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Tolerance Bounds for Log Gamma Regression Models

Robert A. Jones and F. W. Scholz

Boeing Computer Services MS9C-01
Tukwila, WA 98188

Mina Ossiander and Galen R. Shorack

Department of Statistics
University of Washington
Seattle, WA 98195

Finding lower confidence bounds for the quantiles of Weibull populations has received much attention in recent literature. An accurate procedure (based on solving a quadratic equation) is presented in (1.17). It is, in fact, more accurate than the currently available Monte Carlo tables. It extends to any location-scale family; this article shows that it is accurate for all members of the log gamma (K) family with $\frac{1}{2} \le K \le \infty$. The procedure is shown to work well for censored data. It also extends naturally to regression data. An even more accurate procedure (an approximation to the Lawless conditional procedure, in which the "configurations" are replaced by an approximation of their expected values) is presented in (3.1). It involves numerical integration, but the tables are independent of the data. It extends easily to the censored case.

KEY WORDS: Tolerance bounds; Log gamma distribution; Censoring; Closed form approximation; Tables for approximate conditional approximation.

1. APPROXIMATE TOLERANCE BOUNDS FOR REGRESSION MODELS

Let $X_1, \ldots, X_n$ denote independent failure times on $(0, \infty)$, and define $Y_i = \log X_i$ (log failure time). Consider the general regression model

$$Y_i = \mu + W_i \beta + \sigma \varepsilon_i, \quad 1 \le i \le n, \quad \varepsilon_1, \ldots, \varepsilon_n \text{ iid with df } F, \tag{1.1}$$

where F has mean 0 and variance 1, β is an unknown $r \times 1$ parameter vector and W_i is a $1 \times r$ vector of known covariates associated with the ith observation. We agree that

$$D \equiv D_n \equiv \left[\frac{W'W}{n}\right]^{-1}, \quad \text{where } W' \equiv [W_1' \cdots W_n'] \text{ is } r \times n \text{ of rank } r, \tag{1.2}$$

and

$$\text{each column of } W \text{ is orthogonal to } 1 \equiv (1, \ldots, 1)'. \tag{1.3}$$

(The location-scale model is obtained as a special case by setting $\beta = 0$.) For $0 < p < 1$ we let

$$\varepsilon_p \equiv F^{-1}(p) \tag{1.4}$$

denote the pth quantile of ε. Thus an observation taken under conditions W_0 has pth quantile

$$y_p \equiv \mu + W_0 \beta + \sigma \varepsilon_p. \tag{1.5}$$

The tolerance bound problem, which is our main concern in this article, is to find a lower confidence bound on y_p. Lieberman and Miller (1963) solved this problem for normal F, whereas McCool (1980) proposed a Monte Carlo solution for Weibull F.

We suppose now that $\hat{\mu}$, $\hat{\beta}$, $\hat{\sigma}$ are *invariant* estimators of μ, β, σ; that is, $\hat{\mu}(aY + b1 + Wc) = a\hat{\mu}(Y) + b$, $\hat{\beta}(aY + b1 + Wc) = a\hat{\beta}(Y) + c$, and $\hat{\sigma}(aY + b1 + Wc) = a\hat{\sigma}(Y)$. Then

$$\text{the distribution of } (\hat{\mu} - \mu)/\sigma, (\hat{\beta} - \beta)/\sigma, \hat{\sigma}/\sigma \text{ depends only on the distribution of } \varepsilon, \text{ and not on } \mu, \beta, \sigma. \tag{1.6}$$

This invariance holds for the maximum likelihood estimates (MLE's). We also suppose (as is true for MLE's under regularity on F), that

$$\sqrt{n}\begin{bmatrix} \hat{\sigma} - \sigma \\ \hat{\mu} - \mu \\ \hat{\beta} - \beta \end{bmatrix} \underset{d}{\rightarrow} N\left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & 0 \\ 0 & 0 & a_{22} D_\infty \end{bmatrix}\right] \tag{1.7}$$

for some a_{ij}'s depending on F (provided D_n converges to some D_∞). For

$$g \equiv (\log f)', \tag{1.8}$$

the MLE's satisfy (1.7) with covariance matrix given by (see Cox and Hinkley 1968)

$$\sigma^2 \begin{bmatrix} \begin{bmatrix} E(-1-\varepsilon^2 g'(\varepsilon) - 2\varepsilon g(\varepsilon)) & E(-g(\varepsilon) - \varepsilon g'(\varepsilon)) \\ * & E(-g'(\varepsilon)) \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & [E(-g'(\varepsilon))]^{-1} D_\infty \end{bmatrix}. \tag{1.9}$$

The natural estimate of the quantile y_p under conditions W_0 is

$$\hat{Y}_p \equiv \hat{\mu} + W_0 \hat{\beta} + \hat{\sigma}\varepsilon_p. \tag{1.10}$$

As our solution to the *tolerance bound problem*, we seek a constant $B_n \equiv B_n(\gamma, p, W_0, F)$, satisfying

$$\gamma = P(\hat{Y}_p - B_n \hat{\sigma}/\sqrt{n} \le y_p) = P(\sqrt{n}(\hat{Y}_p - y_p)/\hat{\sigma} \le B_n). \tag{1.11}$$

The desired tolerance bound is then $\hat{Y}_p - B_n \hat{\sigma}/\sqrt{n}$. We have phrased our problem in terms of determining B_n, since it is asymptotically stable; that is, B_n converges to some B_∞ in $(-\infty, \infty)$. We use standardized rv's ε in the hope that B_n will turn out to be reasonably stable across various distributions.

To find B_n we rewrite (1.11) as

$$\gamma = P\left(\sqrt{n}\,\frac{(\hat{\mu}-\mu) + W_0(\hat{\beta}-\beta) - \sigma\varepsilon_p}{\hat{\sigma}} \le B_n - \sqrt{n}\varepsilon_p\right) \tag{1.12}$$

$$= P(Z + tA \le \sqrt{n}(\delta - t)), \tag{1.13}$$

where

$$\delta = \varepsilon_p/\tau_0, \qquad \tau_0^2 = a_{11} + a_{22} W_0 D W_0',$$
$$t = ((n-r-1)/n)^{1/2}(\varepsilon_p - B_n/\sqrt{n})/\tau_0,$$
$$A = n^{1/2}\{[(n/(n-r-1))^{1/2}\hat{\sigma}/\sigma] - 1\},$$

and

$$Z = \sqrt{n}((\hat{\mu}-\mu) + W_0(\hat{\beta}-\beta))/(\sigma\tau_0).$$

We will assume (as did Jennett and Welch 1939 in approximating the noncentral t) that $Z + tA$ is approximately normal (the accuracy of this will be considered later). This assumption leads [after solving the appropriate quadratic in t coming from (1.13)] to

$$B_n = \frac{z_\gamma\left(\frac{n}{n-r-1}\right)^{1/2}\tau_0\left\{\begin{matrix}(\delta^2 \operatorname{var}[A] + 2\delta \operatorname{cov}[A,Z] + 1) + 2(1+\delta\operatorname{cov}[A,Z])EA/\sqrt{n} \\ + \{E^2A + z_\gamma^2(\operatorname{cov}^2[A,Z] - \operatorname{var}[A])\}/n\end{matrix}\right\}^{1/2}}{1 + 2EA/\sqrt{n} + (E^2A - z_\gamma^2 \operatorname{var}[A])/n}$$
$$+ \sqrt{n}\left\{\varepsilon_p - \left(\frac{n}{n-r-1}\right)^{1/2}\tau_0\,\frac{\delta + \delta EA/\sqrt{n} + z_\gamma^2\operatorname{cov}[A,Z]/n}{1 + 2EA/\sqrt{n} + (E^2A - z_\gamma^2\operatorname{var}[A])/n}\right\} \tag{1.14}$$

by using the seemingly safe approximations

$$EZ \doteq 0 \quad \text{and} \quad \operatorname{var}[Z] \doteq 1. \tag{1.15}$$

Adding the less obvious approximations

$$EA \doteq 0, \quad \operatorname{var}[A] \doteq a_{00}, \quad \text{and} \quad \operatorname{cov}[A,Z] \doteq a_{01}/\tau_0 \tag{1.16}$$

reduces (1.14) to

$$B_n = z_\gamma\left(\frac{n}{n-r-1}\right)^{1/2} \times \frac{\{(\tau_0^2 + 2\varepsilon_p a_{01} + \varepsilon_p^2 a_{00}) + z_\gamma^2(a_{01}^2 - a_{00}\tau_0^2)/n\}^{1/2}}{1 - z_\gamma^2 a_{00}/n}$$
$$+ \sqrt{n}\left\{\varepsilon_p - \left(\frac{n}{n-r-1}\right)^{1/2}\frac{\varepsilon_p + z_\gamma^2 a_{01}/n}{1 - z_\gamma^2 a_{00}/n}\right\}. \tag{1.17}$$

There is one special case on which we wish to focus our method. Let G denote a gamma (K) random variable (rv) with density $x^{K-1}e^{-x}/\Gamma(K)$ for $x > 0$ so that $L \equiv \log G$ has the log gamma (K) density $\exp(Ky - e^y)/\Gamma(K)$. Let $\varepsilon \equiv (L - EL)/\sqrt{\operatorname{var}[L]}$, and let F denote the df of ε. The cumulant generating function of L is

$$\log E(\exp(tL)) = \log(\Gamma(t+K)/\Gamma(K)) \quad \text{for } t > -K \tag{1.18}$$

Table 1. *Characteristics of the Standardized Log Gamma (K) Densities*

Shape K	*Mean* $E(L) = \psi(K)$	*Standard Deviation* $\sigma(L) = \sqrt{\psi'(K)}$	*Skewness* $\gamma_1(L) = \gamma_1(\varepsilon)$	*Kurtosis* $\gamma_2(L) = \gamma_2(\varepsilon)$	*Median* $m(L)$	*Mode* $\tilde{m}(L)$
.5	−1.963510	2.22144	−1.53514	4.00000	−1.481	−.693
1.0	−.577216	1.28255	−1.13955	2.40001	−.367	.000
2.0	.422784	.80308	−.78025	1.18754	.518	.693
4.0	1.256117	.53275	−.52934	.55695	1.301	1.386
16.0	2.741013	.25396	−.25385	.12886	2.752	2.773
∞	∞	.00000	.00000	.00000	∞	∞

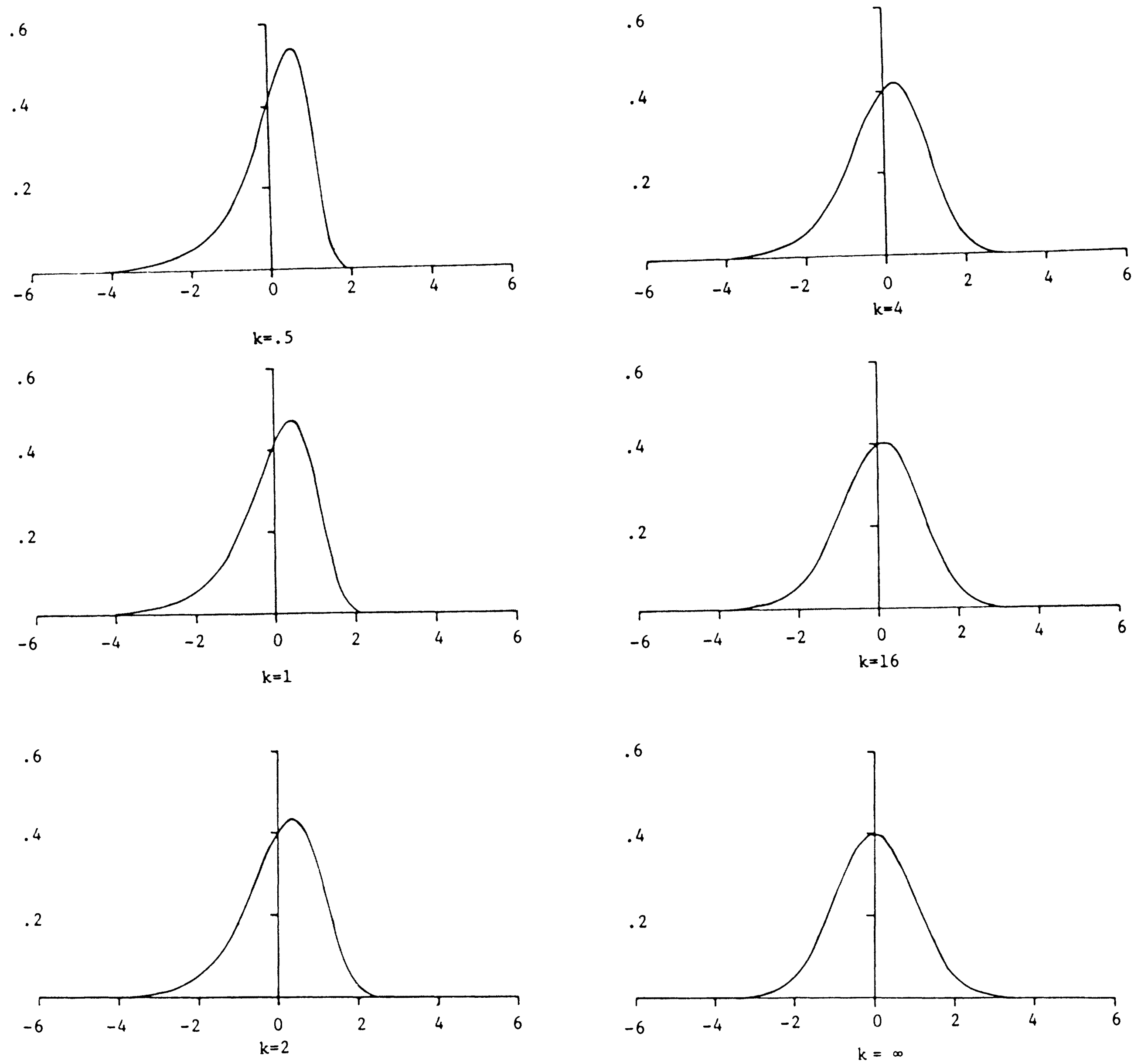

Figure 1. *Graphs of the Standardized Log Gamma (k) Density Functions, f_k.*

so that

$$EL = \psi(K) \quad \text{and} \quad \text{var}\,[L] = \psi'(K), \qquad (1.19)$$

where ψ is the digamma function $\psi(x) \equiv (d/dx) \log \Gamma(x)$. Moreover, (coefficient of skewness) $\equiv \gamma_1 = \psi''(K)/[\psi'(K)]^{3/2}$, (coefficient of kurtosis) $\equiv \gamma_2 = \psi'''(K)/[\psi'(K)]^2$, and the mode is log K. [Of course, when $K = 1$ the rv $L = \log G$ is an extreme value rv; moreover, the original observed X_i is Weibull (b_i, c) with $b_i = \exp\,(\mu + W_i\beta - \sigma\psi(K)/\sqrt{\psi'(K)})$, $c = \sqrt{\psi'(K)}/\sigma$, and density $cx^{c-1}b_i^{-c} \exp\,(-(x/b_i)^c)$ for $x > 0$. The limiting case $K = \infty$ corresponds to $Y_i = \log X_i$ having a normal distribution. See Prentice 1974 for a discussion.] See Table 1 for characteristics of the distribution of L.

Thus we now specialize to the case

$$Y_i = \mu + W_i\beta + \sigma\varepsilon_i,$$

$$1 \le i \le n, \quad \varepsilon_1, \ldots, \varepsilon_n \text{ iid } F_K, \qquad (1.20)$$

where F_K denotes the standardized log gamma (K) distribution. The densities f_K of F_K are shown in Figure 1, and the standardized df's F_K themselves are shown on normal probability paper in Figure 2. The elements $a_{ij} \equiv a_{ij}(K)$ of the matrix of (1.7) and (1.9) are given in Table 2. The rule

approximate the B_n of (1.11) by (1.17) (1.21)

constitutes our proposed solution number one of the tolerance-bound problem. Tables 2 and 3 make it tractable to perform (1.21) for standardized log

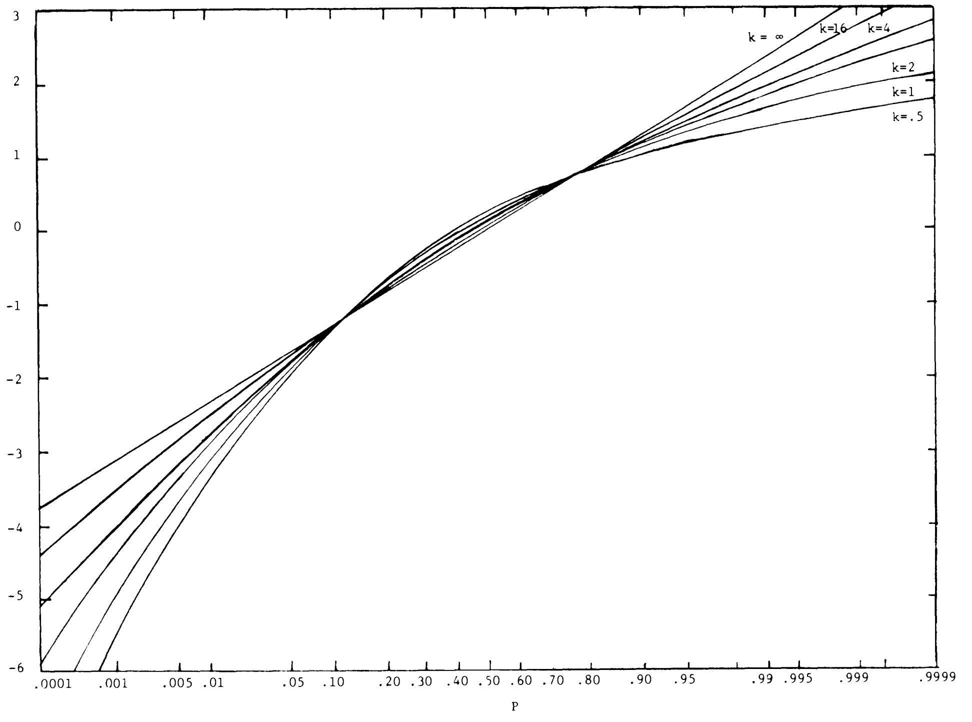

Figure 2. *Graphs of the Standardized Log Gamma (k) Distribution Functions, F_k, on Normal Probability Paper.*

gamma (K) error rv's and MLE estimators. (A somewhat similar procedure was proposed by Bain and Engelhardt 1981; however, it is less accurate and is asymptotically incorrect.)

2. CENSORING

This approach to finding confidence bounds can be extended to location-scale models in which type II censoring has occurred. In this situation the m_1 smallest and m_2 largest of the ordered log lifetimes $Y_{n:1} < \cdots < Y_{n:n}$ have not been observed. Thus the log likelihood is

$$l(\mu, \sigma) = m_1 \log F\left(\frac{Y_{n:m_1+1} - \mu}{\sigma}\right) + m_2 \log\left[1 - F\left(\frac{Y_{n:n-m_2} - \mu}{\sigma}\right)\right] + \sum_{i=m_1+1}^{n-m_2} \log f\left(\frac{Y_{n:i} - \mu}{\sigma}\right) - (n - m_1 - m_2)\log \sigma + \log \frac{n!}{m_1! m_2!}. \quad (2.1)$$

When F is regular, the MLE's $\hat{\mu}$ and $\hat{\sigma}$ again satisfy [recall (1.7)]

$$\sqrt{n}\begin{pmatrix}\hat{\sigma} - \sigma \\ \hat{\mu} - \mu\end{pmatrix} \underset{d}{\rightarrow} N\left(\begin{bmatrix}0\\0\end{bmatrix}, \sigma^2\begin{bmatrix}d_{00} & d_{01}\\ d_{01} & d_{11}\end{bmatrix}^{-1}\right) \equiv N\left(\begin{bmatrix}0\\0\end{bmatrix}, \sigma^2\begin{bmatrix}a_{00} & a_{01}\\ a_{01} & a_{11}\end{bmatrix}\right) \quad (2.2)$$

as $n \to \infty$, where

$$g \equiv (\log f)', \quad h_1 \equiv (\log F)', \quad h_2 \equiv (\log(1 - F))', \quad (2.3)$$

and [with $\varepsilon \equiv (Y - \mu)/\sigma$]

$$d_{11} = -\frac{m_1}{n} Eh_1'(\varepsilon_{n:m_1+1}) - \frac{m_2}{n} Eh_2'(\varepsilon_{n:n-m_2}) - \frac{1}{n}\sum_{i=m_1+1}^{n-m_2} Eg'(\varepsilon_{n:i}), \quad (2.4)$$

$$d_0 = -\frac{m_1}{n} E[h_1(\varepsilon_{n:m_1+1}) + \varepsilon_{n:m_1+1}h_1'(\varepsilon_{n:m_1+1})] - \frac{m_2}{n} E[h_2(\varepsilon_{n:n-m_2}) + \varepsilon_{n:n-m_2}h_2'(\varepsilon_{n:n-m_2})] - \frac{1}{n}\sum_{i=m_1+1}^{n-m_2} E[g(\varepsilon_{n:i}) + \varepsilon_{n:i}g'(\varepsilon_{n:i})], \quad (2.5)$$

Table 2. *Characteristics of the Standardized Log Gamma (K) Densities*

K	a_{00}	a_{01}	a_{11}	a_{22}
.5	.681477	−.613544	.957669	.405285
1.0	.607927	−.473999	.977502	.607927
2.0	.558701	−.347852	.991846	.775273
4.0	.530422	−.248907	.997634	.880831
16.0	.507768	−.124964	.999837	.969082
∞	.500000	.000000	1.000000	1.000000

NOTE: The a_{ij} are required in (1.17).

Table 3. *pth Quantiles of the Standardized Log Gamma (K) Densities*

p	$K = .5$, $C = 1.414$	Extreme Value: $K = 1.0$, $C = 1.000$	$K = 2.0$, $C = .707$	$K = 4.0$, $C = .500$	$K = 16.0$, $C = .250$	Normal: $K = \infty$, $C = .000$
.0001	−7.51707	−6.73118	−5.82340	−5.10185	−4.32675	−3.71902
.0005	−6.06807	−5.47615	−4.81401	−4.30074	−3.74462	−3.29053
.0010	−5.44402	−4.93551	−4.37689	−3.94830	−2.48043	−3.09023
.0050	−3.99501	−3.67907	−3.35088	−3.10335	−2.82417	−2.57583
.0100	−3.37094	−3.13667	−2.90082	−2.72287	−2.51691	−2.32635
.0250	−2.54586	−2.41631	−2.29210	−2.19627	−2.07816	−1.95996
.0500	−1.92137	−1.86580	−1.81477	−1.77193	−1.71226	−1.64485
.1000	−1.29554	−1.30455	−1.31277	−1.31299	−1.30295	−1.28155
.2000	−.66427	−.71945	−.76692	−.79701	−.82451	−.84162
.3000	−.28675	−.35376	−.41078	−.44963	−.49062	−.52440
.4000	−.00929	−.07369	−.12863	−.16787	−.21252	−.25335
.5000	.21732	.16428	.11833	.08378	.04176	.00000
.6000	.41663	.38189	.35048	.32484	.29069	.25335
.7000	.60408	.59479	.58387	.57175	.55130	.52440
.8000	.79521	.82110	.83918	.84723	.84905	.84162
.9000	1.01991	1.10035	1.16496	1.20717	1.24957	1.28155
.9500	1.17771	1.30553	1.41215	1.48671	1.56994	1.64485
.9750	1.29851	1.46781	1.61243	1.71733	1.84054	1.95996
.9900	1.42372	1.64079	1.83056	1.97272	2.14704	2.32635
.9950	1.50111	1.75011	1.97087	2.13933	2.35093	2.57583
.9990	1.64419	1.95693	2.24143	2.46571	2.75953	3.09023
.9995	1.69478	2.03149	2.34058	2.58698	2.91450	3.29053
.9999	1.79500	2.18124	2.54223	2.83636	3.23859	3.71902

NOTE: The values of quantiles $\varepsilon_p = \varepsilon_{K_p}$ are needed in (1.17). This table is adapted from Harter (1964). Here $C = 1/\sqrt{K}$, as in Prentice (1974).

and

$$d_{00} = -\frac{m_1}{n} E[2\varepsilon_{n:m_1+1} h_1(\varepsilon_{n:m_1+1}) + \varepsilon^2_{n:m_1+1} h'_1(\varepsilon_{n:m_1+1})] - \frac{m_2}{n} E[2\varepsilon_{n:n-m_2} h_2(\varepsilon_{n:n-m_2}) + \varepsilon^2_{n:n-m_2} h'_2(\varepsilon_{n:n-m_2})] - \frac{n-m_1-m_2}{n} - \frac{1}{n}\sum_{i=m_1+1}^{n-m_2} E[2\varepsilon_{n:i} g(\varepsilon_{n:i}) + \varepsilon^2_{n:i} g'(\varepsilon_{n:i})]. \tag{2.6}$$

The values of the covariance matrix may be difficult to compute for small n. For some special cases, however, under the assumptions $m_1/n \to q_1$ and $m_2/n \to q_2$ as $n \to \infty$, the asymptotic covariance matrix can be computed from available information. Let $F \equiv F_K$ denote the df of $\varepsilon \equiv (L - EL)/\sqrt{\text{var}\ [L]}$ with $L = \log$ gamma (K) as in Section 1. Then for $K = 1$ (extreme value) and $K = \infty$ (normal), the asymptotic covariance matrices, obtained from Harter (1970), are given in Table 4.

More generally, one can use the observed information by making simple modifications in the preceding formulas. In fact, this is probably preferable.

Table 4. *Asymptotic Covariance Matrix of the MLE's of Log Gamma Parameters*

		$K = 1$			$K = \infty$		
q_1	q_2	a_{00}	a_{01}	a_{11}	a_{00}	a_{01}	a_{11}
.0	.0	.607927	−.473999	.977503	.500000	.000000	1.000000
.0	.1	.767044	−.482759	.979312	.585925	.041136	1.020092
.0	.2	.928191	−.456165	.984094	.688692	.106905	1.062323
.0	.3	1.122447	−.392241	1.005537	.819749	.206568	1.138257
.0	.4	1.372781	−.269610	1.066162	.994759	.359824	1.272656
.0	.5	1.716182	−.042759	1.216920	1.241453	.605233	1.517094
.1	.0	.654702	−.511948	1.008303	.702692	.000000	1.035011
.1	.1	.842250	−.532192	1.011820	.847527	.071658	1.070615
.1	.2	1.639534	−.513785	1.013933	1.041120	.187749	1.140391
.1	.3	1.287741	−.454085	1.028702	1.315918	.379562	1.274494
.1	.4	1.625480	−.325211	1.078433	1.736943	.715075	1.542208
.1	.5	2.124585	−.062235	1.217902	2.458565	1.364988	2.128202

NOTE: The shape parameters are $K = 1$ and $K = \infty$. Proportions q_1 are censored from below, and proportions q_2 are censored from above.

3. AN APPROXIMATE CONDITIONAL PROCEDURE AND THE MONTE CARLO METHOD

The accuracy of the preceding approximation will be examined mainly for the location-scale model, since for the general regression model few standards for comparisons are available. Even for the location-scale model it is difficult to obtain reasonably accurate determinations for the γ-percentile $B_n = B_{n,\gamma,p}$ of the pivot $V_p = \sqrt{n}(\hat{Y}_p - y_p)/\hat{\sigma}$, since the distribution of V_p is typically analytically intractable, the normal error model forming a notable exception.

For the extreme value error model ($K = 1$), Bain (1978) (based on a method of Thomas et al. 1970) has tabulated estimates $\tilde{u}_n$ of $u_{n,\gamma}(1 - p) \equiv \sigma_L B_{n,\gamma,p}$. [The factor $\sigma_L = 1.28255$ is the standard deviation of the density $\exp(y - \exp(y))$; the conversion factor σ_L arises from the fact that Bain deals with the standard extreme value density, whereas we chose the standardized version with mean zero and variance one. Note also the switch from p to $1 - p$.] Bain obtained these estimates $\tilde{u}_n$ from extensive Monte Carlo simulations for various values of γ, p, and n. The accuracy of Bain's tables is variable and in fact for $p = .02$ these estimates appear to be off considerably. This is indicated by the asymptotic values reversing the monotone trend exhibited by $u_{n,\gamma}(1 - p)$ as $n \to \infty$, and is corroborated by our separate simulations.

Another standard for comparison is made possible by modifying a method for constructing conditional tolerance values, due to Lawless (1975, 1982). The basic difference between Lawless's method and the one based on Bain's tables is that for the former the multiplier B_n is no longer constant for a sample of a given size but changes from sample to sample. In fact, Lawless's B_n is a function of certain ancillary statistics $a = (a_1, \ldots, a_n)$ and the confidence level is conditionally (given a) equal to γ and thus also unconditionally equal to γ. The advantage of this conditional approach is that the conditional distribution of the pivot V_p given a is analytically tractable, although a nontrivial computer program is required for the computation of the conditional percentiles $B_n(a)$. The drawback is that $B_n(a)$ cannot be tabulated, since it changes from sample to sample. To bypass this problem we propose to compute the $B_n(a_0)$ for a pseudo extreme value sample of size n with resultant ancillaries a_0. Our second proposal is thus:

Use the resulting value $B_n(a_0)$ as a substitute for the unconditional percentile B_n. (3.1)

As a pseudo sample we propose to take the following approximation to the expected extreme-value order statistics (compare Blom 1958, p. 73):

$$Y_i = \log\left(-\log\left(1 - \frac{i - .5}{n + .25}\right)\right), \qquad i = 1, \ldots, n. \tag{3.2}$$

4. ACCURACY OF THE APPROXIMATIONS

As is evident from Table 5, our proposals (1.21) and (3.1) and the Monte Carlo values of Bain (1978) agree reasonably well as long as $p \geq .05$. For $p < .05$ both of our methods perform better. In order to judge the performance of $\hat{B}_n$ [our approximaton (1.21)], $B_n =$

Table 5. *Extreme-Value Distribution ($K = 1$)*

P, n	.01 $\hat{B}_n$	.01 B_n	.02 $\hat{B}_n$	.02 B_n	.02 $\tilde{B}_n$	.05 $\hat{B}_n$	.05 B_n	.05 $\tilde{B}_n$	.10 $\hat{B}_n$	.10 B_n	.10 $\tilde{B}_n$
					Confidence Level $\gamma = .90$						
15	6.016 (.8824)	6.428 (.8969)	5.203 (.8820)	5.537 (.8968)	4.850 (.8666)	4.131 (.8846)	4.361 (.8960)	4.288 (.8919)	3.319 (.8879)	3.472 (.8958)	3.460 (.8949)
30	5.290 (.8899)	5.511 (.8974)	4.582 (.8909)	4.761 (.8982)	3.972 (.8547)	3.649 (.8922)	3.773 (.8975)	3.702 (.8938)	2.945 (.8908)	3.026 (.8973)	3.043 (.8989)
80	4.737 (.8982)	4.842 (.9031)	4.109 (.8988)	4.195 (.9033)	3.488 (.8592)	3.284 (.8999)	3.343 (.9041)	3.339 (.9038)	2.662 (.9009)	2.701 (.9032)	2.759 (.9066)
∞		4.041		3.515			2.826			2.310	
					Confidence Level $\gamma = .98$						
15	11.70 (.9812)	11.10 (.9782)	10.12 (.9817)	9.564 (.9777)	8.458 (.9657)	8.034 (.9827)	7.535 (.9773)	7.407 (.9765)	6.447 (.9835)	6.001 (.9775)	5.996 (.9774)
30	9.512 (.9794)	9.211 (.9771)	8.240 (.9796)	7.957 (.9769)	6.966 (.9657)	6.562 (.9807)	6.308 (.9776)	6.195 (.9762)	5.291 (.9814)	5.063 (.9789)	5.059 (.9789)
80	8.067 (.9813)	7.919 (.9808)	6.999 (.9813)	6.861 (.9802)	5.873 (.9629)	5.593 (.9821)	5.470 (.9803)	5.382 (.9787)	4.532 (.9817)	4.422 (.9795)	4.433 (.9797)
∞		6.474		5.632			4.530			3.703	

NOTE: $\hat{B}_n$—Approximation (1.21), B_n—Lawless (pseudo sample), $\tilde{B}_n$—Bain; the observed confidence level $\hat{\gamma}$ is in parentheses; 10,000 replications.

Table 6. *Extreme Value Distribution (K = 1), Censored Data*

p, n	γ = .90: .05 $\hat{B}_n$	.05 B_n	.10 $\hat{B}_n$	.10 B_n	.50 $\hat{B}_n$	.50 B_n
	Censoring Pattern $q_1 = 0$, $q_2 = .2$					
10	5.572 (.8889)	6.063 (.9048)	4.340 (.8912)	4.649 (.9058)	1.542 (.9057)	1.482 (.8984)
30	4.207 (.8862)	4.359 (.8944)	3.305 (.8866)	3.397 (.8951)	1.326 (.9042)	1.285 (.8971)
80	3.712 (.8981)	3.783 (.9019)	2.930 (.8993)	2.972 (.9025)	1.256 (.9049)	1.229 (.8998)
∞	3.118		2.483		1.188	
	Censoring Pattern $q_1 = 0$, $q_2 = .5$					
10	7.773 (.8698)	9.211 (.9022)	5.680 (.8773)	6.524 (.9022)	1.581 (.9265)	1.340 (.9002)
30	5.137 (.8834)	5.498 (.8959)	3.816 (.8859)	4.015 (.8954)	1.437 (.9195)	1.273 (.8969)
80	4.340 (.8910)	4.500 (.9000)	3.252 (.8935)	3.336 (.8981)	1.413 (.9171)	1.302 (.8998)
∞	3.475		2.642		1.432	

NOTE: $\hat{B}_n$—Approximation (1.21), B_n—Lawless (pseudo sample); the observed confidence level $\hat{\gamma}$ is in parentheses; 10,000 replications.

$B_n(a_0)$ [our conditional approximation (3.1) based on applying Lawless's method to the pseudo sample], and $\tilde{B}_n = \tilde{u}_n/\sigma_L$ (Bain's Monte Carlo tables) for various values of p, γ, and n, a limited Monte Carlo study was performed. In 10,000 replications, using the uniform random number generator of Schrage (1979), the observed confidence levels were recorded. For a given sample size n, one set of 10,000 replications produced observed confidence levels simultaneously for several nominal confidence levels and several percentiles p. Thus the resulting observed confidence levels are only independent between different sample sizes but not within the same sample size. Table 5 shows that $B_n = B_n(a_0)$ serves as a reasonably accurate proxy for the true $B_{n,\gamma,p}$ no matter what the parameters n, γ, p are. Furthermore (also using Monte Carlo tables not presented here):

4.1 The approximation $\hat{B}_n$ tends to improve with increasing n and is remarkably accurate over the range $p = .01, .02, .05, .10, .50$, and $\gamma = .95, .98, .99$, even for sample sizes as low as $n = 15$.

As the nominal confidence level γ decreases towards .75, one notices a slight deterioration in the observed confidence level for $\hat{B}_n$. Our studies showed that this discrepancy becomes more pronounced for $\gamma = .5$. Indeed, a closer examination of our approximation (1.21) reveals that for $\gamma = .5$, the approximation $EA = 0$, neglecting terms of order $1/n$ is no longer satisfactory. It is conceivable that accounting for the term a/n in the expansion $EA = 1 + a/n + \cdots$ (if available) would further enhance the accuracy of $\hat{B}_n$, particularly for nominal γ values near .5. However, these cases have the least practical application, except for $\gamma = .5$, which arises in consideration of median unbiased estimation.

The approximation $B_n(a_0)$ of (3.1) is obviously too complex for many users who do not have access to tables. However:

4.2 The approximation $B_n(a_0)$ of (3.1) seems to be very accurate in all cases considered. When its tables are available, it would seem to be the method of choice.

After completion of our computational work we became aware of a similar, yet different, way of obtaining proxy values $B_n^*(a_0)$ due to Lawless (1974,

Table 7. *Normal Distribution (K = ∞)*

p, n	.01 $\hat{B}_n$	.01 B_n	.02 $\hat{B}_n$	.02 B_n	.05 $\hat{B}_n$	.05 B_n	.10 $\hat{B}_n$	.10 B_n	.50 $\hat{B}_n$	.50 B_n
	Confidence Level γ = .90									
15	3.538 (.8753)	3.866 (.8981)	3.209 (.8753)	3.499 (.8981)	2.733 (.8760)	2.966 (.8956)	2.337 (.8792)	2.521 (.8941)	1.364 (.8932)	1.392 (.8983)
30	3.153 (.8850)	3.323 (.8961)	2.867 (.8851)	3.017 (.8966)	2.456 (.8866)	2.577 (.8979)	2.116 (.8869)	2.212 (.8975)	1.322 (.8960)	1.334 (.8976)
80	2.853 (.8900)	2.933 (.8968)	2.601 (.8911)	2.672 (.8971)	2.241 (.8921)	2.298 (.8983)	1.945 (.8950)	1.991 (.8990)	1.296 (.8995)	1.301 (.9002)
∞	2.467		2.261		1.967		1.730		1.282	
	Confidence Level γ = .98									
15	6.513 (.9766)	6.703 (.9790)	5.888 (.9766)	6.066 (.9787)	4.982 (.9757)	5.141 (.9780)	4.221 (.9749)	4.365 (.9779)	2.293 (.9756)	2.343 (.9770)
30	5.482 (.9767)	5.563 (.9775)	4.974 (.9757)	5.052 (.9771)	4.241 (.9763)	4.314 (.9773)	3.633 (.9760)	3.700 (.9778)	2.166 (.9771)	2.187 (.9780)
80	4.770 (.9799)	4.801 (.9801)	4.343 (.9800)	4.374 (.9804)	3.731 (.9790)	3.762 (.9802)	3.228 (.9793)	3.258 (.9803)	2.095 (.9800)	2.101 (.9800)
∞	3.955		3.622		3.151		2.772		2.054	

NOTE: $\hat{B}_n$—Approximation (1.21), B_n—Exact by Normal Theory; the observed confidence level $\hat{\gamma}$ is in parentheses; 10,000 replications.

Table 8. *Standardized Log Gamma* ($K = .5$)

p, n	.01	.02	.05	.10	.50
			$\hat{B}_n$, $\gamma = .90$		
20	6.539 (.8840)	5.592 (.8849)	4.347 (.8873)	3.415 (.8876)	1.326 (.8977)
30	6.086 (.8942)	5.208 (.8946)	4.054 (.8950)	3.191 (.8958)	1.266 (.9043)
80	5.421 (.8940)	4.645 (.8943)	3.626 (.8959)	2.865 (.8949)	1.183 (.8962)
∞	4.597	3.947	3.098	2.436	1.091
			$\hat{B}_n$, $\gamma = .99$		
20	15.28 (.9929)	13.07 (.9928)	10.17 (.9940)	7.994 (.9942)	3.040 (.9947)
30	13.30 (.9947)	11.39 (.9946)	8.870 (.9946)	6.984 (.9946)	2.727 (.9940)
80	10.84 (.9918)	9.286 (.9918)	7.253 (.9918)	5.731 (.9920)	2.345 (.9928)
∞	8.343	7.165	5.624	4.475	1.979

NOTE: $\hat{B}_n$—Approximation (1.21); the observed confidence level $\hat{\gamma}$ is in parentheses; 10,000 replications.

1980). Lawless's proxy method works equally well for complete samples but does not extend to type II censored data. However:

4.3 Our modification (3.1) extends easily to the censored case by censoring the pseudo sample.

For censoring fractions of 20% and 50% the performance of $B_n = B_n(a_0)$ and $\hat{B}_n$ is exhibited in Table 6.

For the normal error model ($K = \infty$), the exact percentiles B_n were computed from the noncentral t distribution and they serve as a standard for comparison. Again a limited Monte Carlo study was performed in order to judge the discrepancies between B_n and $\hat{B}_n$ in terms of the observed versus nominal confidence levels. The quality of the approximation $\hat{B}_n$ basically parallels the extreme value case (see Table 7).

A similar study for B_n (without benefit of exact or pseudo exact values to compare to) was conducted for the log gamma (K) family with $K = .5$. The observed confidence levels relative to the nominal levels show roughly the same pattern as observed in the other cases (see Table 8).

Concerning the accuracy of (1.21) in the regression case, we point out that in the normal case, $Z + tA$ in (1.13) is a linear combination of a standard normal and an independent chi random variable, just as in the location-scale case. Hence the approximation (1.21) should do comparably well. This is confirmed in a limited form in Table 9. Another comparison is made possible in the Weibull regression case through some Monte Carlo results of McCool (1980). When the pivot percentiles $u^*(p, \gamma, s)$ as estimated in McCool's table 1 are converted via $B_n(p, \gamma, s) = \sqrt{n}\, u^*(p, \gamma, s)/1.28255$ into our scale, we can compare McCool's converted estimates $\bar{B}_n$ with our approximation $\hat{B}_n$ of (1.21). McCool covers the case $p = .10$, $\gamma = .95, .5, .05$, and $n = 40$. McCool's covariate s attains four levels $s_1, \ldots, s_4$, and 10 observations are taken at each level. These covariates convert in our notation to $w_i = Z_i - \bar{Z}$ with $Z_i = -\log s_i$. We omit $\gamma = .5$ from the comparison for reasons explained previously. Table 9 compares $\hat{B}_n$ with $\bar{B}_n$ in the Weibull case and for the simple linear regression setup $\hat{B}_n$ is compared with the exact values B_n (obtained from the noncentral t distribution) in the normal case. Whereas in the normal case the (regression) approximation quality parallels that of the location-scale case, we seem to be at some odds with McCool's estimates in the Weibull case. Some of that may be explained by the standard deviation of McCool's Monte Carlo estimates. A rough idea (possibly too small) of this standard deviation may be obtained by assuming an approximate normal distribution for the pivot whose distribution was simulated by McCool. Concentrating on the case (of widest discrepancy between $\bar{B}_n$ and $\hat{B}_n$) with $\gamma = .95$ and $s_0 = .75$, we find a standard deviation of .062 for the quantile estimator of the 95th percentile of the pivot distribution. [The standard deviation of the pivot distribution was obtained from $(5.04 - (-4.56))/3.3 = 2.91$; we then calculated $(.05 \times .95)/(10{,}000\, \varphi^2(1.645)/2.91^2) = (.062)^2$.] This seems to indicate that Monte Carlo methods for establishing extreme population percentiles are still rather unre-

Table 9. *Simple Linear Regression* ($p = .10$)

Covariates		Weibull Case				Normal Case			
		$\gamma = .95$		$\gamma = .05$		$\gamma = .95$		$\gamma = .05$	
		$\bar{B}_n$	$\hat{B}_n$	$\bar{B}_n$	$\hat{B}_n$	B_n	$\hat{B}_n$	B_n	$\hat{B}_n$
$s_0 = .75$	$w_0 = .3133$	5.04	5.66	−4.56	−4.07	5.89	5.78	−4.75	−4.78
$s_1 = .87$	$w_1 = .1649$	4.05	4.51	−3.20	−2.92	4.02	3.92	−2.89	−2.90
$s_2 = .99$	$w_2 = .0356$	3.71	3.98	−2.55	−2.39	2.99	2.89	−1.88	−1.87
$s_3 = 1.09$	$w_3 = -.0606$	3.78	4.03	−2.62	−2.45	3.11	3.01	−1.99	−1.99
$s_4 = 1.18$	$w_4 = -.1399$	4.04	4.36	−3.00	−2.77	3.75	3.65	−2.62	−2.64

NOTE: $\bar{B}_n$—McCool's estimate, $\hat{B}_n$—Approximation (1.21), B_n—exact (normal case); 10 observations at each s_i, (w_i) $i = 1, \ldots, 4$, $n = 40$.

liable, even for 10,000 replications. In this same vein, recall the difficulties with Bain's Monte Carlo values for percentiles well out in the tail.

5. AN EXAMPLE

Failure strengths of ceramic specimens (Norton NC-132 HP-Si_3N_4) from each of three billets (N, A, and B) are presented in Table 10. Each specimen was tested with a flexural strength (four-point) test with a crosshead speed of .005 mm per minute and a stress rate of 20 MPa per minute at temperature 1204°c (Larsen et al. 1981, p. 188).

Let Y_{ij} represent the log failure time of the jth observation in the ith billet for $1 \le i \le 3$, $1 \le j \le 10$; with $i = 1, 2, 3$ denoting billets N, A, B, respectively. Two possible models for the data are as follows:

Model A. $Y_{ij} = \mu + a_i + \sigma\varepsilon_{ij}(K)$ with side condition $a_1 = 0$

Model B. $Y_{ij} = \mu + \sigma\varepsilon_{ij}(K)$,

where the $\varepsilon_{ij}(K)$ are iid standardized log gamma (K) for $1 \le i \le 3, 1 \le j \le 10$.

When Model A is assumed and $\hat{\mu}, \hat{a}_2, \hat{a}_3, \hat{\sigma}$ are then calculated for varying values of K, we find that $K = 1$ very nearly maximizes the likelihood equation. Assuming $K = 1$ is equivalent to assuming that exp (Y_{ij}) follows a Weibull distribution with modulus $b_i =$ exp $(\mu + a_i - \sigma\psi(1)/\sqrt{\psi'(1)})$ and characteristic strength $c = \sqrt{\psi'(1)}/\sigma$. (The Weibull distribution is commonly used to model failure strength of ceramic materials).

Assuming Model A with $K = 1$, we find $\hat{\mu} = 6.52875$, $\hat{a}_2 = .03735$, $\hat{a}_3 = .11320$, and $\hat{\sigma} = .11221$. In the setup of Sections 2 and 3, we have $n = 30$; $r = 2$; the matrix $D = 3I_2$, where I_2 is the 2×2 identity matrix; $a_{00} = a_{22} = .607927$; $a_{01} = -.473999$; $a_{11} = .977503$; and $\varepsilon_{.1} = -1.305$. Then for all three billets, $B_{30}(\gamma = .95,\ p = .1) \doteq 4.374$ from (1.21); so $\hat{Y}_{.1}(N) = 6.38232$ with approximate 95% tolerance bound $\hat{Y}_{.1}(N) - B_{30}\hat{\sigma}/\sqrt{30} = 6.29271$, $\hat{Y}_{.1}(A) = 6.41967$ with 95% tolerance bound 6.33006, and $\hat{Y}_{.1}(B) = 6.49552$ with 95% tolerance bound 6.40591.

When Model B is assumed and $\hat{\mu}(K)$ and $\hat{\sigma}(K)$ are calculated for varying values of K, we find $K = 1$ is close to a local maximum of the likelihood equation, although the value of K yielding the global maximum appears to be quite large. Assuming $K = 1$, we find $\hat{\mu} = 6.57244$, $\hat{\sigma} = .13373$, and $\hat{Y}_{.1} = 6.39792$. The approximate values of $B_{30}(\gamma = .95,\ p = .1)$ are 3.971 [from (1.21)], 3.943 [from (3.1)], or 4.008 (from Bain 1978); these yield 95% tolerance bounds for $y_{.1}$ of 6.30096 [from (1.21)], 6.30165 [from (3.1)], or 6.30006 (from Bain 1978).

Assuming Model B with $K = \infty$, we find $\hat{\mu} = \bar{y} = 6.57994$, $\hat{\sigma} = \sqrt{(n-1)/ns} = .10768$, and $\hat{Y}_{.1} = 6.44189$. Now $B_{30}(\gamma = .95, p = .1) \doteq 2.793$, yielding an approximate 95% tolerance bound for $y_{.1}$ of 6.38698.

If Model B is assumed, probability plots suggest the log-normal model ($K = \infty$) is more appropriate here. The Weibull estimates of location and scale are strongly influenced by the highest order statistics, and they give conservative estimates of the lower percentiles.

Assume Model B with $K = 1$, and trim the six highest order statistics. Then, in the setup of Section 2, we have $q_1 = .0$ and $q_2 = .2$. We find $\hat{\mu} = 6.57043$, $\hat{\sigma} = .09210$, and $\hat{Y}_{.1} = 6.45024$. Then $B_{30}(\gamma = .95, p = .1) \doteq 4.545$, yielding an approximate 95% tolerance bound for $y_{.1}$ of 6.37382. These values agree more closely with those given by Model B with $K = \infty$ than by Model A with $K = 1$.

A note is in order on the comparative merits of the different models considered. Often the estimate of interest in this situation is the estimate of the tolerance bound for an overall percentile of the data. In this case Model B would be more appropriate. However, comparative box plots of the billets suggest that Model A may more accurately represent the random variation of the data. This is reinforced by calculating the likelihood ratio for Model A, $K = 1$, and Model B, $K = 1$, which has a value of 3.45193.

Comparison of estimates of tolerance bounds for various values of K (while tabulating the likelihood of the sample for these different values of K) provides an approach to robustness (see Fraser 1976). See Prentice (1974) for another approach.

Table 10. *Failure Strength of $Si_3\ N_4$ Specimens*

Billet		
N	*A*	*B*
640	522	658
660	629	676
670	632	696
681	712	696
696	730	721
707	748	735
737	759	761
741	768	828
766	781	875
771	826	917

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