Weibull Reliability Analysis


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Wallodi Weibull

Societies and worked to the last day of his remarkable life. He died on October 12, 1979, in Annecy, France.

Perhaps the greatest tribute to anyone's work is the extent to which it is used and cited by subsequent researchers. The table and graph below, supplied by Mr. Goran Weibull, show the rapidly increasing use of the Weibull distribution and the wide range of fields in which it has been applied. Although the tabulation indicates the variety of applications, it underestimates the total number of references. Another 600 have appeared since the tabulations were prepared.

As some of you know, I am of Hungarian origin. In the Hungarian language, "Wallodi", spelled "Valodi", means "the Real Thing". Wallodi Weibull was the real thing.
Seminal Paper

A STATISTICAL THEORY OF THE STRENGTH OF MATERIALS

by

W. WEIBULL

Professor, Royal Technical University, Stockholm

STOCKHOLM 1939

GENERALSTABENS LITOGRAFISKA ANSTALTS FÖRLAG

STOCKHOLM
The Weibull Distribution

• **Weibull distribution**, useful **uncertainty model** for
  
  – **wearout failure time** $T$
    
    when governed by wearout of weakest subpart
  
  – **material strength** $T$
    
    when governed by embedded flaws or weaknesses,

• It has often been found useful based on empirical data (e.g. Y2K)

• It is also theoretically founded on the **weakest link principle**

\[
T = \min (X_1, \ldots, X_n),
\]

with $X_1, \ldots, X_n$ statistically independent random strengths or failure times of the $n$ “links” comprising the whole.

The $X_i$ must have a natural finite lower endpoint, e.g., link strength $\geq 0$ or subpart time to failure $\geq 0$. 

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Weibull Reliability Analysis—FWS-5/1999—4
Theoretical Basis

- Under weak conditions **Extreme Value Theory** shows\(^1\) that for large \(n\)

\[
P(T \leq t) \approx 1 - \exp \left( - \frac{t - \tau}{\alpha} \right)^\beta \quad \text{for} \quad t \geq \tau, \alpha > 0, \beta > 0
\]

- The above approximation has very much the same spirit as the **Central Limit Theorem** which under some weak conditions on the \(X_i\) asserts that the distribution of \(T = X_1 + \ldots + X_n\) is approximately bell-shaped normal or Gaussian

- Assuming a Weibull model for \(T\), material strength or cycle time to failure, amounts to treating the above approximation as an equality

\[
F(t) = P(T \leq t) = 1 - \exp \left( - \frac{t - \tau}{\alpha} \right)^\beta \quad \text{for} \quad t \geq \tau, \alpha > 0, \beta > 0
\]

Weibull Reproductive Property

If $X_1, \ldots, X_n$ are statistically independent

with $X_i \sim \text{Weibull}(\alpha_i, \beta)$ then

$$P(\min(X_1, \ldots, X_n) > t) = P(X_1 > t) \times \cdots \times P(X_n > t)$$

$$= \prod_{i=1}^{n} \exp \left( - \left[ \frac{t}{\alpha_i} \right]^\beta \right) = \exp \left( - \sum_{i=1}^{n} \left[ \frac{t}{\alpha_i} \right]^\beta \right)$$

$$= \exp \left( - \left[ \frac{t}{\alpha} \right]^\beta \right)$$

Hence $T = \min(X_1, \ldots, X_n) \sim \text{Weibull}(\alpha, \beta)$ with

$$\alpha = \left( \sum_{i=1}^{n} \alpha_i^{-\beta} \right)^{-1/\beta}$$
Weibull Parameters

The Weibull distribution may be controlled by 2 or 3 parameters:

- **the threshold parameter** $\tau$

  $T \geq \tau$ with probability 1

  $\tau = 0 \implies$ 2-parameter Weibull model.

- **the characteristic life or scale parameter** $\alpha > 0$

  $$P(T \leq \tau + \alpha) = 1 - \exp \left( - \left[ \frac{\alpha}{\alpha} \right]^\beta \right) = 1 - \exp(-1) = .632$$

  regardless of the value $\beta > 0$

- **the shape parameter** $\beta > 0$, usually $\beta \geq 1$
2-Parameter Weibull Model

- We focus on analysis using the 2-parameter Weibull model
- Methods and software tools much better developed
- Estimation of $\tau$ in the 3-parameter Weibull model leads to complications
- When a 3-parameter Weibull model is assumed, it will be stated explicitly
Relation of $\alpha$ & $\beta$ to Statistical Parameters

- The expectation or mean value of $T$

$$\mu = E(T) = \int_0^\infty t \, f(t) \, dt = \alpha \Gamma(1 + 1/\beta)$$

with $$\Gamma(t) = \int_0^\infty \exp(-x) \, x^{t-1} \, dx$$

- The variance of $T$

$$\sigma^2 = E(T - \mu)^2 = \int_0^\infty (t - \mu)^2 \, f(t) \, dt = \alpha^2 \left[ \Gamma(1 + 2/\beta) - \Gamma^2(1 + 1/\beta) \right]$$

- $p$-quantile $t_p$ of $T$, i.e., by definition $P(T \leq t_p) = p$

$$t_p = \alpha \left[ - \log(1 - p) \right]^{1/\beta}, \text{ for } p = 1 - \exp(-1) = .632 \quad \Rightarrow \quad t_p = \alpha$$
Weibull Density

- The cumulative distribution function $F(t) = P(T \leq t)$ is just one way to describe the distribution of the random quantity $T$

- The density function $f(t)$ is another representation ($\tau = 0$)

\[
f(t) = F'(t) = \frac{dF(t)}{dt} = \frac{\beta}{\alpha} \left( \frac{t}{\alpha} \right)^{\beta-1} \exp \left( - \left[ \frac{t}{\alpha} \right]^\beta \right) \quad t \geq 0
\]

\[
P(t \leq T \leq t + dt) \approx f(t) \, dt
\]

\[
F(t) = \int_0^t f(x) \, dx
\]
Weibull Density & Distribution Function

Weibull density $\alpha = 10000$, $\beta = 2.5$

total area under density $= 1$

cumulative distribution function

Weibull Reliability Analysis—FWS-5/1999—11
Weibull Densities: Effect of $\tau$

$\alpha = 1000, \quad \beta = 2.5$

The figure shows the effect of varying $\tau$ on the Weibull density distribution, with $\tau = 0$, $\tau = 1000$, and $\tau = 2000$. The parameters $\alpha = 1000$ and $\beta = 2.5$ are fixed.
Weibull Densities: Effect of $\alpha$

![Graph showing Weibull densities for different $\alpha$ values with $\tau = 0$, $\beta = 2.5$]
Weibull Densities: Effect of $\beta$

$\tau = 0, \quad \alpha = 1000$

cycles

probability density

$\beta = 0.5, 1, 2, 4, 7$
A third representation of the Weibull distribution is through the hazard or failure rate function

\[ \lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{\beta}{\alpha} \left( \frac{t}{\alpha} \right)^{\beta-1} \]

- \( \lambda(t) \) is increasing \( t \) for \( \beta > 1 \) (wearout)
- \( \lambda(t) \) is decreasing \( t \) for \( \beta < 1 \)
- \( \lambda(t) \) is constant for \( \beta = 1 \) (exponential distribution)

\[
P(t \leq T \leq t + dt | T \geq t) = \frac{P(t \leq T \leq t + dt)}{P(T \geq t)} \approx \frac{f(t) \, dt}{1 - F(t)} = \lambda(t) \, dt
\]

\[ F(t) = 1 - \exp \left( - \int_0^t \lambda(x) \, dx \right) \quad \text{and} \quad f(t) = \lambda(t) \exp \left( - \int_0^t \lambda(x) \, dx \right) \]
Exponential Distribution

- The exponential distribution is a special case: \( \beta = 1 \) & \( \tau = 0 \)

\[
F(t) = P(T \leq t) = 1 - \exp\left(-\frac{t}{\alpha}\right) \quad \text{for} \quad t \geq 0
\]

- This distribution is useful when parts fail due to random external influences and not due to wear out

- Characterized by the memoryless property,
  a part that has not failed by time \( t \) is as good as new,
  past stresses without failure are water under the bridge

- Good for describing lifetimes of electronic components,
  failures due to external voltage spikes or overloads
Unknown Parameters

• Typically will not know the Weibull distribution: $\alpha, \beta$ unknown

• Will only have sample data $\implies$ estimates $\hat{\alpha}, \hat{\beta}$
  get estimated Weibull model for failure time distribution
  $\implies$ double uncertainty
  uncertainty of failure time & uncertainty of estimated model

• Samples of failure times are sometimes very small,
  only 7 fuse pins or 8 ball bearings tested until failure,
  long lifetimes make destructive testing difficult

• Variability issues are often not sufficiently appreciated
  how do small sample sizes affect our confidence in
  estimates and predictions concerning future failure experiences?
Estimation Uncertainty

A Weibull Population: Histogram for N = 10,000 & Density

- Characteristic life = 30,000
- Shape = 2.5

Graph showing the histogram and density for a Weibull distribution with N = 10,000, Characteristic life of 30,000 cycles, and shape parameter of 2.5. The graph compares the true model with the estimated model from 9 data points.
Weibull Parameters & Sample Estimates

characteristic life $\alpha = 30,000$

shape parameter $\beta = 4$

$p = P(T < t)$

t = p-quantile

parameter estimates from three samples of size $n = 10$

$25390 \quad 3.02$

$33860 \quad 4.27$

$29410 \quad 5.01$
• Using the quantile relationship $t_p = \alpha \left[ -\log(1 - p) \right]^{1/\beta}$

one can generate a Weibull random sample of size $n$ by

– generating a random sample $U_1, \ldots, U_n$

from a uniform $[0, 1]$ distribution

– and computing $T_i = \alpha \left[ -\log(1 - U_i) \right]^{1/\beta}$, $i = 1, \ldots, n$.

– Then $T_1, \ldots, T_n$ can be viewed as a random sample of size $n$

from a Weibull population or Weibull distribution

with parameters $\alpha$ & $\beta$.

• Simulations are useful in gaining insight on estimation procedures
Graphical Methods

- Suppose we have a complete Weibull sample of size $n$: $T_1, \ldots, T_n$

- Sort these values from lowest to highest: $T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(n)}$

- Recall that the $p$-quantile is $t_p = \alpha \left[ - \log(1 - p) \right]^{1/\beta}$

- Compute $t_{p_1} < \ldots < t_{p_n}$ for $p_i = (i - .5)/n$, $i = 1, \ldots, n$

- Plot the points $(T_i, t_{p_i})$, $i = 1, \ldots, n$ and expect these points to cluster around main diagonal
Weibull Quantile-Quantile Plot: Known Parameters
Weibull QQ-Plot: Unknown Parameters

- Previous plot requires knowledge of the unknown parameters $\alpha$ & $\beta$

- Note that

\[
\log(t_p) = \log(\alpha) + w_p/\beta, \quad \text{where} \quad w_p = \log[-\log(1 - p)]
\]

- Expect points $(\log[T(i)], w_{pi}), i = 1, \ldots, n$, to cluster around line with slope $1/\beta$ and intercept $\log(\alpha)$

- This suggests estimating $\alpha$ & $\beta$ from a fitted least squares line
Maximum Likelihood Estimation

• If $t_1, \ldots, t_n$ are the observed sample values one can contemplate
  the probability of obtaining such a sample or of values nearby, i.e.,

$$P (T_1 \in [t_1 - dt/2, t_1 + dt/2], \ldots, T_n \in [t_n - dt/2, t_n + dt/2])$$

$$= P (T_1 \in [t_1 - dt/2, t_1 + dt/2]) \times \cdots \times P (T_n \in [t_n - dt/2, t_n + dt/2])$$

$$\approx f_{\alpha, \beta}(t_1) dt \times \cdots \times f_{\alpha, \beta}(t_n) dt$$

where $f(t) = f_{\alpha, \beta}(t)$ is the Weibull density with parameters $(\alpha, \beta)$

• Maximum likelihood estimation maximizes this probability
  over $\alpha \& \beta \implies$ maximum likelihood estimates (m.l.e.s) $\hat{\alpha}$ and $\hat{\beta}$
General Remarks on Estimation

- MLEs tend to be optimal in large samples (lots of theory)

- Method is very versatile in extending to many other data scenarios
  censoring and covariates

- Least squares method applied to QQ-plot is not entirely appropriate
  tends to be unduly affected by stray observations
  not as versatile to extend to other situations
Weibull Plot: \( n = 20 \)

- True model, Weibull(100, 3)
- M.l.e. model, Weibull(108, 3.338)
- Least squares model, Weibull(107.5, 3.684)
Weibull Plot: $n = 100$

- True model, Weibull(100, 3)
- MLE model, Weibull(102.8, 2.981)
- Least squares model, Weibull(102.2, 3.144)
Tests of Fit (Graphical)

• The Weibull plots provide an informal diagnostic for checking the Weibull model assumption

• The anticipated linearity is based on the Weibull model properties

• Strong nonlinearity indicates that the model is not Weibull

• Sorting out nonlinearity from normal statistical point scatter takes a lot of practice and a good sense for the effect of sample size on the variation in point scatter

• Formal tests of fit are available for complete samples\(^2\) and also for some other censored data scenarios

Formal Goodness-of-Fit Tests

- Let $F_{\hat{\alpha}, \hat{\beta}}(t)$ be the fitted Weibull distribution function.
- Let $F_n(t) = \frac{\#\{T_i \leq t; i=1,\ldots,n\}}{n}$ be the empirical distribution function.
- Compute a discrepancy metric $D$ between $F_{\hat{\alpha}, \hat{\beta}}$ and $F_n$,
  
  $$D_{KS}(F_{\hat{\alpha}, \hat{\beta}}, F_n) = \sup_t \left| F_{\hat{\alpha}, \hat{\beta}}(t) - F_n(t) \right| \quad \text{Kolmogorov-Smirnov}$$
  
  $$D_{CvM}(F_{\hat{\alpha}, \hat{\beta}}, F_n) = \int_0^\infty \left( F_{\hat{\alpha}, \hat{\beta}}(t) - F_n(t) \right)^2 f_{\hat{\alpha}, \hat{\beta}}(t) \, dt \quad \text{Cramer-von Mises}$$
  
  $$D_{AD}(F_{\hat{\alpha}, \hat{\beta}}, F_n) = \int_0^\infty \frac{\left( F_{\hat{\alpha}, \hat{\beta}}(t) - F_n(t) \right)^2}{F_{\hat{\alpha}, \hat{\beta}}(t)(1 - F_{\hat{\alpha}, \hat{\beta}}(t))} f_{\hat{\alpha}, \hat{\beta}}(t) \, dt \quad \text{Anderson-Darling}$$

- The distributions of $D$, when sampling from a Weibull population, are known and $p$-values of observed values $d$ of $D$ can be calculated.

$$p = P(D \geq d) \quad \Rightarrow \quad \text{BCSLIB: HSPFIT}$$
Kolmogorov-Smirnov Distance

K-S distance

n = 10

Weibull Reliability Analysis—FWS-5/1999—30
Weibull Plots: \( n = 10 \)
Weibull Plots: \( n = 20 \)
Weibull Plots: \( n = 50 \)

\[ p(\text{KS}) = 0.7 \]
\[ p(\text{CvM}) = 0.41 \]
\[ p(\text{AD}) = 0.43 \]

\[ n = 50 \]

\[ p(\text{KS}) = 0.22 \]
\[ p(\text{CvM}) = 0.35 \]
\[ p(\text{AD}) = 0.32 \]

\[ n = 50 \]

\[ p(\text{KS}) = 0.15 \]
\[ p(\text{CvM}) = 0.23 \]
\[ p(\text{AD}) = 0.12 \]

\[ n = 50 \]

\[ p(\text{KS}) = 0.76 \]
\[ p(\text{CvM}) = 0.49 \]
\[ p(\text{AD}) = 0.52 \]

\[ n = 50 \]

\[ p(\text{KS}) = 0.88 \]
\[ p(\text{CvM}) = 0.66 \]
\[ p(\text{AD}) = 0.72 \]

\[ n = 50 \]

\[ p(\text{KS}) = 0.27 \]
\[ p(\text{CvM}) = 0.22 \]
\[ p(\text{AD}) = 0.23 \]

\[ n = 50 \]

\[ p(\text{KS}) = 0.85 \]
\[ p(\text{CvM}) = 0.56 \]
\[ p(\text{AD}) = 0.62 \]

\[ n = 50 \]

\[ p(\text{KS}) = 0.18 \]
\[ p(\text{CvM}) = 0.035 \]
\[ p(\text{AD}) = 0.039 \]
Weibull Plots: \( n = 100 \)
Estimates and Confidence Bounds/Intervals

- For a target $\theta$
  \[
  \theta = \alpha, \quad \theta = \beta, \quad \theta = t_p = \alpha [-\log(1 - p)]^{1/\beta}, \quad \text{or} \quad \theta = P_{\alpha,\beta}(T \leq t)
  \]
  one gets corresponding m.l.e $\hat{\theta}$ by replacing $(\alpha, \beta)$ by $(\hat{\alpha}, \hat{\beta})$

- Such estimates vary around the target due to sampling variation

- Capture the estimation uncertainty via confidence bounds

- For $0 < \gamma < 1$ get $100\gamma\%$ lower/upper confidence bounds $\hat{\theta}_{L,\gamma}$ & $\hat{\theta}_{U,\gamma}$
  \[
  P(\hat{\theta}_{L,\gamma} \leq \theta) = \gamma \quad \text{or} \quad P(\theta \leq \hat{\theta}_{U,\gamma}) = \gamma
  \]

- For $\gamma > .5$ get a $100(2\gamma - 1)\%$ confidence interval by $[\hat{\theta}_{L,\gamma}, \hat{\theta}_{U,\gamma}]$
  \[
  P(\hat{\theta}_{L,\gamma} \leq \theta \leq \hat{\theta}_{U,\gamma}) = 2\gamma - 1 = \gamma^*
  \]
Confidence Bounds & Sampling Variation, \( n = 10 \)

L: Lawless Exact Method (RAP); B: Bain Exact Method (Tables); M: MLE Approximate Method

Weibull Reliability Analysis—FWS-5/1999—36
Confidence Bounds & Sampling Variation, \( n = 10 \)

L: Lawless Exact Method (RAP); B: Bain Exact Method (Tables); M: MLE Approximate Method
Confidence Bounds & Effect of \( n \)

Weibull Reliability Analysis—FWS-5/1999—38
Confidence Bounds & Effect of \( n \)
Incomplete Data or Censored Failure Times

- **Type I censoring** or time censoring:
  units are tested until failure or until a prespecified time has elapsed

- **Type II censoring** or failure censoring:
  only the $r$ lowest values of the total sample of size $n$ become known
  this shows up when we put $n$ units on test simultaneously and
  terminate the test after the first $r$ units have failed

- **Interval censoring, inspection data, grouped data**:
  $i^{th}$ unit is only known to have failed between two known
  inspection time points, i.e., failure time $T_i$ falls in $(s_i, e_i)$
  only bracketing intervals $(s_i, e_i)$, $i = 1, \ldots, n$, become known
Incomplete Data or Censored Failure Times (continued)

- **Random right censoring:**
  units are observed until failed or removed from observation due to other causes (different failure modes, competing risks)

- **Multiple right censoring:**
  units are put into service at different times and times to failure or censoring are observed

- Data can also combine several of the above censoring phenomena

- It is important that the censoring mechanism should **not** correlate with the (potential) failure times, i.e., no censoring of anticipated failures

- **All data (censored & uncensored) should enter analysis**
Type I or Time Censored Data: $n = 10$

Weibull Reliability Analysis—FWS-5/1999—42
Type II or Failure Censored Data: \( n = 10 \)
Interval Censored Data: \( n = 10 \)
Multiply Right Censored Data: $n = 10$
Nonparametric MLEs of CDF

- For complete, type I & II censored data the nonparametric MLE is
  \[ \hat{F}(t) = \frac{\#\{T_i \leq t; i = 1, \ldots, n\}}{n}, \]
  \(-\infty < t < \infty \) for complete sample,
  \(-\infty < t \leq t_0 \) for type I censored (at \( t_0 \)) sample,
  \(-\infty < t \leq T_{(r)} \) for type II censored (at \( T_{(r)} \), \( r^{th} \) smallest failure time)

- For multiply right censored data with failures at \( t_1^* < \ldots < t_k^* \) the nonparametric MLE (Kaplan-Meier or Product Limit Estimator) is
  \[ \hat{F}(t) = 1 - \left[ (1 - \hat{p}_1)\delta_1(t) \cdots (1 - \hat{p}_k)\delta_k(t) \right], \quad \delta_i(t) = \begin{cases} 1 & \text{for } t_i^* \leq t \\ 0 & \text{for } t_i^* > t \end{cases} \]
  where \( \hat{p}_i = d_i/n_i \), \( n_i = \# \) units known to be at risk just prior to \( t_i^* \)
  and \( d_i = \# \) units that failed at \( t_i^* \)

- The nonparametric MLE for interval censored data is complicated
Empirical CDF (complete samples)

0 5000 15000 25000
0.0 0.4 0.8
n = 10

empirical and true
and Weibull mle
distribution functions

0 5000 15000 25000
0.0 0.4 0.8
n = 30

0 5000 15000 25000
0.0 0.4 0.8
n = 50

0 5000 15000 25000
0.0 0.4 0.8
n = 100

Weibull Reliability Analysis—FWS-5/1999—47
Nonparametric MLE for Type I Censored Data

![Diagram showing nonparametric MLE of CDF, Weibull MLE of CDF, and true CDF for different sample sizes (n = 10, n = 30, n = 50, n = 100).]
Nonparametric MLE for Type II Censored Data

Weibull Reliability Analysis—FWS-5/1999—49
Kaplan-Meier Estimates (multiply right censored samples)

Kaplan-Meier CDF
Weibull mle for CDF
true CDF

Kaplan-Meier CDF
Weibull mle for CDF
true CDF

Kaplan-Meier CDF
Weibull mle for CDF
true CDF

Kaplan-Meier CDF
Weibull mle for CDF
true CDF
Nonparametric MLE for Interval Censored Data

superimposed is the true Weibull(10000,2.5) that generated the 1,000 interval censored data cases inspection points roughly 3000 cycles apart
Nonparametric MLE for Interval Censored Data

superimposed is the true Weibull(10000,2.5)
that generated the 10,000 interval censored data cases
inspection intervals randomly generated from same Weibull
Application to Y2K Questionnaire Return Data

Weibull Reliability Analysis—FWS-5/1999—53
Plotting Positions for Weibull Plots (Censored Case)

- For complete samples we plotted \(\log[T(i)], \log[\log(1 - p_i)]\) with
  \[p_i = 
  \frac{i - .5}{n} = 
  \frac{1}{2} \left[ F(T(i)) + F(T(i-1)) \right] = 
  \frac{1}{2} \left[ \frac{i}{n} + \frac{i - 1}{n} \right]\]

- For type I & II & multiply right censored samples
  we plot in analogy \(\log[T(i)], \log[\log(1 - p_i)]\), where
  \[T^*_1 < \ldots < T^*_k\] are the \(k\) distinct observed failure times and
  \[p_i = \frac{1}{2} \left[ F(T^*_i) + F(T^*_i - 1) \right]\]
  and \(F(t)\) is the nonparametric MLE of \(F(t)\) for the censored sample
Weibull Plot, Multiply Right Censored Sample $n = 50$

- True model, Weibull(100, 3)
- M.L.E. model, Weibull(96.27, 2.641) $n = 50$
- Least squares model, Weibull(96.3, 2.541) $n = 50$
Confidence Bounds for Censored Data

- For complete & type II censored data RAP computes exact coverage confidence bounds for $\alpha$, $\beta$, $t_p$, and $P(T \leq t)$.

- WEIBREG and commercial software (Weibull++, WeibullSMITH, common statistical packages) compute approximate confidence bounds for above targets, based on large sample m.l.e. theory

- RAP & WEIBREG are Boeing code, runs within DOS mode of Windows95 (interface clumsy, but job gets done), contact me

- Boeing has a site license for Weibull++ Version 4 in the Puget Sound area
  contact David Twigg (425) 717-1221, david.twigg@pss.boeing.com

- There is a Weibull++ Version 5 out
Weibull Plot With Confidence Bound

\[ \alpha = 102.4 \ [82.52, 143.8] \ 95\% \]

\[ \beta = 2.524 \ [1.352, 3.696] \ 95\% \]

\[ n = 30, \ r = 13 \]
alpha = 89.79 [77.4, 109.4] 95%
beta = 3.674 [2.15, 5.198] 95%
n = 30, r = 13
Weibull Regression Model & Analysis

Recall $t_p = \alpha[-\log(1-p)]^{1/\beta}$ or

$$\log(t_p) = \log(\alpha) + w_p/\beta \quad \text{with} \quad w_p = \log[-\log(1 - p)]$$

- Often we deal with failure data collected under different conditions
  - different part types
  - different environmental conditions
  - different part users

- Regression model on $\log(\alpha)$ (multiplicative on $\alpha$)

$$\log(\alpha_i) = b_1 Z_{i1} + \ldots + b_p Z_{ip}$$

where $Z_{i1}, \ldots, Z_{ip}$ are known covariates for $i^{\text{th}}$ unit

- Now we have $p + 1$ unknown parameters $\beta, b_1, \ldots, b_p$

- parameter estimates & confidence bounds using MLEs
Accelerated Life Testing: Inverse Power Law

- Units last too long under normal usage conditions
- Increase “stress” to accelerate failure time
- Increase voltage and accelerate life via inverse power law

\[
T(\text{Volt}) = T(\text{Volt}_U) \left( \frac{\text{Volt}}{\text{Volt}_U} \right)^c, \quad \text{where usually } \ c < 0
\]

this means

\[
\alpha(\text{Volt}) = \alpha(\text{Volt}_U) \left( \frac{\text{Volt}}{\text{Volt}_U} \right)^c
\]

or

\[
\log [\alpha(\text{Volt})] = \log [\alpha(\text{Volt}_U)] + c \log (\text{Volt}) - c \log (\text{Volt}_U)
\]

\[
= b_1 + b_2 Z
\]

with

\[
b_1 = \log [\alpha(\text{Volt}_U)] - c \log (\text{Volt}_U), \quad b_2 = c, \quad \text{and } Z = \log (\text{Volt})
Accelerated Life Testing: Arrhenius Model

- Another way of accelerating failure in processes involving chemical reaction rates is to increase the temperature.

- Arrhenius proposed the following acceleration model with temperature measured in Kelvin \( \text{temp} K = \text{temp} °C + 273.15 \)

\[
T(\text{temp}) = T(\text{temp}_U) \exp \left[ \frac{\kappa}{\text{temp}} - \frac{\kappa}{\text{temp}_U} \right]
\]

or

\[
\log[\alpha(\text{temp})] = \log[\alpha(\text{temp}_U)] + \frac{\kappa}{\text{temp}} - \frac{\kappa}{\text{temp}_U}
\]

\[
= b_1 + b_2 Z
\]

with

\[
b_1 = \log[\alpha(\text{temp}_U)] - \frac{\kappa}{\text{temp}_U}, \quad b_2 = \kappa, \quad \text{and} \quad Z = \text{temp}^{-1}
\]
Pooling Data With Different $\alpha$’s

- Suppose we have three groups of failure data
  \[
  T_1, \ldots, T_{n_1} \sim W(\tilde{\alpha}_1, \beta), \quad T_{n_1+1}, \ldots, T_{n_1+n_2} \sim W(\tilde{\alpha}_2, \beta), \\
  T_{n_1+n_2+1}, \ldots, T_{n_1+n_2+n_3} \sim W(\tilde{\alpha}_3, \beta)
  \]

- We can analyze the whole data set of $N = n_1 + n_2 + n_3$ values jointly using the following model for $\alpha_j$ and dummy covariates $Z_{1,j}$, $Z_{2,j}$ & $Z_{3,j}$

\[
\log(\alpha_j) = b_1 Z_{1,j} + b_2 Z_{2,j} + b_3 Z_{3,j}, \quad j = 1, 2, \ldots, N
\]

where $b_1 = \log(\tilde{\alpha}_1)$, $b_2 = \log(\tilde{\alpha}_2) - \log(\tilde{\alpha}_1)$, and $b_3 = \log(\tilde{\alpha}_3) - \log(\tilde{\alpha}_1)$

- $Z_{1,j} = 1$ for $j = 1, \ldots, N$, $Z_{2,j} = 1$ for $j = n_1 + 1, \ldots, n_1 + n_2$, & $Z_{2,j} = 0$ else,

- $Z_{3,j} = 1$ for $j = n_1 + n_2 + 1, \ldots, N$, & $Z_{3,j} = 0$ else.

- Advantage: Smaller estimation error
Pooling Data (continued)

\[
\begin{align*}
\log(\alpha_1) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 0 = \log(\tilde{\alpha}_1) \\
\vdots & \quad \vdots \\
\log(\alpha_{n_1}) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 0 = \log(\tilde{\alpha}_1) \\
\vdots & \quad \vdots \\
\log(\alpha_{n_1+n_2}) &= b_1 \cdot 1 + b_2 \cdot 1 + b_3 \cdot 0 = \log(\tilde{\alpha}_1) + [\log(\tilde{\alpha}_2) - \log(\tilde{\alpha}_1)] = \log(\tilde{\alpha}_2) \\
\vdots & \quad \vdots \\
\log(\alpha_{n_1+n_2+1}) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 1 = \log(\tilde{\alpha}_1) + [\log(\tilde{\alpha}_3) - \log(\tilde{\alpha}_1)] = \log(\tilde{\alpha}_3) \\
\vdots & \quad \vdots \\
\log(\alpha_N) &= b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 1 = \log(\tilde{\alpha}_1) + [\log(\tilde{\alpha}_3) - \log(\tilde{\alpha}_1)] = \log(\tilde{\alpha}_3)
\end{align*}
\]
Accelerated Life Testing: Model, Data & MLEs
Analysis for Data from Exponential Distribution

- $T \sim \mathcal{E}(\theta)$ Exponential with mean $\theta$, i.e., $\mathcal{E}(\theta) = \mathcal{W}(\alpha = \theta, \beta = 1)$

- It suffices to get confidence bounds for $\theta$ since all other quantities of interest are explicit and monotone functions of $\theta$

$$F_\theta(t_0) = P_\theta(T \leq t_0) = 1 - \exp(-t_0/\theta)$$

$$R_\theta(t_0) = P_\theta(T > t_0) = \exp(-t_0/\theta)$$

and

$$t_p(\theta) = \theta \left[ -\log(1 - p) \right]$$
Complete & Type II Censored Exponential Samples

- $T \sim \mathcal{E}(\theta)$ Exponential with mean $\theta$, $\mathcal{E}(\theta) = \mathcal{W}(\alpha = \theta, \beta = 1)$

- For complete exponential samples $T_1, \ldots, T_n \sim \mathcal{E}(\theta)$ or for a type II censored sample of this type, i.e., with observed $r$ lowest values $T_{(1)} \leq \ldots \leq T_{(r)}$, $1 \leq r \leq n$, one gets $100\gamma\%$ lower confidence bounds for $\theta$ by computing

$$\hat{\theta}_{L,\gamma} = \frac{2 \times TTT}{\chi^2_{2r,\gamma}},$$

where $TTT = T_{(1)} + \cdots + T_{(r)} + (n - r)T_{(r)} = \text{Total Time on Test}$, and $\chi^2_{2r,\gamma} = \gamma$-quantile of the $\chi^2_{2r}$ distribution. Get this in Excel via $= \text{GAMMAINV}(\gamma, r, 1)$ or $= \text{CHIINV}(1 - \gamma, 2r)/2$

- These bounds have exact confidence coverage properties
Multiply Right Censored Exponential Data

- Here we define $TTT = T_1 + \ldots + T_n$ as the total time on test but now $T_i$ is either the observed failure time of the $i^{th}$ part or the observed right censoring time of the $i^{th}$ part.
- The number $r$ of observed failures is random here, $0 \leq r \leq n$.

- Approximate $100\gamma\%$ lower confidence bounds for $\theta$ are computed as

$$\hat{\theta}_{L,\gamma} = \frac{2 \times TTT}{\chi_{2r+2,\gamma}^2}$$

- This also holds for $r = 0$, i.e., no failures at all over exposure period.

Weibull Reliability Analysis—FWS-5/1999—68
Weibull Shortcut Methods for Known $\beta$

- $T \sim \mathcal{W}(\alpha, \beta)$ (Weibull) $\Rightarrow T^\beta \sim \mathcal{E}(\theta)$ Exponential with mean $\theta = \alpha^\beta$

- As $100\gamma\%$ lower confidence bound for $\alpha$ compute

$$\hat{\alpha}_{L,\gamma} = \left( \frac{2 \times TTT(\beta)}{\chi^2_{f,\gamma}} \right)^{1/\beta}$$

where for complete or type II censored samples we take $f = 2r$ and

$$TTT = T_{(1)}^\beta + \cdots + T_{(r)}^\beta + (n - r)T_{(r)}^\beta$$

and for multiply right censored data we take $f = 2r + 2$ and

$$TTT = T_1^\beta + \cdots + T_n^\beta.$$  

- Sensitivity should be studied by repeating analyses for several $\beta$'s.
A- and B-Allowables, ABVAL Program

- My first Weibull work led to the program ABVAL (1983) supporting Cecil Parsons and Ron Zabora w.r.t. MIL-HDBK-5
- ABVAL computes A- and B-Allowables based on samples from the 3-parameter Weibull distribution.
  - The A-Allowable is a 95% lower confidence bound for 99% of the population, i.e., for the .01-quantile.
  - The B-Allowable is a 95% lower confidence bound for 90% of the population, i.e., for the .10-quantile.
- Hybrid approach of using Mann-Fertig threshold estimate combined with maximum likelihood estimates for $\alpha$ and $\beta$, took lot of simulation and tuning and is very specialized in its capabilities
- It is now a method in MIL-HDBK-5, I still maintain software
Selected References


  — (1985), “Weibull analysis of reliability data with few or no failures,” *Journal of Quality Technology* 17, 140-146

  — (1996) “Maximum likelihood estimation for type I censored Weibull data including covariates.” *ISSTECH-96-022*
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