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# Characterization of the Weibull distribution

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*Abstract:* Let $F$ be a cumulative distribution function with quantiles $x_F(u) = x(u) = F^{-1}(u) = \inf\{x: F(x) \geq u\}$ for $0 < u < 1$ and let $C = \{(u, v, w): 0 < u < v < w < 1, \log(1-u)\log(1-w) = (\log(1-v))^2\}$. For the three parameter Weibull distribution function, defined for $\alpha > 0$, $\beta > 0$ and $\tau \in R$ by $G(x) = 1 - \exp(-((x-\tau)/\alpha)^\beta)$ for $x \geq \tau$ and $G(x) = 0$ for $x < \tau$, it is known that for some fixed $t$, namely $t = \tau$, the following relation holds between its quantiles, $x_G(u) = x(u)$: $x(u)x(w) - x^2(v) = t(x(u) + x(w) - 2x(v))$ for all $(u, v, w) \in C$. We prove that this quantile relationship characterizes the three parameter Weibull distribution in the sense that a random variable $X$ with c.d.f. $F$, satisfying this quantile relationship is either degenerate or $X \sim G$ with $\tau = t$.

*Keywords:* Weibull, Characterization, Quantile, Functional equation, Test of fit.

## Introduction

Several characterizations of the Weibull distribution have been given, c.f. [3]–[6], [8]–[12]. However, all these concern the characterization of the two parameter Weibull distribution, i.e., assuming a lower threshold of zero. Here a characterization of the three parameter Weibull distribution is given in terms of relationships between particular triads of quantiles as delineated by the set $C$ below. These relationships stipulate that a certain function of the quantile triad is always proportional to a second function of the same triad, the proportionality factor remaining constant over all such triads. These quantile relationships for the three parameter Weibull distribution are well known, c.f. [7] p. 261, and were investigated in [1] as basis for estimating the Weibull parameters. Of course, this characterization of the three parameter Weibull distribution is easily specialized to the case of a two parameter Weibull distribution. We conclude the paper with some thoughts on how to utilize this characterization in a test of fit test for the three parameter Weibull distribution.

## Characterization Theorem

In order to state the characterization theorem the following notation is introduced. Let

$$C = \{(u, v, w): 0 < u < v < w < 1, \log(1-u)\log(1-w) = (\log(1-v))^2\}$$

and let $F$ be a cumulative distribution function with quantiles $x_F(u) = x(u) = F^{-1}(u) = \inf\{x: F(x) \geq u\}$ for $0 < u < 1$. For the three parameter Weibull distribution function, defined for $\alpha > 0$, $\beta > 0$ and $\tau \in R$ by

$$G(x) = 1 - \exp\left(-\left(\frac{x-\tau}{\alpha}\right)^\beta\right) \text{ for } x \geq \tau$$

and $G(x) = 0$ for $x < \tau$, it is known that for some fixed $t$, namely $t = \tau$, the following relation holds between its quantiles, $x(u) = x_G(u)$:

$$x(u)x(w) - x^2(v) = t(x(u) + x(w) - 2x(v)) \text{ for all } (u, v, w) \in C. \quad (1)$$

The following theorem states that this relationship actually characterizes the three parameter Weibull distribution.

**Characterization Theorem.** *Any random variable $X$ with cumulative distribution function $F(x)$ and quantiles $x_F(u) = x(u)$ satisfying the relationships* (1) *is either degenerate or $X$ has a three parameter Weibull distribution with $\tau = t$.*

Of course the degenerate case could be subsumed in the Weibull model with $\alpha \geq 0$.

**Proof.** The proof consists of the following four steps.

1. The support of $F$ cannot be $(-\infty, \infty)$.
2. The support is finite only in the degenerate case.
3. Assuming that the support is $[a, \infty)$ or $(-\infty, a]$ it follows that $a = t$.
4. Finally, it is shown that the quantile relationship (1) translates into a linearity relation from which the Weibull characterization follows.

**Proof of 1:** This follows by contradiction upon dividing the relation (1) by $x(u)$ $x(w)$ and letting $u \to 0$ and $w \to 1$ while holding $v$ fixed.

**Proof of 2:** Suppose $F$ has finite support $[a, b]$. Let $Y = X - a$ with corresponding quantiles $y(u)$. The quantile relation (1) translates to

$$y(u)y(w) - y^2(v) = (t-a)(y(u) + y(w) - 2y(v)) \text{ for all } (u, v, w) \in C.$$

Writing $s = t - a$ and letting $u \to 0$ and $w \to 1$ while holding $v$ fixed, with $y(v) = y$, leads to the following equation

$$-y^2 = s(b - a - 2y) \text{ with solutions } y = s \pm \sqrt{s^2 - (b-a)s}\,.$$

For any $s$ this equation yields at most one solution $y \in [0, b-a]$. This implies the degenerate case of the characterization.

**Proof of 3:** Dividing the relationship (1) by $x(w)$ (or $x(u)$, whichever becomes unbounded) and letting $u \to 0$, $w \to 1$ while $v$ is fixed one obtains $t = a$.

**Proof of 4:** Proceeding as in step 2, the quantile relation becomes

$$y(u)\,y(w) - y^2(v) = 0 \text{ for all } (u, v, w) \in C.$$

Let $h(z) = \log(y(\rho^{-1}(z)))$ for all $z \in R$, where $\rho(p) = \log(-\log(1-p))$. For all $(u, v, w) \in C$ one now has

$$h(\rho(u)) + h(\rho(w)) = 2h(\rho(v)) \text{ and } \rho(u) + \rho(w) = 2\rho(v).$$

This implies the following functional equation

$$h\left(\frac{z_1 + z_3}{2}\right) = \frac{h(z_1) + h(z_3)}{2} \text{ for all } z_1,\ z_2 \in R.$$

Since $h(z)$ is bounded on any finite interval it follows (see [2], p. 91) that $h$ is convex, concave and continuous, thus linear, i.e., $h(z) = A + Bz$ with $B > 0$ since $h(z)$ is strictly increasing. Hence

$$y(p) = \exp(h(\rho(p))) = \exp(A + B\rho(p)) = \exp(A)(-\log(1-p))^B,$$

which is the $p$-quantile of a two parameter Weibull distribution with $\alpha = \exp(A)$, and $\beta = 1/B$. Hence $x(p) = y(p) + \tau$ is the $p$-quantile of $G(x)$.

## Test of Fit Considerations

Replacing quantiles by sample quantiles and examining the characterizing proportionality property through some correlation metric one could easily devise a test of fit statistic for the three parameter Weibull distribution. Of course it is desirable to construct a metric for which the null distribution is independent of all three Weibull parameters. So far we were only successful in constructing a location and scale invariant metric.

To describe this metric consider the random sample $X_1, \ldots, X_n$ and denote by $X_{(1)} \leq \ldots \leq X_{(n)}$ the corresponding order statistics. Select a triplet of order statistics $X_{(i)}$, $X_{(j)}$ and $X_{(k)}$, where $i < j < k$ are chosen such that $u_i = i/(n+1)$, $v_j = j/(n+1)$ and $w_k = k/(n+1)$ approximately conform to the restrictions stipulated in $C$. There may be $N$ such triplets. Let $U_l = X_{(i)}X_{(k)} - X_{(j)}^2$ and $V_l = X_{(i)} + X_{(k)} - 2X_{(j)}$ for $l = 1, \ldots, N$. Since under the three parameter Weibull model we expect for some $t$ that $U_l \approx tV_l$ for all $l$ we are led to the following location and scale invariant test of fit metric:

$$R = \frac{\sum_l (U_l - \hat{t}V_l)^2}{\left(\sum_l V_l^2\right)^2}.$$

Here $\hat{t}$ represents the least squares estimator for the proportionality constant $t$, i.e.,

$$\hat{t} = \frac{\sum_l U_l V_l}{\sum_l V_l^2}.$$

If the corresponding correlation coefficient is denoted by

$$\hat{\rho} = \hat{t} \cdot \sqrt{\frac{\sum_l V_l^2}{\sum_l U_l^2}} = \frac{\sum_l U_l V_l}{\sqrt{\sum_l U_l^2 \sum_l V_l^2}},$$

then one easily shows the following simple relation between $R$ and $\hat{\rho}$:

$$R = \frac{\sum_l U_l^2}{\left(\sum_l V_l^2\right)^2}\left(1 - \hat{\rho}^2\right).$$

To what extent the null distribution of this metric varies with the Weibull shape parameter still needs to be investigated through simulation and asymptotic methods.

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