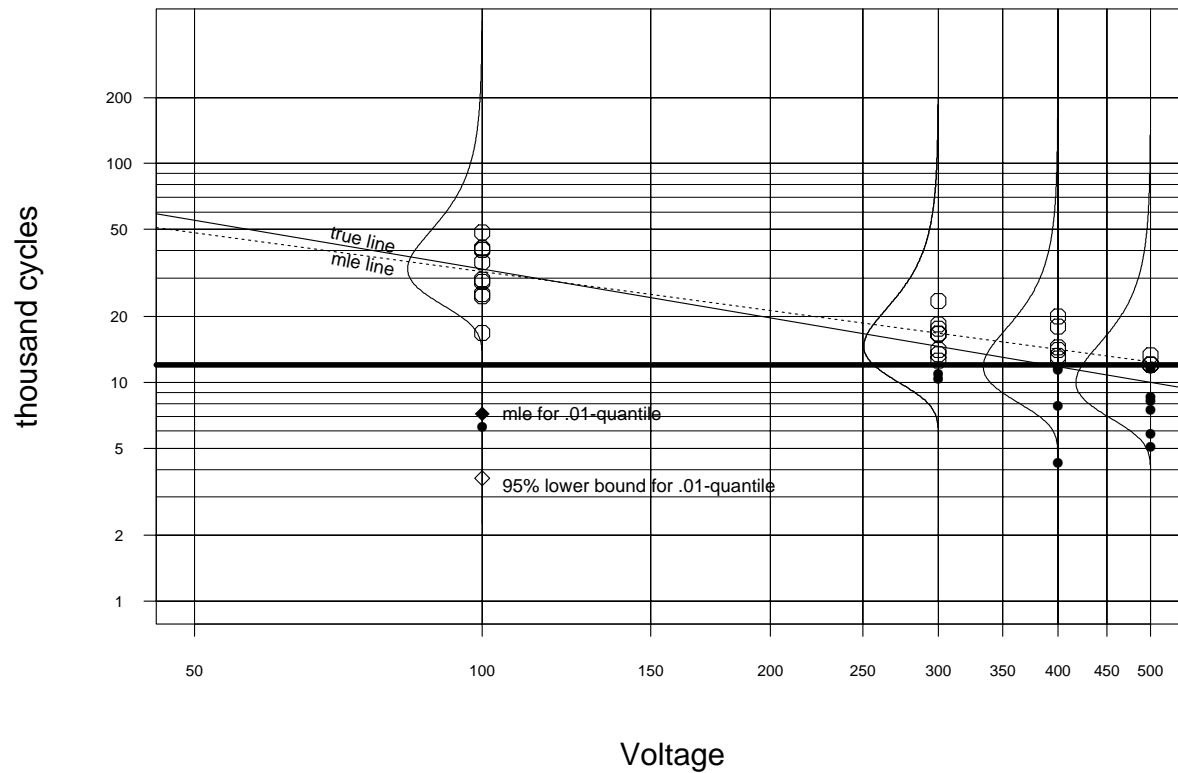


$$\begin{aligned}\log(\alpha_1) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 0 = \log(\tilde{\alpha}_1) \\ &\quad \vdots \qquad \qquad \qquad \quad \vdots \\ \log(\alpha_{n_1}) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 0 = \log(\tilde{\alpha}_1) \\ &\quad \vdots \qquad \qquad \qquad \quad \vdots \\ \log(\alpha_{n_1+1}) &= b_1 \cdot 1 + b_2 \cdot 1 + b_3 \cdot 0 = \log(\tilde{\alpha}_1) + [\log(\tilde{\alpha}_2) - \log(\tilde{\alpha}_1)] = \log(\tilde{\alpha}_2) \\ &\quad \vdots \qquad \qquad \qquad \quad \vdots \\ \log(\alpha_{n_1+n_2}) &= b_1 \cdot 1 + b_2 \cdot 1 + b_3 \cdot 0 = \log(\tilde{\alpha}_1) + [\log(\tilde{\alpha}_2) - \log(\tilde{\alpha}_1)] = \log(\tilde{\alpha}_2) \\ &\quad \vdots \qquad \qquad \qquad \quad \vdots \\ \log(\alpha_{n_1+n_2+1}) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 1 = \log(\tilde{\alpha}_1) + [\log(\tilde{\alpha}_3) - \log(\tilde{\alpha}_1)] = \log(\tilde{\alpha}_3) \\ &\quad \vdots \qquad \qquad \qquad \quad \vdots \\ \log(\alpha_N) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 1 = \log(\tilde{\alpha}_1) + [\log(\tilde{\alpha}_3) - \log(\tilde{\alpha}_1)] = \log(\tilde{\alpha}_3)\end{aligned}$$

Accelerated Life Testing: Model & Data



Accelerated Life Testing: Model, Data & MLEs



Analysis for Data from Exponential Distribution



- $T \sim \mathcal{E}(\theta)$ Exponential with mean θ , i.e., $\mathcal{E}(\theta) = \mathcal{W}(\alpha = \theta, \beta = 1)$
- It suffices to get confidence bounds for θ since all other quantities of interest are explicit and monotone functions of θ

$$F_{\theta}(t_0) = P_{\theta}(T \leq t_0) = 1 - \exp(-t_0/\theta)$$

$$R_{\theta}(t_0) = P_{\theta}(T > t_0) = \exp(-t_0/\theta)$$

and

$$t_p(\theta) = \theta [-\log(1 - p)]$$

Complete & Type II Censored Exponential Samples



- $T \sim \mathcal{E}(\theta)$ Exponential with mean θ , $\mathcal{E}(\theta) = \mathcal{W}(\alpha = \theta, \beta = 1)$
- For complete exponential samples $T_1, \dots, T_n \sim \mathcal{E}(\theta)$
or for a type II censored sample of this type,
i.e., with observed r lowest values $T_{(1)} \leq \dots \leq T_{(r)}$, $1 \leq r \leq n$,
one gets $100\gamma\%$ lower confidence bounds for θ by computing

$$\hat{\theta}_{L,\gamma} = \frac{2 \times TTT}{\chi_{2r,\gamma}^2},$$

where $TTT = T_{(1)} + \dots + T_{(r)} + (n - r)T_{(r)} = \mathbf{Total\ Time\ on\ Test}$,
and $\chi_{2r,\gamma}^2 = \gamma$ -quantile of the χ_{2r}^2 distribution
get this in Excel via = GAMMAINV(γ , r , 1) or = CHIINV($1 - \gamma$, $2r$)/2

- These bounds have exact confidence coverage properties

Multiply Right Censored Exponential Data



- Here we define $TTT = T_1 + \dots + T_n$ as the total time on test but now T_i is either the observed failure time of the i^{th} part or the observed right censoring time of the i^{th} part

The number r of observed failures is random here, $0 \leq r \leq n$

- *Approximate* $100\gamma\%$ lower confidence bounds for θ are computed as

$$\hat{\theta}_{L,\gamma} = \frac{2 \times TTT}{\chi_{2r+2,\gamma}^2}$$

- This also holds for $r = 0$, i.e., no failures at all over exposure period

Weibull Shortcut Methods for Known β



- $T \sim \mathcal{W}(\alpha, \beta)$ (Weibull) $\Rightarrow T^\beta \sim \mathcal{E}(\theta)$ Exponential with mean $\theta = \alpha^\beta$
- As $100\gamma\%$ lower confidence bound for α compute

$$\hat{\alpha}_{L,\gamma} = \left(\frac{2 \times TTT(\beta)}{\chi_{f,\gamma}^2} \right)^{1/\beta}$$

where for complete or type II censored samples we take $f = 2r$ and

$$TTT = T_{(1)}^\beta + \cdots + T_{(r)}^\beta + (n - r)T_{(r)}^\beta$$

and for multiply right censored data we take $f = 2r + 2$ and

$$TTT = T_1^\beta + \cdots + T_n^\beta .$$

- Sensitivity should be studied by repeating analyses for several β 's

A- and B-Allowables, ABVAL Program



- My first Weibull work led to the program ABVAL (1983) supporting Cecil Parsons and Ron Zabora w.r.t. MIL-HDBK-5
- ABVAL computes A- and B-Allowables based on samples from the **3-parameter Weibull distribution**.
 - The A-Allowable is a 95% lower confidence bound for 99% of the population, i.e., for the .01-quantile.
 - The B-Allowable is a 95% lower confidence bound for 90% of the population, i.e., for the .10-quantile.
- Hybrid approach of using Mann-Fertig threshold estimate combined with maximum likelihood estimates for α and β , took lot of simulation and tuning and is very specialized in its capabilities
- It is now a method in MIL-HDBK-5, I still maintain software

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