Actuator Tolerancing Case Study

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1 Problem Description

The following geometric problem arose in an actuator design situation and was posed to me by Neal Huynh who also provided the illustration in Figure 1. We have a triangle with legs A, R and B. The angle between A and R is denoted by θ. A and R are subject to some manufacturing variation with tolerance specifications $A \in A_0 \pm T_A$ and $R \in R_0 \pm T_R$. The leg B, representing the actuator, can be adjusted such that the angle θ agrees exactly with a specified value θ_0. Once $\theta = \theta_0$ is achieved the actuator is in its neutral position. From there B can extend or contract by an amount $\pm\Delta$ thus changing the angle θ to a maximum and minimum value $\theta_{\max}$ and $\theta_{\min}$, respectively. Setting $A = A_0$ and $R = R_0$ will result in nominal values for $\theta_{\max}$ and $\theta_{\min}$, denoted by $\theta_{\max,0}$ and $\theta_{\min,0}$, respectively.

The question of interest is: How much variation of $\theta_{\max}$ and $\theta_{\min}$ around $\theta_{\max,0}$ and $\theta_{\min,0}$ can we expect due to the variations in A and R over their respective tolerance ranges $A_0 \pm T_A$ and $R_0 \pm T_R$?

Any actual dimensions used in this case study are fictitious and are used for illustration purposes only. Any referenced R functions can be found in the R work space posted at

`http : //www.stat.washington.edu/fritz/Stat498B.html`.

The statistical analysis platform R is freely available from `http://cran.r-project.org/`.

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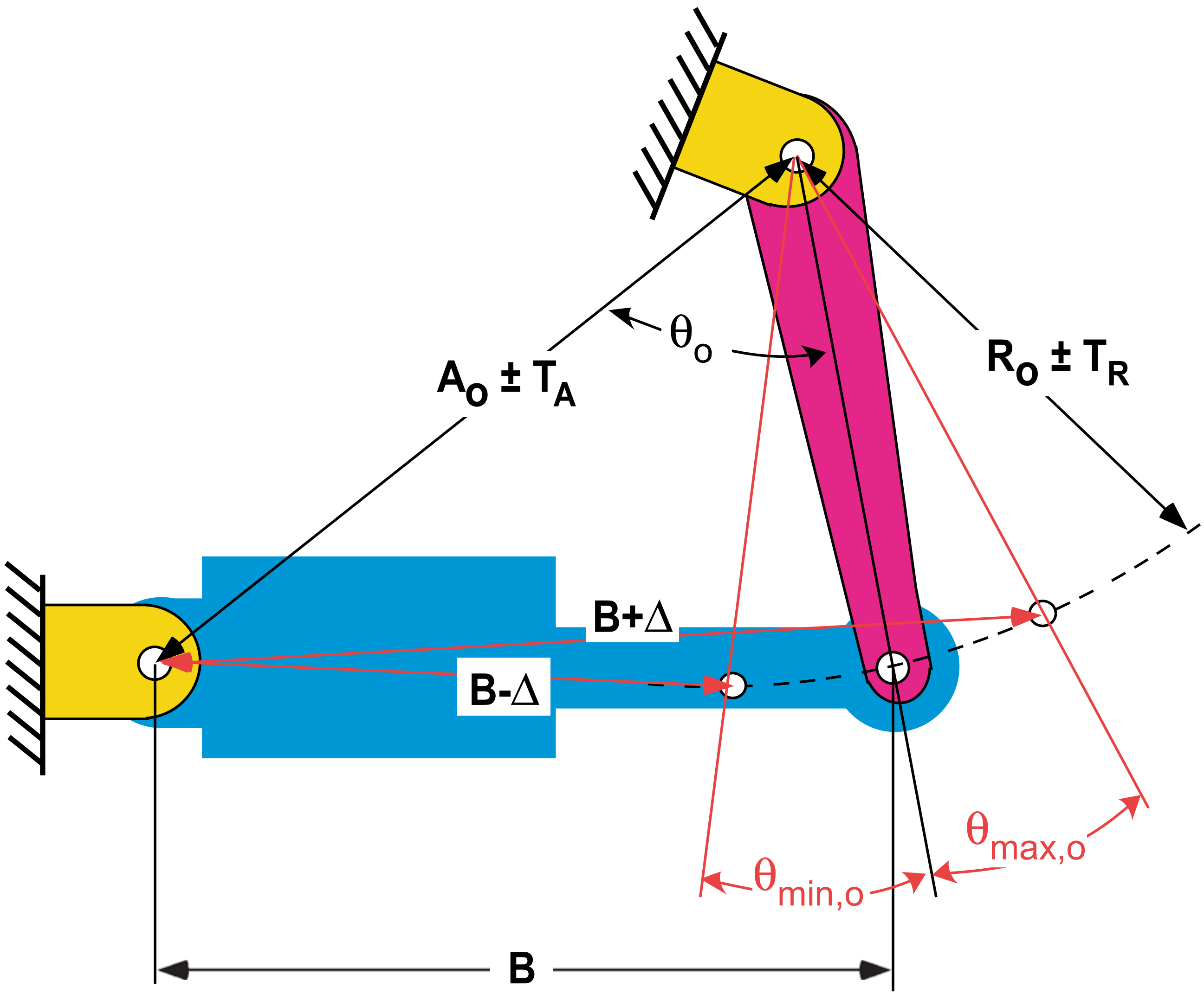


Figure 1: Actuator Geometry

2 Geometric Considerations

Given A, R and θ_0 the length of the (neutral position) actuator length can be expressed as

$$B = B(A, R) = \sqrt{A^2 + R^2 - 2AR\cos(\theta_0)}\ .$$

When the actuator is extended/contracted by the amount $x = \pm\Delta$ from the neutral position we get the following expression for θ_x

$$\theta_x = 2\arctan\left(\sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}}\right)\ ,$$

where $B_x = B(A, R) + x$ and $s_x = (A + R + B_x)/2$. Note that θ_Δ corresponds to $\theta_{\max}$ and $\theta_{-\Delta}$ corresponds to $\theta_{\min}$. We see from the above expressions that θ_x is affected by A and R in a variety of ways, either directly and indirectly through $B(A, R)$. To express this we will also write more explicitly

$$\theta_{\max} = \theta_{\max}(A, R) \qquad \text{and} \qquad \theta_{\min} = \theta_{\min}(A, R)\ .$$

3 Statistical Tolerancing via Simulation

The simplest way of dealing with the variation behavior of $\theta_\Delta = \theta_{\max}$ and $\theta_{-\Delta} = \theta_{\min}$ due to variation in A and R is through simulation. Since R allows vectorized calculations one can generate two N-vectors of A and R values from $\mathcal{N}(\mu_A, (T_A/3)^2)$ and $\mathcal{N}(\mu_R, (T_R/3)^2)$, respectively, calculate the correspondingly adjusted $B = B(A, R)$ vector and from that the N-vector of $\theta_{\max}$ and $\theta_{\min}$, respectively.

Here $\mu_A = A_0$ and $\mu_R = R_0$ follows the usual convention of assuming that the variations in A and R are centered on the respective tolerance intervals, i.e., we don't assume mean shifts. Furthermore, the choice of $\sigma_A = T_A/3$ and $\sigma_R = T_R/3$ also corresponds to the usual practice of interpreting the $\pm T$ tolerance range as a $\pm 3\sigma$ range of a normal distribution.

This simulation was carried out by the R function `theta.simNN` using $N = 10^6$. This function produced the plot shown in Figure 2. It took just a few seconds to run. The vertical bars on either side of the shown histograms give the $\pm 3\sigma$ limits and the respective estimates of the 3σ value are shown as T_1 and T_2, respectively. The respective σ's are estimated by taking the sample standard deviations of the N simulated $\theta_{\max}$ and $\theta_{\min}$ values.

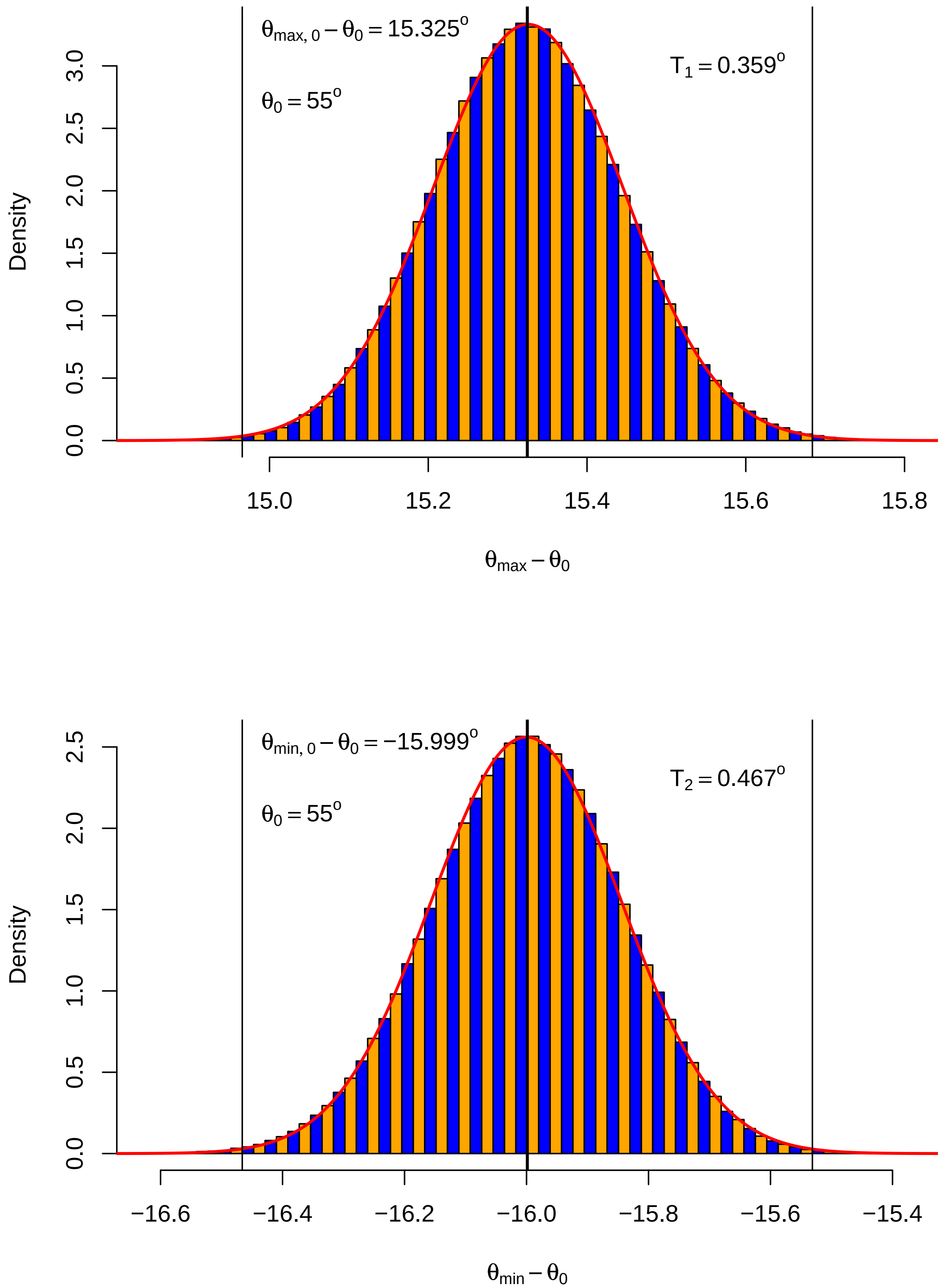


Figure 2: Simulation of 1,000,000 Deviations of $\theta_{\max}$ and $\theta_{\min}$ from θ_0

$\Delta = 1.6$, $\quad A \sim \mathcal{N}(12.8, (.12/3)^2) \quad$ and $\quad R \sim \mathcal{N}(6, (.14/3)^2)$

4 Statistical Tolerancing via RSS

Here RSS stands for "Root Sum Square" which is a "simple" analytical formula for arriving at very good approximations for T_1 and T_2. These formulae are given by

$$T_1 = \sqrt{a^2_{\max,A} \times T_A^2 + a^2_{\max,R} \times T_R^2} \qquad \text{and} \qquad T_2 = \sqrt{a^2_{\min,A} \times T_A^2 + a^2_{\min,R} \times T_R^2}\,,$$

where

$$a_{\max,A} = \frac{\partial \theta_{\max}}{\partial A}, \quad a_{\max,R} = \frac{\partial \theta_{\max}}{\partial R}, \quad a_{\min,A} = \frac{\partial \theta_{\min}}{\partial A}, \quad \text{and} \quad a_{\min,R} = \frac{\partial \theta_{\min}}{\partial R}$$

and all derivatives are evaluated at the nominal values (A_0, R_0) of (A, R).

The basis of these RSS formulae is the linearization of $\theta_x(A, R)$ in the neighborhood of (A_0, R_0), i.e.,

$$\theta_x(A,R) = \theta_x(A_0,R_0) + (A - A_0) \times \left.\frac{\partial \theta_x}{\partial A}\right|_{A=A_0, R=R_0} + (R - R_0) \times \left.\frac{\partial \theta_x}{\partial R}\right|_{A=A_0, R=R_0}\,,$$

which is then taken as an approximation for $\theta_x(A, R)$ near $(A, R) = (A_0, R_0)$. The quality of this approximations depends on the smoothness of either function $\theta_x = \theta_\Delta = \theta_{\max}(A, R)$ and $\theta_x = \theta_{-\Delta} = \theta_{\max}(A, R)$ with respect to A and R at (A_0, R_0), and it also depends on the tolerances T_A and T_R of the contributing variation terms, because that determines over what range one needs to approximate the respective functions. All this assumes of course that θ_x is differentiable near $(A, R) = (A_0, R_0)$. We point this out since there are tolerance situation where differentiability is an issue and in that case the RSS paradigm does not work.

The interpretation of T_1 and T_2 as obtained by the RSS formula is that they represent 3σ values for $\theta_{\max}$ and $\theta_{\min}$, respectively, just as the input tolerances represented 3σ values for the contributing variation terms A and R. The basic principle used here is that the variance (the square of the standard deviation σ) of a sum is the sum of the variances when the summands are uncorrelated (or even stronger, when they are statistically independent). Furthermore, the variance of a constant is zero and $\sigma^2(a \times A) = a^2 \times \sigma^2(A)$, where a is a constant factor and A is a random variable.

In order to apply the RSS formula one needs to evaluate the required derivatives. We will not give explicit formulas for $a_{\max,A}, \ldots, a_{\min,R}$ but will instead present the stepping stones for evaluating them. We will do this simultaneously for $x = \pm\Delta$. The first stepping stone is

$$\frac{\partial \theta_x}{\partial A} = \frac{2}{1 + \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \frac{\partial}{\partial A} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}}$$

and

$$\frac{\partial \theta_x}{\partial R} = \frac{2}{1 + \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \frac{\partial}{\partial R} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \, .$$

Next we have

$$\frac{\partial}{\partial A} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} = \left\{ 2\sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \right\}^{-1} \frac{\partial}{\partial A} \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}$$

and

$$\frac{\partial}{\partial R} \sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} = \left\{ 2\sqrt{\frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)}} \right\}^{-1} \frac{\partial}{\partial R} \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)} \, .$$

We also have the following list of derivative expressions

$$\frac{\partial B_x}{\partial A} = \frac{A - R\cos(\theta_0)}{\sqrt{A^2 + R^2 - 2AR\cos(\theta_0)}} \quad \text{and} \quad \frac{\partial B_x}{\partial R} = \frac{R - A\cos(\theta_0)}{\sqrt{A^2 + R^2 - 2AR\cos(\theta_0)}}$$

$$\frac{\partial (s_x - A)}{\partial A} = \frac{1}{2}\left(\frac{A - R\cos(\theta_0)}{B} - 1\right) \quad \text{and} \quad \frac{\partial (s_x - R)}{\partial A} = \frac{1}{2}\left(\frac{A - R\cos(\theta_0)}{B} + 1\right)$$

$$\frac{\partial (s_x - A)}{\partial R} = \frac{1}{2}\left(\frac{R - A\cos(\theta_0)}{B} + 1\right) \quad \text{and} \quad \frac{\partial (s_x - R)}{\partial R} = \frac{1}{2}\left(\frac{R - A\cos(\theta_0)}{B} - 1\right)$$

$$\frac{\partial s_x}{\partial A} = \frac{1}{2}\left(\frac{A - R\cos(\theta_0)}{B} + 1\right) \quad \text{and} \quad \frac{\partial s_x}{\partial R} = \frac{1}{2}\left(\frac{R - A\cos(\theta_0)}{B} + 1\right)$$

$$\frac{\partial (s_x - B_x)}{\partial A} = \frac{1}{2}\left(1 - \frac{A - R\cos(\theta_0)}{B}\right) \quad \text{and} \quad \frac{\partial (s_x - B_x)}{\partial R} = \frac{1}{2}\left(1 - \frac{R - A\cos(\theta_0)}{B}\right) \, .$$

$$\begin{aligned} &\frac{\partial}{\partial A} \frac{(s_x - A)(s_x - R)}{s_x(s_x - B_x)} \\ &\quad = \frac{1}{s_x^2(s_x - B_x)^2} \left\{ \left[(s_x - R)\frac{\partial}{\partial A}(s_x - A) + (s_x - A)\frac{\partial}{\partial A}(s_x - R) \right] s_x(s_x - B_x) \right. \\ &\qquad \left. -(s_x - A)(s_x - R) \left[(s_x - B_x)\frac{\partial}{\partial A} s_x + s_x \frac{\partial}{\partial A}(s_x - B_x) \right] \right\} \end{aligned}$$

$$\frac{\partial}{\partial R}\frac{(s_x-A)(s_x-R)}{s_x(s_x-B_x)}$$

$$= \frac{1}{s_x^2(s_x-B_x)^2}\left\{\left[(s_x-R)\frac{\partial}{\partial R}(s_x-A)+(s_x-A)\frac{\partial}{\partial R}(s_x-R)\right]s_x(s_x-B_x)\right.$$
$$\left.-(s_x-A)(s_x-R)\left[(s_x-B_x)\frac{\partial}{\partial R}s_x+s_x\frac{\partial}{\partial R}(s_x-B_x)\right]\right\} .$$

Rather than just using these expressions as they are, it is advisable to simplify them somewhat to avoid significance loss in the calculations. Thus we obtained the following reduced expressions:

$$(s_x-R)\frac{\partial}{\partial A}(s_x-A)+(s_x-A)\frac{\partial}{\partial A}(s_x-R) = \frac{R}{2}[1-\cos(\theta_0)]+\frac{x}{2B}[A-R\cos(\theta_0)]$$

$$(s_x-B_x)\frac{\partial}{\partial A}s_x+s_x\frac{\partial}{\partial A}(s_x-B_x) = \frac{R}{2}[1+\cos(\theta_0)]-\frac{x}{2B}[A-R\cos(\theta_0)]$$

$$(s_x-R)\frac{\partial}{\partial R}(s_x-A)+(s_x-A)\frac{\partial}{\partial R}(s_x-R) = \frac{A}{2}[1-\cos(\theta_0)]+\frac{x}{2B}[R-A\cos(\theta_0)]$$

$$(s_x-B_x)\frac{\partial}{\partial R}s_x+s_x\frac{\partial}{\partial R}(s_x-B_x) = \frac{A}{2}[1+\cos(\theta_0)]-\frac{x}{2B}[R-A\cos(\theta_0)] .$$

The R function `deriv.theta` produced the following derivatives for $A_0 = 12.8$, $R_0 = 6$, $\theta_0 = 55°$, and $\Delta = 1.6$

$$\frac{\partial\theta_{\max}}{\partial A} = -.00006636499 \qquad \text{and} \qquad \frac{\partial\theta_{\min}}{\partial A} = -.004038650$$

and

$$\frac{\partial\theta_{\max}}{\partial R} = -0.04473785 \qquad \text{and} \qquad \frac{\partial\theta_{\min}}{\partial R} = 0.05810921 .$$

The RSS calculation using normal variation for A and R then gives the following values for T_1 and T_2 based on $T_A = .12$ and $T_R = .14$

$$T_1 = 0.3588609 \qquad \text{and} \qquad T_2 = 0.4669441 ,$$

which agree remarkably well with the simulated quantities shown in Figure 2.

Note that the derivatives of $\theta_{\max}$ and $\theta_{\min}$ with respect to A are smaller than the derivatives with respect to R by at least an order of magnitude. This will play an important role later on when we consider other distributions governing the variation of A and R.

5 Numerical Differentiation

Clearly, the derivation of the derivatives was quite laborious although the R code for their calculation is relatively compact. However, these derivatives are quite useful in understanding the variation propagation in the tolerance analysis. We therefore point out an obvious alternative approach, namely that of numerical differentiation. This involves simply the evaluation of the function θ_x that is involved in the simulation approach anyway. The respective derivatives are approximated numerically at $(A, R) = (A_0, R_0)$ by calculating the following respective difference quotients for very small values of δ

$$\left.\frac{\partial\theta_x}{\partial A}\right|_{A=A_0,R=R_0} \approx \frac{\theta_x(A_0+\delta, R_0) - \theta_x(A_0, R_0)}{\delta}$$

$$\left.\frac{\partial\theta_x}{\partial R}\right|_{A=A_0,R=R_0} \approx \frac{\theta_x(A_0, R_0+\delta) - \theta_x(A_0, R_0)}{\delta} \; .$$

For $\delta = .00001$ we got the following numerical derivatives using the R function `deriv.numeric`.

$$\left.\frac{\partial\theta_{\max}}{\partial A}\right|_{A=A_0,R=R_0} \approx -.00006636269 \qquad \text{and} \qquad \left.\frac{\partial\theta_{\min}}{\partial A}\right|_{A=A_0,R=R_0} \approx -.004038651$$

and

$$\left.\frac{\partial\theta_{\max}}{\partial R}\right|_{A=A_0,R=R_0} \approx -0.04473777 \qquad \text{and} \qquad \left.\frac{\partial\theta_{\min}}{\partial R}\right|_{A=A_0,R=R_0} \approx 0.05810908 \; .$$

These agree very well with the derivatives obtained previously.

6 RSS for Non-Normal Variation Contributors

The RSS method for statistical tolerancing is a reexpression of the fact that the variance of a sum of independent random variables, $Y = X_1 + \ldots + X_n$, is the sum of the variances, i.e.

$$\sigma_Y^2 = \sigma_{X_1}^2 + \ldots + \sigma_{X_n}^2$$

Furthermore, if the summands are reasonably well behaved, i.e., the variance of none of them dominates the variances of the others, which is expressed by the condition that

$$\max\left\{\frac{\sigma_{X_1}^2}{\sigma_{X_1}^2 + \ldots + \sigma_{X_n}^2}, \ldots, \frac{\sigma_{X_n}^2}{\sigma_{X_1}^2 + \ldots + \sigma_{X_n}^2}\right\} \quad \text{is small,}$$

then one can appeal to the Central Limit Theorem (CLT) and expect that the distribution of the aggregate Y is approximately normally distributed. Thus most of the variation of Y will fall within $\pm 3\sigma_Y$ of the mean μ_Y of Y and thus we equate $T_Y = 3\sigma_Y$ to express the tolerance for the sum.

When the contributing terms X_i are normally distributed with mean μ_{X_i} and variance $\sigma^2_{X_i}$ it is similarly customary to equate $T_i = 3\sigma_{X_i}$ for the specified tolerances of X_i around their respective nominal values μ_{X_i}. Then the above variance addition formula yields

$$T_Y^2 = (3\sigma_Y)^2 = (3\sigma_{X_1})^2 + \ldots + (3\sigma_{X_n})^2 = T_1^2 + \ldots + T_n^2$$

or

$$T_Y = \sqrt{T_1^2 + \ldots + T_n^2}\ ,$$

which is the RSS formula for simple additive tolerance stacking of variation.

More commonly Y is a linear combination of input variation terms X_i, i.e.,

$$Y = a_0 + a_1 X_1 + \ldots + a_n X_n \quad \text{with known constants } a_0, a_1, \ldots, a_n\ .$$

Then the above formulation of the CLT translates to approximate normality for Y provided that

$$\max\left\{\frac{a_1^2\sigma^2_{X_1}}{a_1^2\sigma^2_{X_1} + \ldots + a_n^2\sigma^2_{X_n}}, \ldots, \frac{a_n^2\sigma^2_{X_n}}{a_1^2\sigma^2_{X_1} + \ldots + a_n^2\sigma^2_{X_n}}\right\} \quad \text{is small,}$$

i.e., none of the $a_i^2\sigma_i^2$ terms dominates the others. Again most of the Y variation will fall within $\pm 3\sigma_Y$ of its mean $\mu_Y = a_0 + a_1\mu_{X_1} + \ldots + a_n\mu_{X_n}$. Invoking again the addition property for the variance of a sum of independent summands and also the scaling property $\sigma^2_{a_i X_i} = a_i^2\sigma^2_{X_i}$ we have

$$\sigma_Y^2 = \sigma^2_{a_1 X_1} + \ldots + \sigma^2_{a_n X_n} = a_1^2\sigma^2_{X_1} + \ldots + a_n^2\sigma {X_n}^2\ .$$

For normal X_i we again equate $3\sigma_{X_i} = T_i$ and get the more general RSS tolerance stacking formula

$$T_Y = 3\sigma_Y = \sqrt{a_1^2(3\sigma_{X_1})^2 + \ldots + a_n^2(3\sigma_{X_n})^2} = \sqrt{a_1^2T_1^2 + \ldots + a_n^2T_n^2} \qquad (1)$$

applicable for linear approximations to smooth functions. Of course, for normally distributed X_i we do not need to invoke the CLT to conclude that Y has a normal distribution as well.

The contributing terms X_i are not always normally distributed and sometimes exhibit other variation behavior. One such variation behavior (common in tool wear situations and also in sorting of electrical components by properties) is described by a uniform distribution over the $\pm T_i$ range around the nominal value for X_i. Of course, other distributions for the X_i variations are possible. Because of the CLT effect the RSS method can be extended to cover such situations as well. All it takes is to calculate the correction factor $c_i = 3\sigma_{X_i}/T_i$ which amounts to calculating the standard deviation of the specified distribution covering the tolerance range. For a uniform distribution over the $\pm T_i$ range this factor is $\sqrt{3} = 1.732$. Such factors are calculated for various distributions in *Tolerance Stack Analysis Methods, A Critical Review*[1], by Fritz Scholz (1995).

Making use of such adjustment factors, equation (1) then becomes

$$T_Y = 3\sigma_Y = \sqrt{a_1^2(3\sigma_{X_1})^2 + \ldots + a_n^2(3\sigma_{X_n})^2} = \sqrt{c_1^2 a_1^2 T_1^2 + \ldots + c_n^2 a_n^2 T_n^2}\,. \qquad (2)$$

Figure 3 illustrates the CLT effect when adding 3 independent uniform random variables U_1, U_2, U_3 over the range $(-1, 1)$, respectively. Superimposed in the case of U_1 and $Y = U_1 + U_2 + U_3$ are the normal densities with the same standard deviation, $1/\sqrt{3}$ and 1, respectively. While the normal density does not fit well with the uniform distribution, it does quite well with the distribution of $Y = U_1 + U_2 + U_3$.

Figure 4 shows the corresponding situation when adding just two uniform random variables. The CLT effect is certainly weaker here since we just get a triangular distribution for Y. This would roughly be the situation for the actuator problem if A and R were uniformly distributed over equal width intervals. However, in the linearization of $\theta_x(A, R)$ around (A_0, R_0) the coefficients corresponding to A and R are of different orders of magnitude and since the tolerance ranges for A and R were somewhat comparable this means that the variation of R will swamp the variation of A. Thus the R distribution will dominate the distribution of $\theta_x(A, R)$.

When the contributing terms A and R both have a normal distribution then any linear combination of them also has a normal distribution. Thus one would then expect that the target distributions of $\theta_{\max}(A, R)$ and $\theta_{\max}(A, R)$ are also approximately normally distributed. Here the dominant variation in R has no effect on the shape of the $Y = \theta_x$ distribution. This was clearly illustrated in Figure 2.

[1] `http://www.stat.washington.edu/fritz/Stat498B.html`

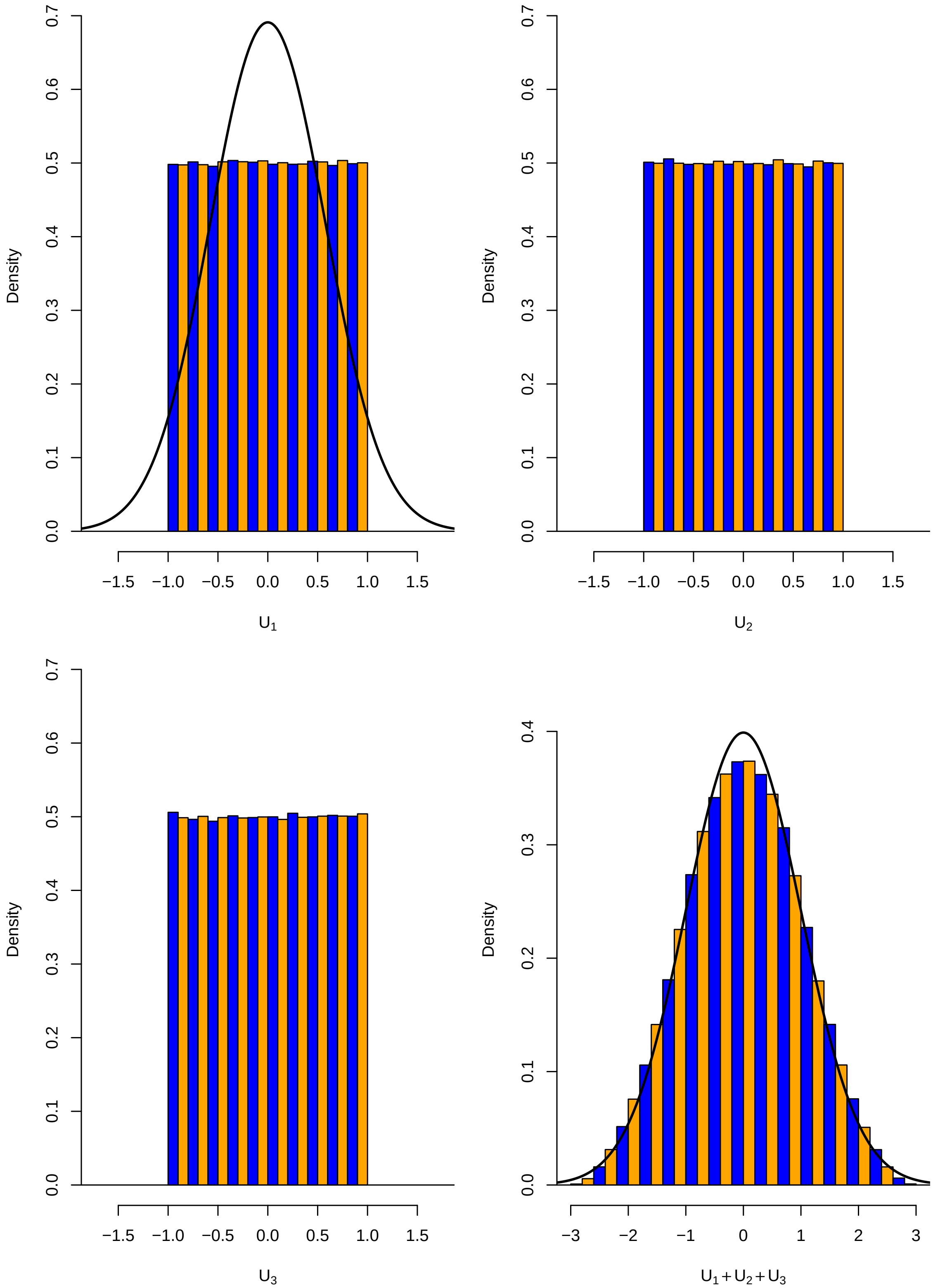


Figure 3: Simulation of Uniform Deviations U_1, U_2, U_3
and their Corresponding Sums $Y = U_1 + U_2 + U_3$
$N = 1,000,000$ Simulations Each

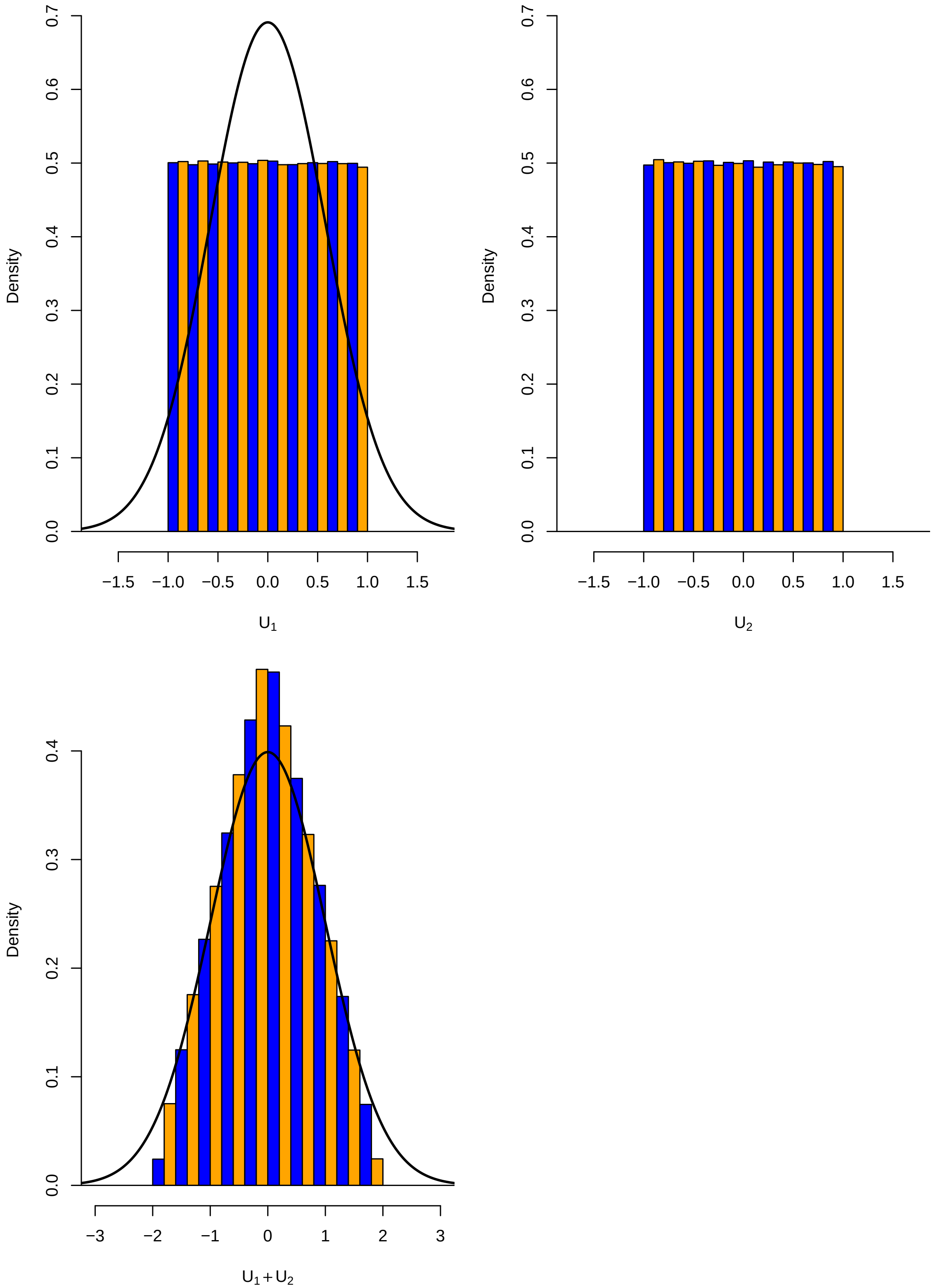


Figure 4: Simulation of Uniform Deviations U_1, U_2
and their Corresponding Sums $Y = U_1 + U_2$
$N = 1,000,000$ Simulations Each

It is quite straightforward to change the distributions describing the variation in A and R. For example, to have $A \sim \mathcal{U}(A_0 - T_A, A_0 + T_A)$ and $R \sim \mathcal{N}(A_0, (T_A/3)^2)$, only lines 4 and 5 in the `theta.simNN` code need to be changed. The resulting function is called `theta.simUN`. The result is shown in Figure 5 and the distribution shape for θ_x is again well approximated by a normal distribution. The reason for this is that the dominant variation part R has a normal distribution.

The functions `theta.simNU` and `theta.simUU` simulate the distribution combinations $A \sim \mathcal{N}(A_0, (T_A/3)^2)$, $R \sim \mathcal{U}(R_0 - T_R, R_0 + T_R)$ and $A \sim \mathcal{U}(A_0 - T_A, A_0 + T_A)$, $R \sim \mathcal{U}(R_0 - T_R, R_0 + T_R)$, respectively. The results are shown in Figures 6 and 7. The shape of the θ_x distribution is no longer normal, but seems closer to a uniform distribution. This is caused by the dominant uniform behavior of R. However, a linear combination of uniform random variables should have a triangular or more generally a trapezoidal density. This is not the case here.

Something else must have affected the distribution of θ_x. The only other factor that could account for this is that the linearization is not very good, i.e., a quadratic component is having some influence over the R variation range. To probe this we ran `theta.simUU` again, this time with tolerances tightened by a factor .1. This produced the plot in Figure 8 which shows a histogram that looks like a very narrow shouldered trapezoid, confirming our conjecture.

Applying the RSS formula (2) while assuming a uniform distribution for both A and R we get

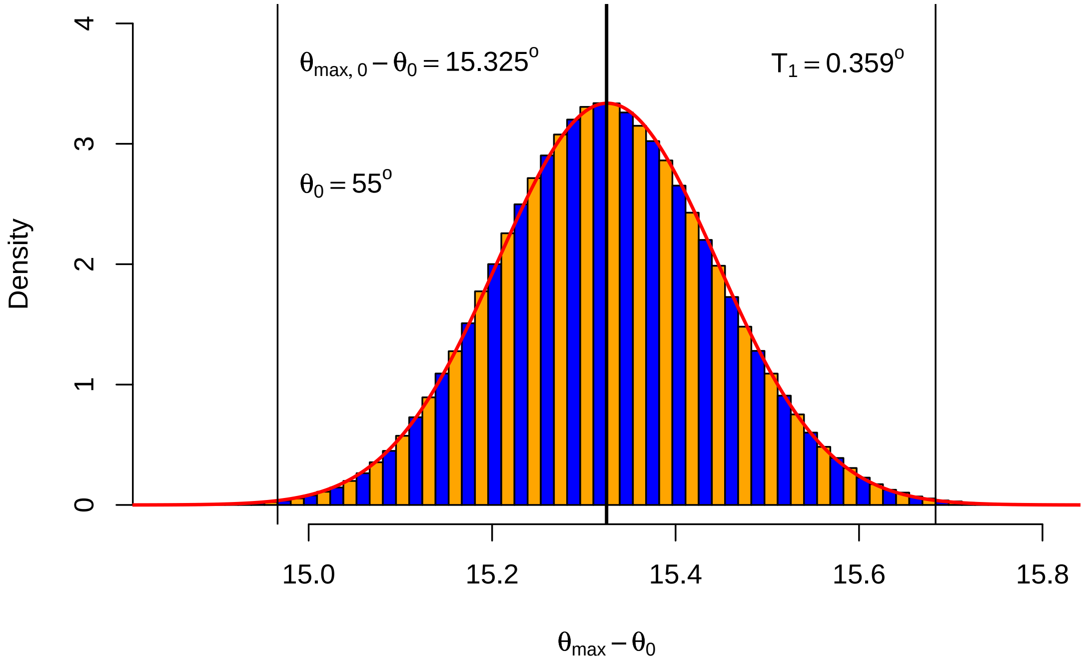
$$T_1 = \sqrt{(-.00006636269)^2 \times 3 \times .12^2 + (-.04473777)^2 \times 3 \times .14^2} \times \frac{360}{2\pi} = 0.6215642^\circ$$

and

$$T_2 = \sqrt{(-.004038651)^2 \times 3 \times .12^2 + (.05810908)^2 \times 3 \times .14^2} \times \frac{360}{2\pi} = 0.8087691^\circ$$

where we used the inflation factor $c = \sqrt{3}$ and the numerically obtained derivatives in both cases. These two values are in reasonable agreement with the values .622° and .81° shown in Figure 7.

This understanding of the variation behavior of $\theta_{\max}$ and $\theta_{\min}$ with respect to the input variations of A and R was definitely influenced by understanding the underlying linear approximations and the relative magnitude of the derivatives since they influence the variances of the summed random terms in the linear combination. The simulation approach would not have given us an easy understanding of the visible effects.



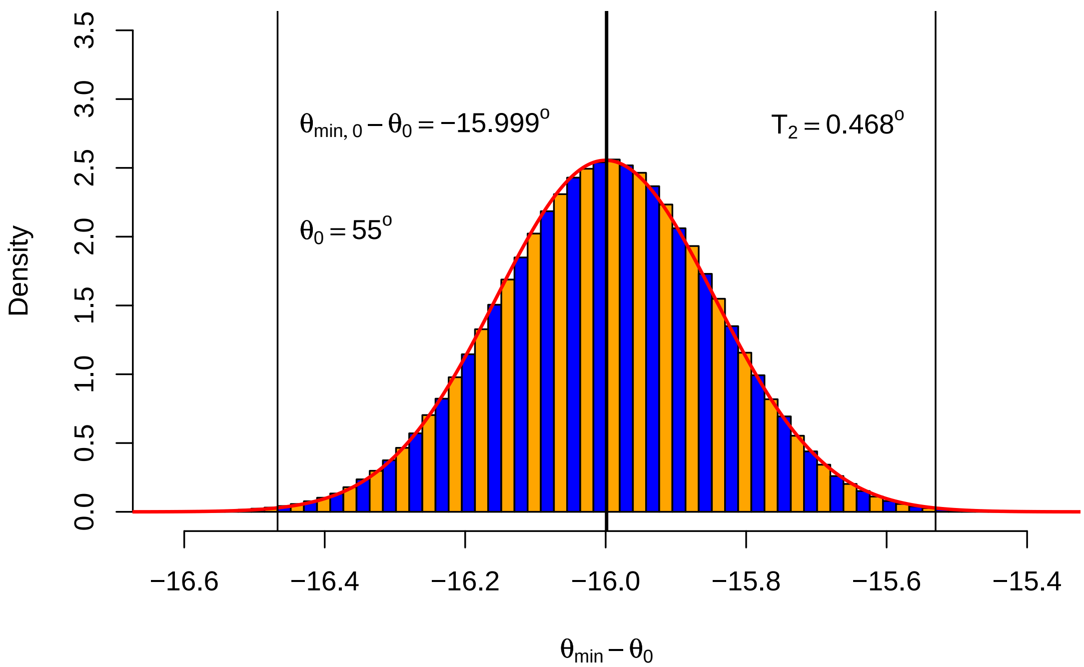


Figure 5: Simulation of 1,000,000 Deviations of $\theta_{\max}$ and $\theta_{\min}$ from θ_0

$\Delta = 1.6$, $\quad A \sim \mathcal{U}(12.8 - .12, 12.8 + .12) \quad$ and $\quad R \sim \mathcal{N}(6, (.14/3)^2)$

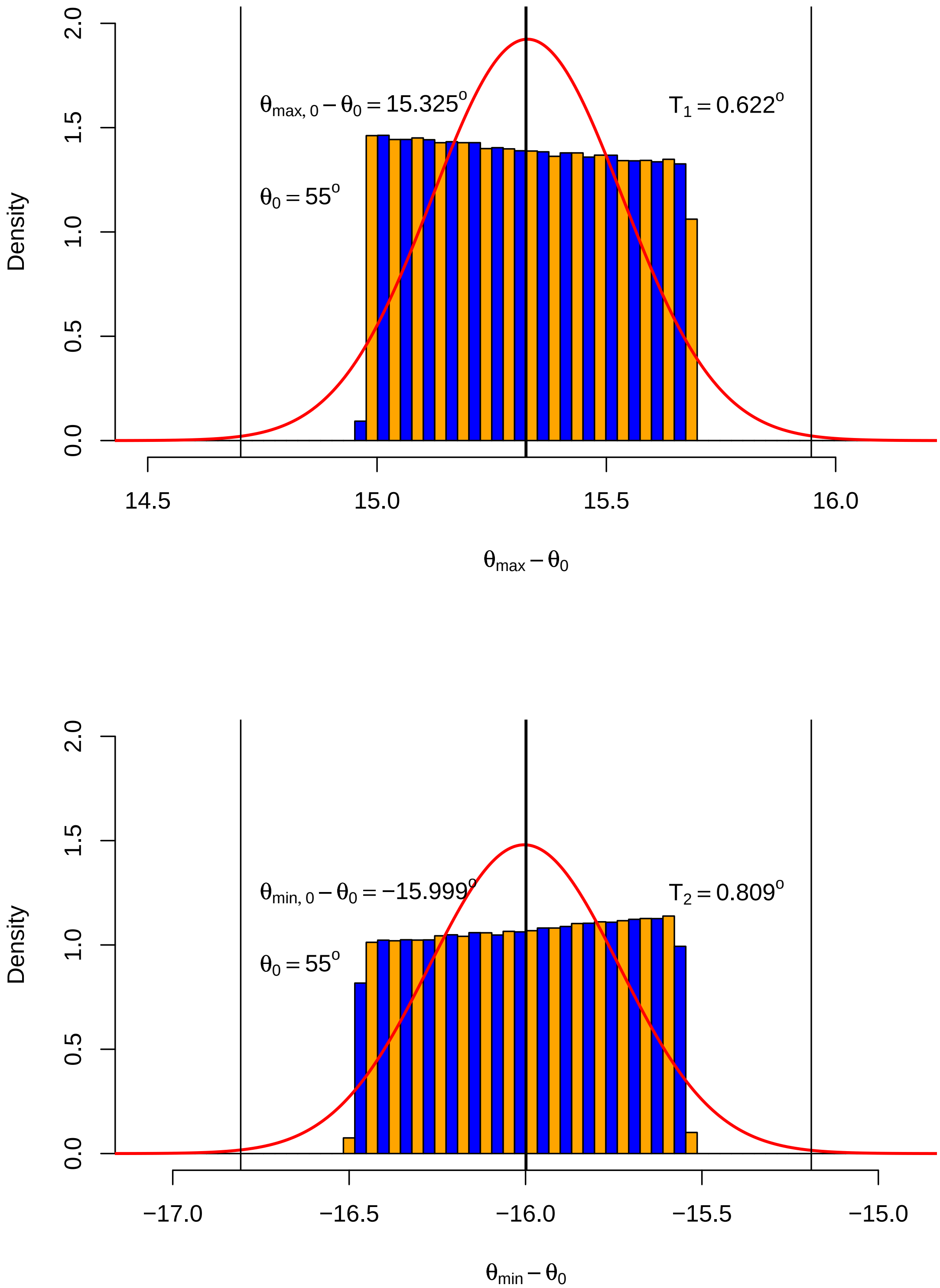
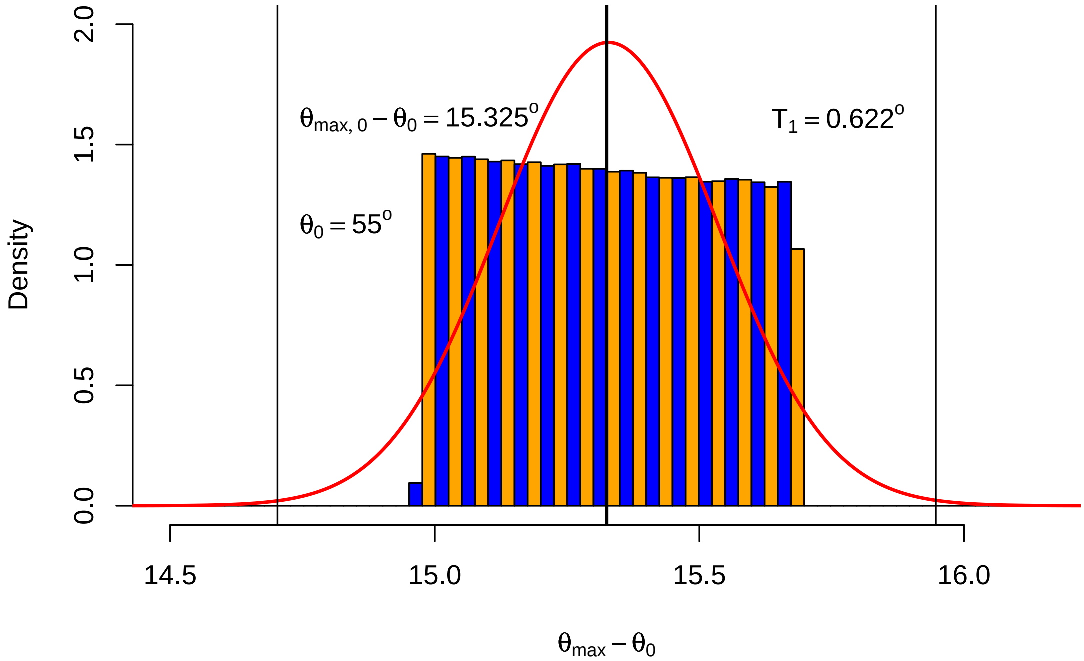


Figure 6: Simulation of 1,000,000 Deviations of $\theta_{\rm max}$ and $\theta_{\rm min}$ from θ_0

$\Delta = 1.6\ , \quad A \sim \mathcal{N}(12.8, (.12/3)^2) \quad \text{and} \quad R \sim \mathcal{U}(6 - .14, 6 + .14)$



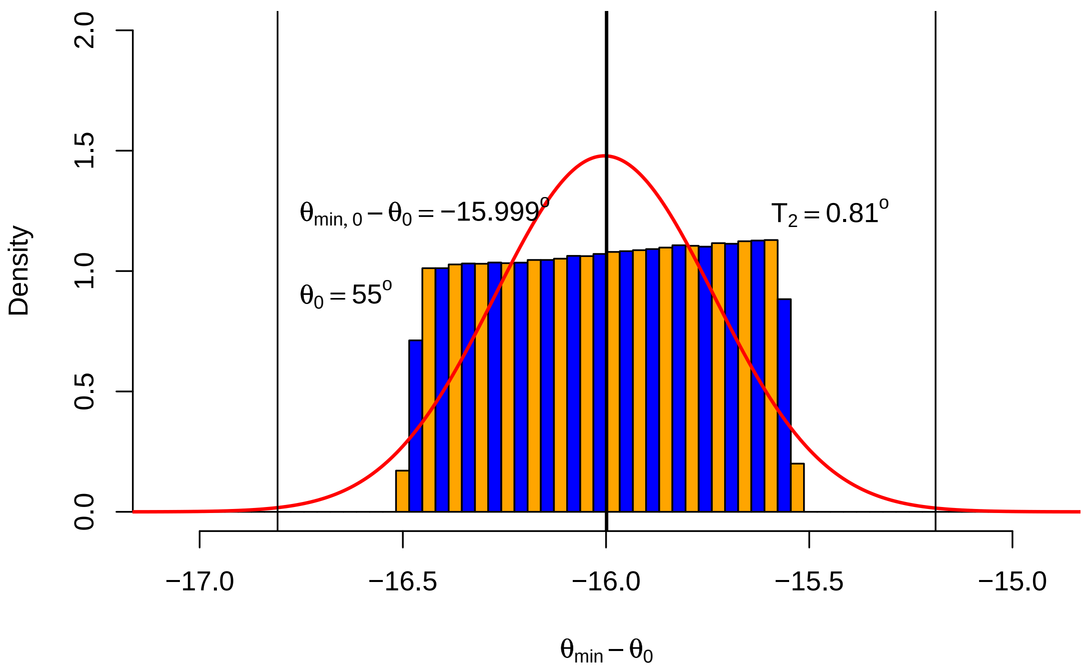


Figure 7: Simulation of 1,000,000 Deviations of $\theta_{\max}$ and $\theta_{\min}$ from θ_0

$\Delta = 1.6$, $\quad A \sim \mathcal{U}(12.8 - .12, 12.8 + .12) \quad$ and $\quad R \sim \mathcal{U}(6 - .14, 6 + .14)$

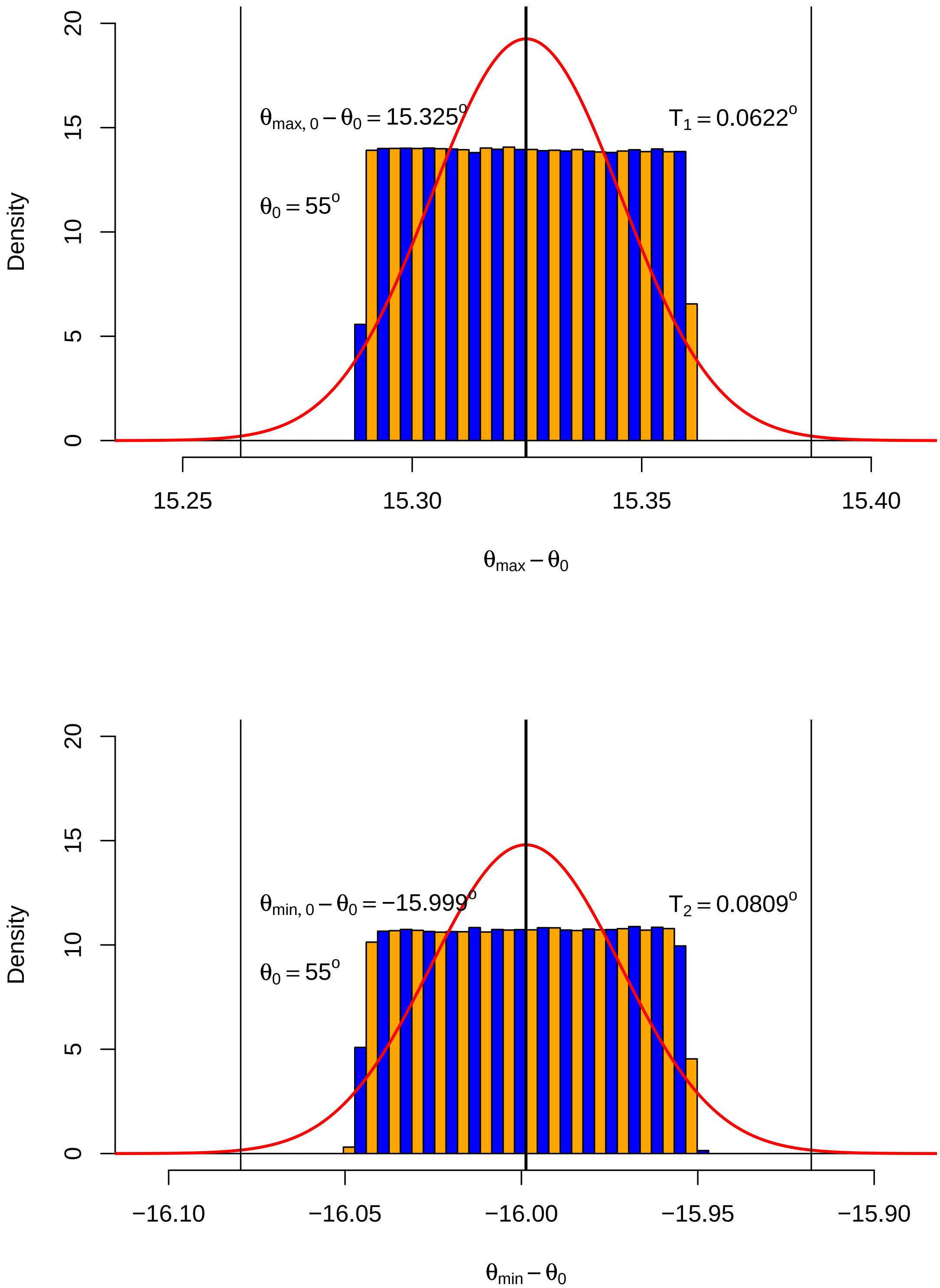


Figure 8: Simulation of 1,000,000 Deviations of $\theta_{\max}$ and $\theta_{\min}$ from θ_0

$\Delta = 1.6$, $A \sim \mathcal{U}(12.8 - .012, 12.8 + .012)$ and $R \sim \mathcal{U}(6 - .014, 6 + .014)$

It should also be clear from Figures 6 and 7 that the normal distribution does not give a good portrayal of the variation of $Y = \theta_x$. Thus a $T_Y = \pm 3\sigma_Y$ is not an adequate measure for the Y variation. The CLT certainly has not yet come into play. In this particular case $\pm 3\sigma_Y$ is much too wide. This is due to the fact that R is dominant and when it is not normally distributed then a tolerance $T_Y = \pm 3\sigma_Y$ is inappropriate. In that case only simulation will give an adequate description of the $Y = \theta_x$ variation.

7 Further Extensions

So far we have considered only two contributors in the tolerance analysis, namely A and R. This can easily extend to three by setting tolerances on the actual θ that is achieved during the adjustment process when trying to aim for θ_0. Denote this achieved value by $\theta_0^\star$. Thus we may want to specify that $\theta_0^\star$ should fall within $\theta_0 \pm T_\theta$ after the adjustment process. One can then ask about the tolerance ranges of $\theta_{\max}$ and $\theta_{\min}$.

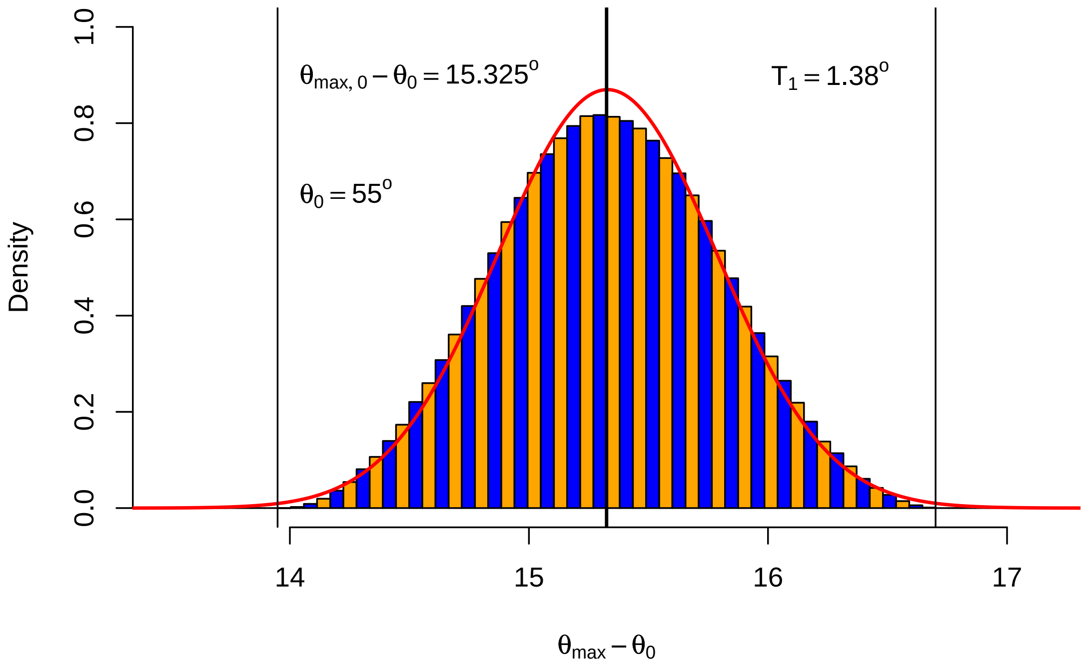
A further extension (to four contributors) concerns a possible uncertainty in the amount of extension or retraction $\pm\Delta$, i.e., one may want to specify $\Delta_0 \pm T_\Delta$ for the range of possible realized values Δ.

Given the rather lengthy derivations of the previous formal derivatives it seems appropriate to deal with these extended problems only via simulation and/or linearization using numerical derivatives. We show in Figure 9 the simulation result as produced by `theta.simUUUU`, using uniform distributions for all 4 contributors. Here the CLT seems to have taken effect to some extent.

8 Final Comments

This actuator example has been very instructive. It showed

- the importance of dominant variability by a single input
- the effect of the CLT when sufficiently many contributing inputs are involved
- the importance of simulation
- the importance of derivatives
- the effect of the variability ranges on the linearization approximation quality.



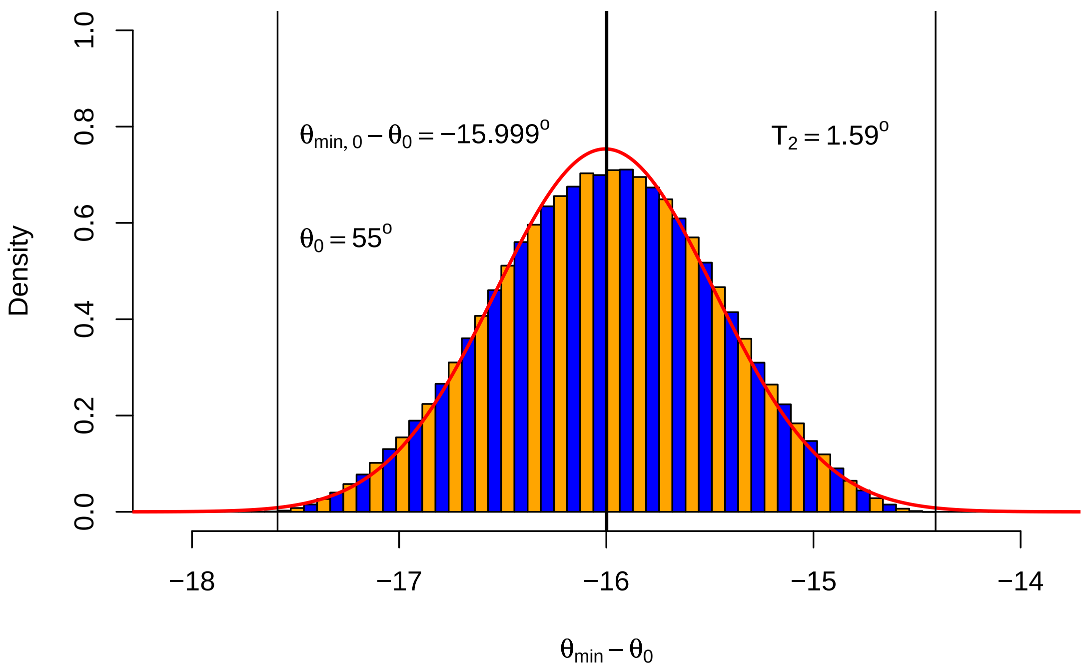


Figure 9: Simulation of 1,000,000 Deviations of $\theta_{\max}$ and $\theta_{\min}$ from $\theta_0 = 55°$
$A \sim \mathcal{U}(12.8 - .22, 12.8 + .22)$ and $R \sim \mathcal{U}(6 - .15, 6 + .15)$
$\Delta \sim \mathcal{U}(1.6 - .05, 1.6 + .05)$ and $\theta_0^\star \sim \mathcal{U}(55 - .5, 55 + .5)$