Weibull Probability Paper

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This note discusses and illustrates the use of Weibull probability paper for complete samples and for the special kind of censoring known as type II censoring. In the latter case the observed lifetimes consist of the r lowest values of the sample, the remaining unobserved lifetimes all being higher. Three blanks for 1 Cycle Log, 2 Cycle Log and 3 Cycle Log scales on the measurement abscissa are provided, together with two illustrative examples involving a full sample of size $n = 10$ and one that treats the lowest five values of that sample as a type II censored sample.

1 Introduction

In characterizing the distribution of life lengths or failure times of certain devices one often employs the Weibull distribution. This is mainly due to its weakest link properties, but other reasons are its increasing[1] failure rate with device age and the variety of distribution shapes that the Weibull density offers. The increasing failure rate accounts to some extent for fatigue failures.

Weibull plotting is a graphical method for informally checking on the assumption of the Weibull distribution model and also for estimating the two Weibull parameters. The method of Weibull plotting is explained and illustrated here only for complete and type II censored samples of failure times. In the latter case only the r lowest lifetimes of a sample of size n are observed. This data scenario is useful when n items (e.g., ball bearings) are simultaneously put on test in a common test bed and cycled until the first r fail, where r is a specified integer $2 \le r \le n$. The requirement $r \ge 2$ is needed at a minimum in order to get some sense of spread in the lifetime data, or in order to fit a line in the Weibull probability plot, since there are an infinite number of lines through a single point. The case $r = n$ leads back to the complete sample situation. Other types of censoring (right censoring, interval censoring) are not considered here, although they could also benefit from using Weibull probability paper.

It is assumed that the two-parameter Weibull distribution is a reasonable model for describing the variability in the failure time data. If T represents the generic failure time of a device, then the Weibull distribution function of T is given by

$$F_T(t) = P(T \le t) = 1 - \exp\left(-\left[\frac{t}{\alpha}\right]^{\beta}\right) \quad \text{for } t \ge 0 \,.$$

[1]for Weibull shape parameter $\beta > 1$

The parameter α is called the scale parameter or *characteristic life*. The latter term is motivated by the property $F_T(\alpha) = 1 - \exp(-1) \approx .632$, regardless of the shape parameter β. There are many ways for estimating the parameters α and β. One of the simplest is through the method of Weibull plotting, which used to be very popular due to its simplicity, graphical appeal, and its informal check on the Weibull model assumption. Such plotting and the accompanying calculations could all be done by hand for small to moderately sized samples. The availability of software and fast computing has changed all that. Thus this note is mainly a link to the past.

2 Weibull Plotting

The basic idea behind Weibull plotting is the relationship between the p-quantiles t_p of the Weibull distribution and p for $0 < p < 1$. The p-quantile t_p is defined by the following property

$$p = F_T(t_p) = P(T \le t_p) = 1 - \exp\left(-\left[\frac{t_p}{\alpha}\right]^\beta\right)$$

which leads to

$$t_p = \alpha\left[-\log_e(1-p)\right]^{1/\beta}$$

or taking decimal logs[2] on both sides

$$y_p = \log_{10}(t_p) = \log_{10}(\alpha) + \frac{1}{\beta}\log_{10}\left[-\log_e(1-p)\right] \ . \qquad (1)$$

Thus $\log_{10}(t_p)$, when plotted against $w(p) = \log_{10}[-\log_e(1-p)]$ should follow a straight line pattern with intercept $a = \log_{10}(\alpha)$ and slope $b = 1/\beta$. Thus $\alpha = 10^a$ and $\beta = 1/b$.

Plotting $w(p)$ against $y_p = \log_{10}(t_p)$, as is usually done in a Weibull plot, one should see the following linear relationship

$$w(p) = \beta\left[\log_{10}(t_p) - \log_{10}(\alpha)\right] \qquad (2)$$

with slope $B = \beta$ and intercept $A = -\beta\log_{10}(\alpha)$. Thus $\beta = B$ and $\alpha = 10^{-A/B}$.

In place of the unknown $\log_{10}$-quantiles $\log_{10}(t_p)$ one uses the corresponding sample quantiles. For a complete sample, $T_1, \ldots, T_n$, these are obtained by ordering these T_i from smallest to largest to get $T_{(1)} \le \ldots \le T_{(n)}$ and then associate with $p_i = (i - .5)/n$ the p_i-quantile estimate or i^{th} sample quantile $T_{(i)}$. These sample quantiles tend to vary around the respective population quantiles t_{p_i}. For large sample sample sizes and for $p_i = (i - .5)/n \approx p$ with $0 < p < 1$ this variation diminishes (i.e., the sample quantile $T_{(i)}$ converges to t_p in a sense not made precise here). For p_i close to 0 or

[2] The explicit notation $\log_{10}$ and $\log_e$ is used to distinguish decimal and natural logs.

1 the sample quantiles $T_{(i)}$ may still exhibit high variability even in large samples. Thus one has to be careful in interpreting extreme sample values in Weibull plots.

The idea of Weibull plotting for a complete sample is to plot $w(p_i) = \log_{10}[-\log_e(1-p_i)]$ against $\log_{10}(T_{(i)})$. Due to the variation of the $T_{(i)}$ around t_{p_i} one should, according to equation (2), then see a roughly linear pattern. The quality of this linear pattern should give us some indication whether the assumed Weibull model is reasonable or not. For small samples such "linear" pattern can be quite ragged, even when the samples come from a Weibull distribution. Thus one should not read too much into apparent deviations from linearity. A formal test of fit is the more prudent way to proceed.

For type II censored samples, where we only have the r lowest values $T_{(1)} \leq \ldots \leq T_{(r)}$, one simply plots only w_{p_i} against $\log_{10}(T_{(i)})$ for $i = 1, \ldots, r$, i.e., the censored values are not shown. They make their presence felt only through the denominator n in $p_i = (i - .5)/n$.

This Weibull plotting is facilitated by Weibull probability paper with a $\log_{10}$-transformed abscissa with untransformed labels and a transformed ordinate scale given by $w(p) = \log_{10}[-\log_e(1-p)]$ with labels in terms of p. Sometimes this scale is labeled in percent (i.e., in terms of $100p\%$). Three blank specimens of such Weibull probability paper are given at the end of this note. They distinguish themselves by the number of $\log_{10}$ cycles (1, 2, or 3) that are provided on the abscissa in order to simultaneously accommodate 1, 2, or 3 orders of magnitude.

For each plotting point $(\log_{10}(T_{(i)}), w(p_i))$ one locates or interpolates the label value of $T_{(i)}$ on the abscissa and the label value p_i on the ordinate, i.e., there is no need for the user to perform the transformations $\log_{10}(T_{(i)})$ and $w(p_i) = \log_{10}[-\log_e(1-p_i)]$.

Some authors suggest to use of $p_i' = (i - .3)/(n + .4)$ in place of p_i, others use $i/(n+1)$. All three choices for p_i give values strictly between 0 and 1, i.e., $0 < p_i < 1$, in order to yield finite values for w_{p_i}. For large n there is little difference between these choices of p_i and for small n the inherent variability in Weibull samples makes any preference between the three methods somewhat questionable.

3 Weibull Paper Scales

The three blanks of Weibull probability paper cover 1, 2, and 3 orders of magnitude on the abscissa, namely from 1 to 10, from 1 to 100 and from 1 to 1000. If the observed life times cover a range from 50 to 4000, one can simply change time units to tens and use the three $\log_{10}$ cycle paper from 5 to 400, which would accommodate such data. If the ranges are very large, even after scaling for a proper time unit, one may be tempted to use Weibull paper covering more orders of magnitude. Such higher order $\log_{10}$ scale Weibull paper is not given here, mainly because the graduation lines would become quite crowded and because there is a simple transformation device around that difficulty. It is based on the following power transformation property of the Weibull distribution.

If $T \sim \mathcal{W}(\alpha, \beta)$ (i.e., T has a Weibull distribution with parameters α and β), then

$$T' = T^a \sim \mathcal{W}(\alpha^a, \beta/a) = \mathcal{W}(\alpha', \beta') \ ,$$

since

$$\begin{aligned} P(T' \le y) = P(T^a \le y) &= P(T \le y^{1/a}) = 1 - \exp\left(-\left[\frac{y^{1/a}}{\alpha}\right]^\beta\right) \\ &= 1 - \exp\left(-\left[\frac{y}{\alpha^a}\right]^{\beta/a}\right) = 1 - \exp\left(-\left[\frac{y}{\alpha'}\right]^{\beta'}\right) \end{aligned}$$

Thus one can always bring the scale of the failure times up or down (but mainly down) into the proper range by an appropriate power transformation. After estimating (α', β') one can easily transform back to (α, β) using the known value a, namely $\alpha = \alpha'^{1/a}$ and $\beta = a\beta'$.

For example, if in a sample the minimum and maximum are $T_{(1)} = 5$ and $T_{(n)} = 800000$ respectively, it would require 6 orders of magnitude to accommodate the full sample, namely within $[1, 1000000]$. However taking $T^{1/2} = \sqrt{T}$ would give $T'_{(1)} = \sqrt{T_{(1)}} = 2.24$ and $T'_{(n)} = \sqrt{T_{(n)}} = \sqrt{800000} = 894.43$ and now the full transformed sample can be accommodated on an interval $[1, 1000]$, i.e., on 3 cycle $\log_{10}$ Weibull paper. On the other hand, if $T_{(1)} = .5$ and $T_{(n)} = 800000$ respectively, the above transformation does not quite work since $\sqrt{.5} = 0.71$. Here one may try $a = 1/3$ and find $(.5)^{1/3} = 0.794$ and $800000^{1/3} = 92.83$. Expressing these values in units of 1/10 we get new values 7.94 and 928.3 which again can be accommodated by 3 cycle $\log_{10}$ Weibull paper.

4 Two Example Plots

Two example usages of Weibull probability paper are shown, one for a complete sample of size $n = 10$ and the other based on a type II censored sample of the lowest five values of the previous sample. Both are shown prior to the Weibull probability paper blanks.

The complete sample consists of `7,12.1,22.8,23.1,25.7,26.7,29.0,29.9,39.5,41.9` drawn from a $\mathcal{W}(30, 3)$ distribution and rounded to one decimal. The Weibull plot for this example shows three lines. The red line corresponds to maximum likelihood estimates (m.l.e.) of α and β. These estimates are given as $\widehat{\alpha}_{\text{MLE}} = 28.9$ and $\widehat{\beta}_{\text{MLE}} = 2.8$ near the bottom of the plot. The other two lines (green and orange) represent least squares fits (the formulas for which will be given later). These two least squares fits differ with respect to which axis variable is fitted as a linear function of the other axis variable. Here $\widehat{\alpha}_{\text{LS1}} = 29.3$ and $\widehat{\beta}_{\text{LS1}} = 2.33$, shown near the bottom of the plot, refers to the least squares estimates of the parameters α and β when the abscissa is viewed as a linear function of the ordinate, while $\widehat{\alpha}_{\text{LS2}} = 29.8$ and $\widehat{\beta}_{\text{LS2}} = 2.18$ refers to the least squares estimates of the same parameters when the ordinate is viewed as a linear function of the abscissa.

The process for finding the m.l.e.'s is more complicated and is usually accomplished via computer, using any of several available software programs. The solution process is described elsewhere.

We point out the sensitivity of the least squares fits to the two lower extreme values. This is not surprising since the method of least squares, as applied here, treats all data equally. It does not know that the data come from a Weibull distribution and that the values are ordered and thus correlated. The method of maximum likelihood employs the fact that the data come from a Weibull model and knows how to properly weigh the various observations, i.e., stragglers as they show up in this example Weibull probability plot will not be given undue influence. Trying to fit a line by eye that allows for such stragglers is inherently somewhat subjective.

As pointed out above, the plotted $\log_{10}(T_{(i)})$ values are monotone increasing and thus correlated. Such properties are not consistent with the usual data model invoked in least squares fitting. One should view least squares fitting in this case as just a simple way of fitting a line through data, but one should not rely on any of the usual good properties that accompany least squares estimates. In particular, one should not attempt to construct confidence intervals around the fitted line that are based on normal variation models.

Since $p = 0.632$ yields $w(p) = 0$ or $\log_{10}(T) - \log_{10}(\alpha) = 0$ one can read off an estimate for α from the abscissa scale where the fitted line intercepts the ordinate level 0.632. For this purpose Weibull paper shows a level line at the ordinate $p = 0.632$.

The scale to the left of the ordinate scale runs from zero to ten and is a nomographic device for reading off the estimated shape parameter associated with the lines fitted to the plotted data. To obtain it one draws a line parallel to the respective fitted line and through the solid dot at the upper right end of that nomographic shape scale, see the illustration.

The second example does the same for the type II censored sample consisting of the five lowest observations in the previous sample. The least squares estimates do not account for the omitted or censored sample values, except through the n in $p_i = (i - .5)/n$. It is clear that any straggling variation behavior in the low sample values will have an even stronger effect on the least squares estimates in the type II censored situation. Here we have as maximum likelihood estimates $\widehat{\alpha}_{\text{MLE}} = 30.7$ and $\widehat{\beta}_{\text{MLE}} = 2.43$, while for the least squares estimates we get $\widehat{\alpha}_{\text{LS1}} = 37.5$ and $\widehat{\beta}_{\text{LS1}} = 1.77$ and $\widehat{\alpha}_{\text{LS2}} = 39.1$ and $\widehat{\beta}_{\text{LS2}} = 1.69$, respectively.

5 Least Squares Estimates

The least squares calculations, when fitting the ordinate $w(p_i) = \log_{10}(-\log_e(1 - p_i))$ (with $p_i = (i - .5)/n$) as linear function of $Y_{(i)} = \log_{10}(T_{(i)})$, use the following formulas

$$\widehat{B} = \widehat{\beta}_{\text{LS2}} = \frac{\sum_{i=1}^{r} w(p_i)(Y_{(i)} - \overline{Y})}{\sum_{i=1}^{r}(Y_{(i)} - \overline{Y})^2} \qquad \text{with} \quad \overline{Y} = \frac{1}{r}\sum_{i=1}^{n} Y_{(i)}$$

and

$$\widehat{A} = -\widehat{\beta}_{\text{LS2}} \log_{10}(\widehat{\alpha}_{\text{LS2}}) = \overline{w} - \widehat{\beta}_{\text{LS2}}\,\overline{Y} \quad \text{or} \quad \widehat{\alpha}_{\text{LS2}} = 10^{\overline{Y}-\overline{w}/\widehat{\beta}_{\text{LS2}}} \qquad \text{with} \quad \overline{w} = \frac{1}{r}\sum_{i=1}^{r} w(p_i)\ .$$

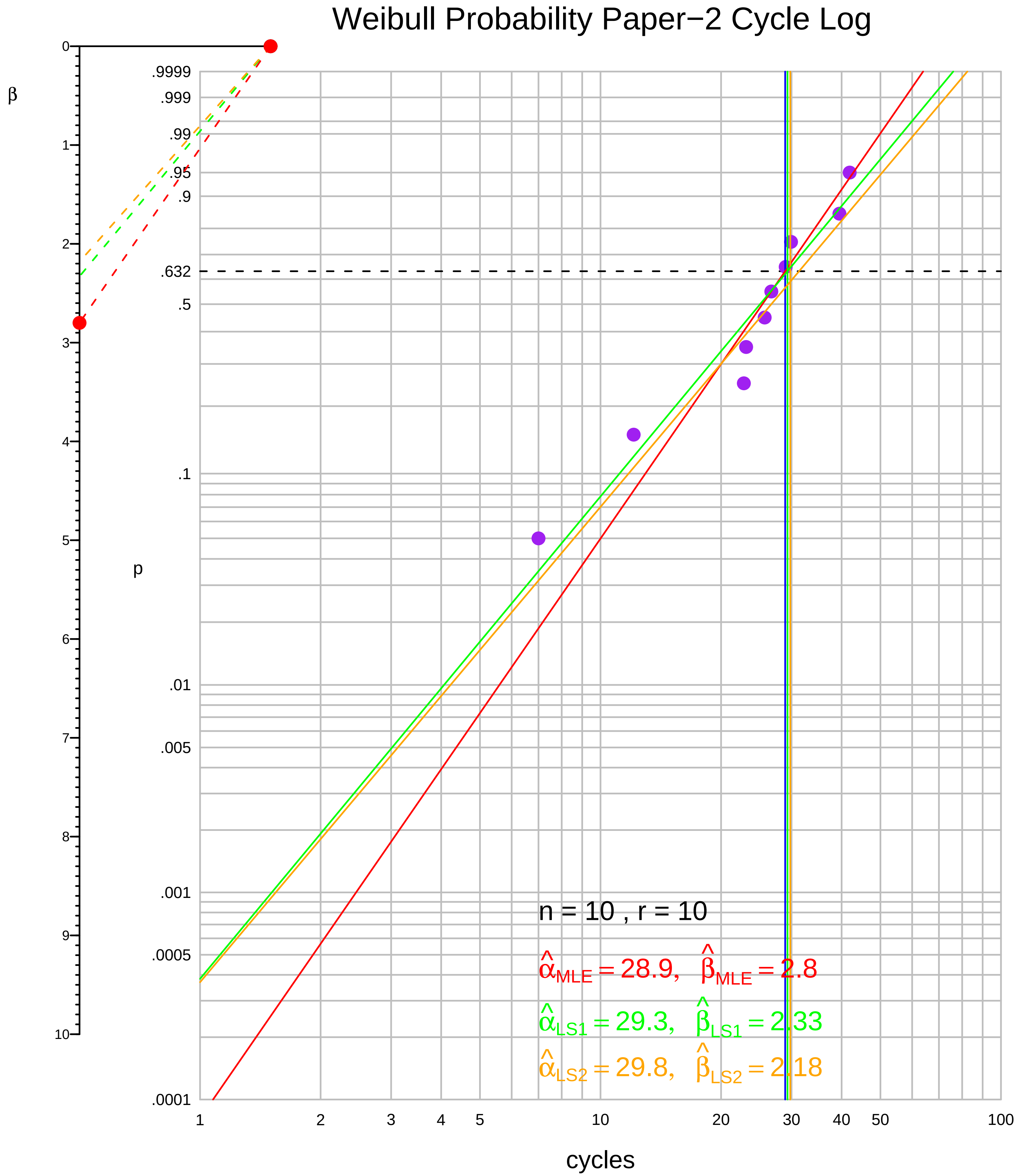
Here $r = n$ when a full sample is involved.
Since the variability is in the $Y_{(i)}$ and not in the $w(p_i)$ one may prefer doing the least squares calculations with abscissa and ordinate reversed, i.e., according to the model (1). In that case one obtains

$$\widehat{b} = \frac{1}{\widehat{\beta}_{\text{LS1}}} = \frac{\sum_{i=1}^{r}(w(p_i) - \overline{w})Y_{(i)}}{\sum_{i=1}^{r}(w(p_i) - \overline{w})^2} \qquad \text{with} \quad \overline{w} = \frac{1}{r}\sum_{i=1}^{r} w(p_i)$$

and

$$\widehat{a} = \log_{10}(\widehat{\alpha}_{\text{LS1}}) = \overline{Y} - \overline{w}/\widehat{\beta}_{\text{LS1}} \quad \text{or} \quad \widehat{\alpha}_{\text{LS1}} = 10^{\overline{Y}-\overline{w}/\widehat{\beta}_{\text{LS1}}} \qquad \text{with} \quad \overline{Y} = \frac{1}{r}\sum_{i=1}^{r} Y_{(i)}\ .$$

Again, $r = n$ when a full sample is involved.



Weibull Probability Paper−2 Cycle Log
β
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p
n = 10 , r = 10
α̂MLE = 28.9, β̂MLE = 2.8
α̂LS1 = 29.3, β̂LS1 = 2.33
α̂LS2 = 29.8, β̂LS2 = 2.18
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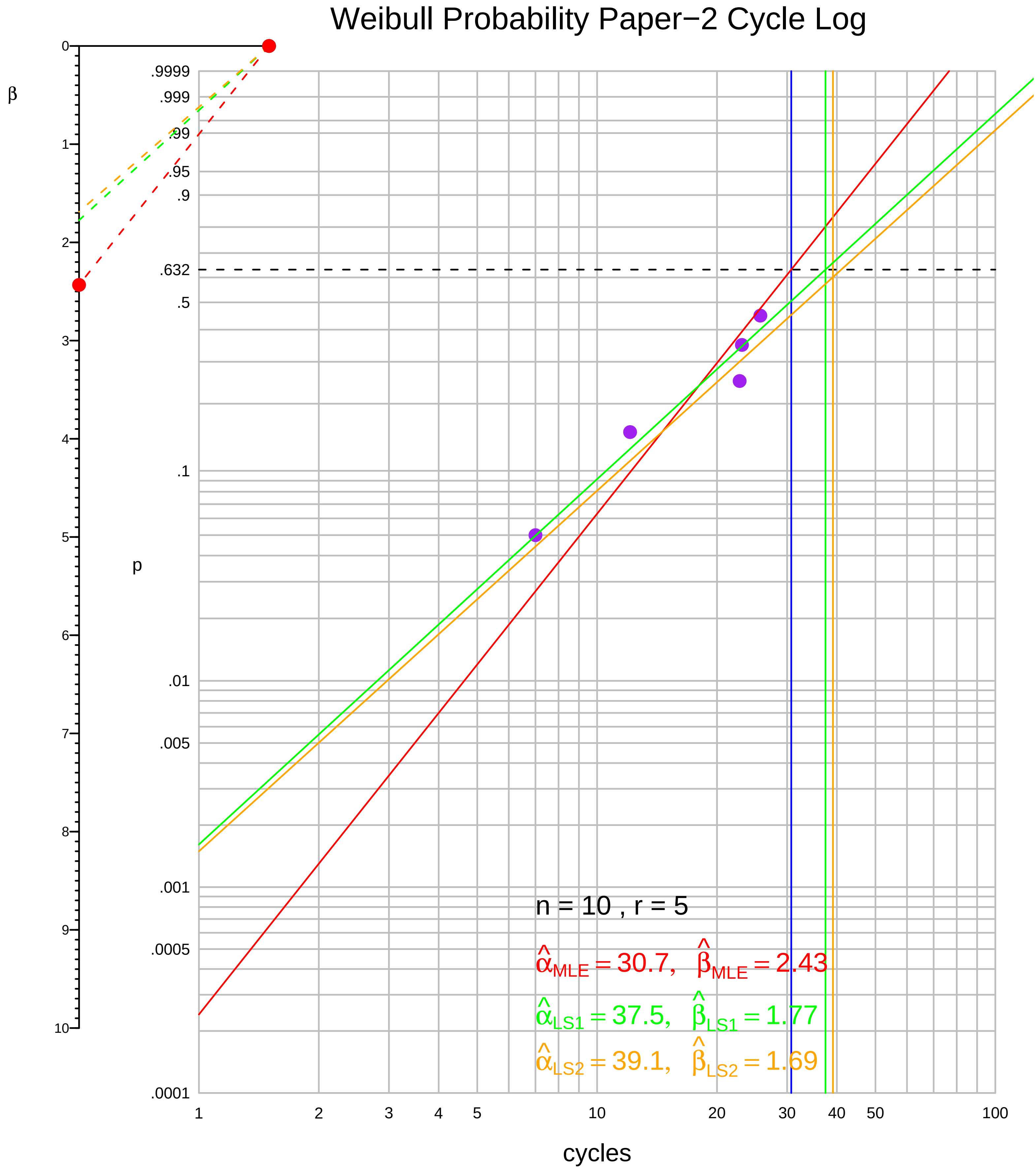
cycles


Weibull Probability Paper−2 Cycle Log
β
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p
n = 10 , r = 5
$\hat{\alpha}_{MLE} = 30.7$, $\hat{\beta}_{MLE} = 2.43$
$\hat{\alpha}_{LS1} = 37.5$, $\hat{\beta}_{LS1} = 1.77$
$\hat{\alpha}_{LS2} = 39.1$, $\hat{\beta}_{LS2} = 1.69$
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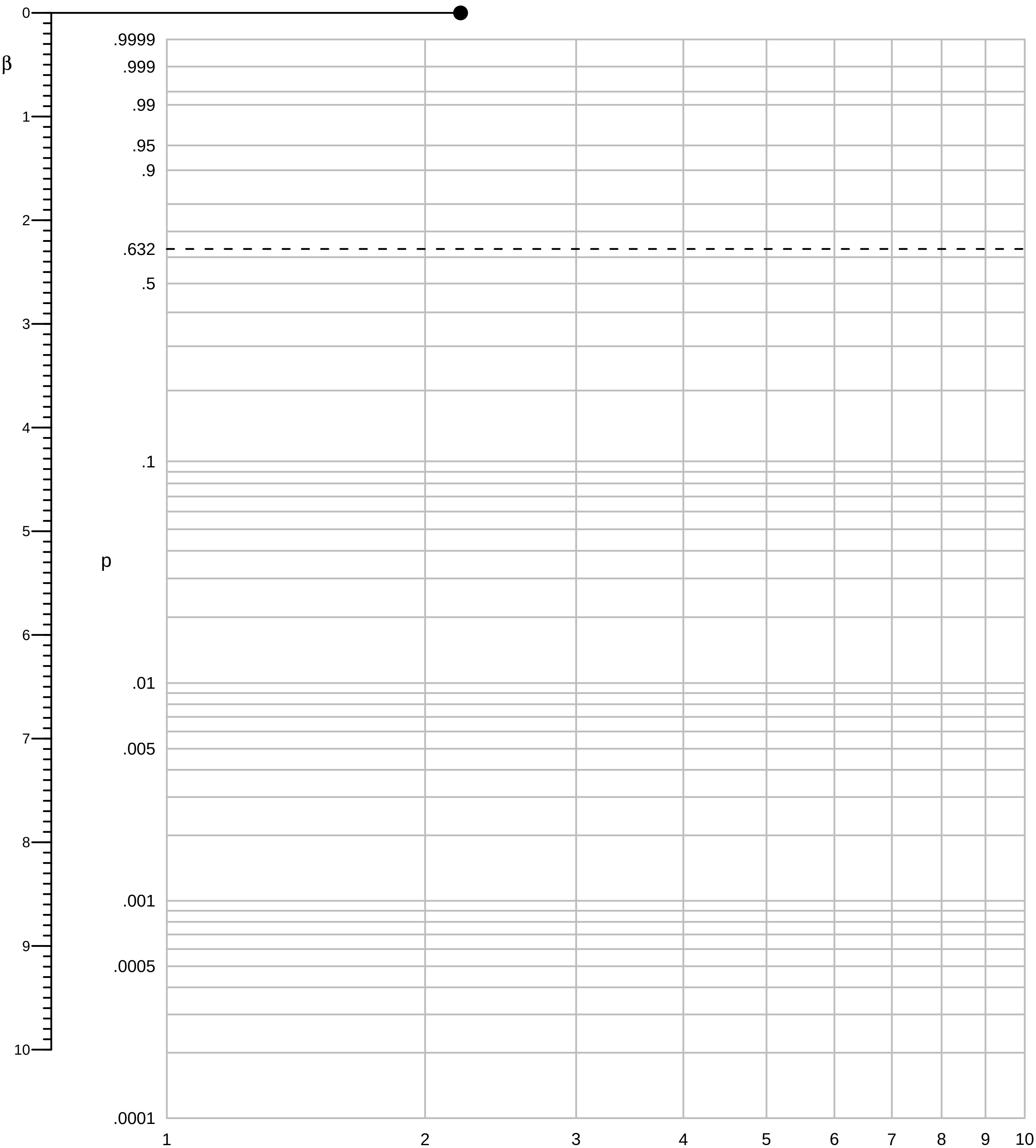
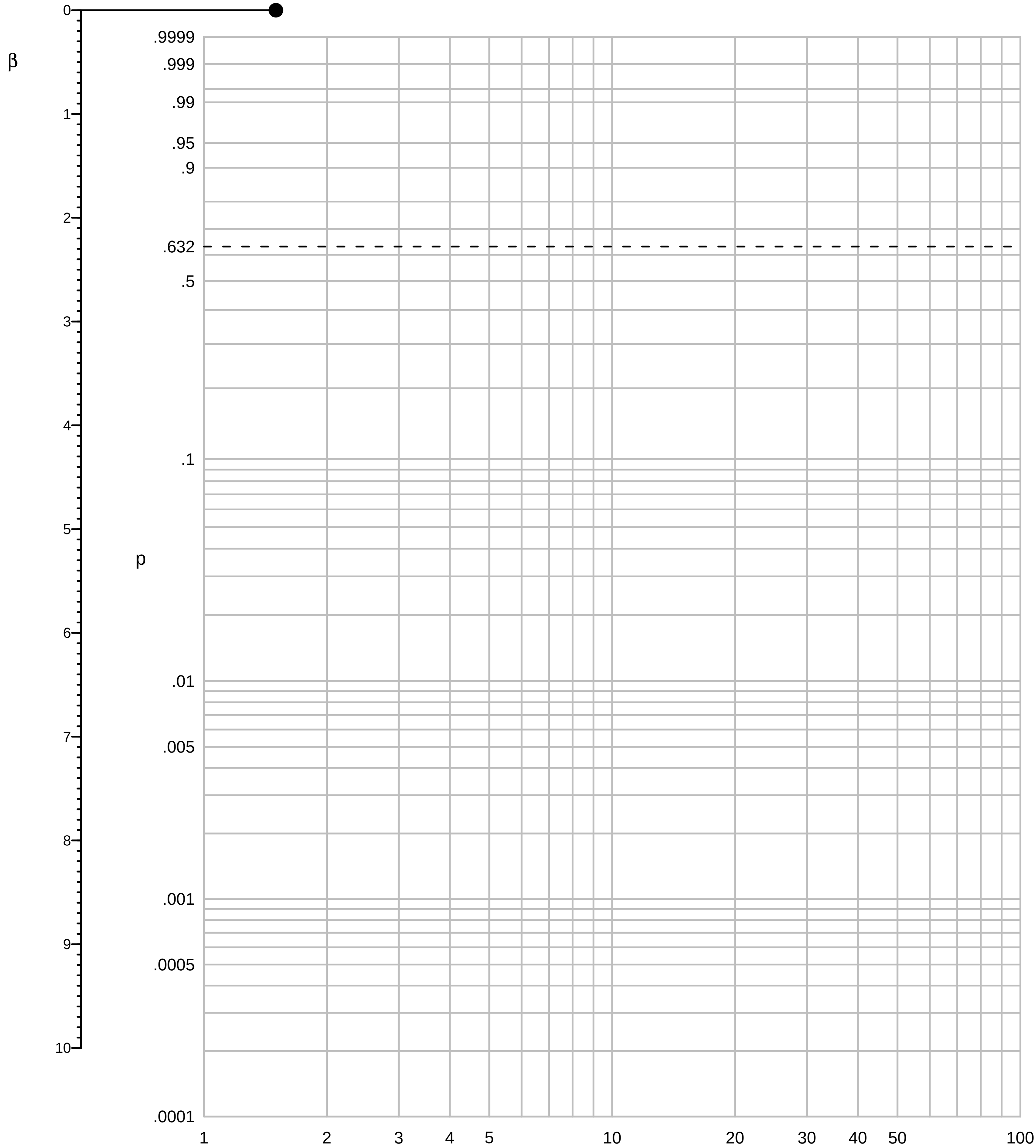
Weibull Probability Paper−1 Cycle Log



Weibull Probability Paper−2 Cycle Log



Weibull Probability Paper−3 Cycle Log