Class Notes 3-8-2019

- **Exponential Distribution as a Limiting Geometric Distribution:**
Let us model the life ending external events as independent Bernoulli trials by dividing the time axis into disjoint segments $[(k-1)/n, k/n)$, $k = 1, 2, 3, \ldots$ where n can be arbitrarily large. Assume that n is large enough so that $\lambda/n < 1$. Assume that each such interval has probability λ/n of experiencing such a life ending event, and the occurrences of such events are independent from interval to interval. Let $T_n = k/n$ when the first event occurs in $[(k-1)/n, k/n)$. Then

$$P\left(T_n = \frac{k}{n}\right) = \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)^{k-1} \quad \text{for } k = 1, 2, 3, \ldots$$

i.e., $nT_n \sim \text{Geo}(\lambda/n)$. Taking the limit as $n \to \infty$ we find

$$\lim_{n\to\infty} P(T_n > t) = e^{-\lambda t} \quad \text{for } t \geq 0$$

i.e., the limiting distribution of T_n as the time intervals get shorter and shorter is an exponential distribution with mean $1/\lambda$.
Proof: Let $\lfloor x \rfloor$ denote the largest integer $\leq x$, i.e., x rounded down to the next integer. Then

$$\begin{aligned} P(T_n > t) &= P(nT_n > nt) = \sum_{k:k>nt} \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)^{k-1} = \sum_{k=\lfloor nt \rfloor + 1}^{\infty} \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)^{k-1} = \left(1 - \frac{\lambda}{n}\right)^{\lfloor nt \rfloor} \\ &= \left(1 - \frac{\lambda}{n}\right)^{nt}\left(1 - \frac{\lambda}{n}\right)^{\lfloor nt \rfloor - nt} \longrightarrow e^{-\lambda t} \quad \text{as } n \to \infty \end{aligned}$$

- **The Failure Rate of a Failure Time Random Variable:**
If T represents the random time to failure of a mechanism or part then the infinitesimal probability of failure in the time increment $(t, t+dt)$ given that $T > t$ is

$$P(T \in (t, t+dt) | T > t) = \frac{P(T \in (t, t+dt)}{P(T > t))} \approx \frac{f_T(t)dt}{1 - F_T(t)}$$

and $r(t) = r_T(t) = f_T(t)/(1 - F_T(t))$ is called the *failure rate function* of T.
If F_T is differentiable and $F_T(t) = 0$ for $t < 0$ then

$$r_T(t) = \frac{d}{dt}\left[-\ln(1 - F_T(t))\right]$$

and we can regain the cdf of T from $r(t)$ via integration as follows

$$\int_0^s r(t)dt = \int_0^s \frac{d}{dt}\left[-\ln(1 - F_T(t))\right] dt = -\ln(1 - F_T(t))\big|_0^s = -\ln(1 - F_T(s))$$

and thus

$$1 - e^{-\int_0^s r(t)dt} = F_T(s) \quad \text{for } s \geq 0 \text{ and } F_T(s) = 0 \text{ for } s < 0$$

- **The Failure Rate Function of the Exponential Distribution:**

$$r_T(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

i.e., the exponential distribution has a constant failure rate, again manifesting its no aging property over time.

- **The Weibull Distribution:**
A positive random variable T is said to have a Weibull distribution if for positive parameters α and β it has the following density

$$f_T(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^\beta} I_{[0,\infty)}(t)$$

Its cdf is easier to memorize as

$$F_T(t) = 1 - e^{-\left(\frac{t}{\alpha}\right)^\beta} \qquad \text{for } t \geq 0 \text{ and } F(t) = 0 \text{ for } t < 0$$

Such random variables are useful in modeling the random lifetime of some part or organism or the breaking strength of some material specimen under increasing load, both very important at Boeing. Problem 2 of HW 8 has you examine a very characteristic property of the Weibull distribution.

When $\beta = 1$ the Weibull distribution becomes exponential with parameter $\lambda = 1/\alpha$.

Based of the Weibull pdf and cdf we immediately get its failure rate function as

$$r(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} I_{[0,\infty)}(t)$$

which is increasing in t when $\beta > 1$ and decreasing when $\beta < 1$. In the first case we see the effects of aging with time and the second case we may interpret as hardening (getting better) with time.

The Weibull distribution is very popular with mechanical engineers. One reason for that may be that Weibull's original paper was rejected by a statistical journal and published in the Journal of Applied Mechanics, see the link in Problem 2 of HW 8.

Transformations of Random Variables, Section 5.2

- **Transformation of a Discrete Random Variable:**
Example:
Let X have possible values $\{-1, 0, 1, 2\}$ with $P(X = k) = \frac{1}{4}$ for all 4 values of k. Let $Y = X^2$. Find the pmf of Y. The possible values of Y are $\{0, 1, 4\}$ with

$$\begin{aligned} P(Y = 0) &= P(X^2 = 0) = P(X = 0) = \frac{1}{4} \\ P(Y = 1) &= P(X^2 = 1) = P(X = 1) + P(X = -1) = \frac{1}{2} \\ P(Y = 4) &= P(X^2 = 4) = P(X = 2) + P(X = -2) = \frac{1}{4} \end{aligned}$$

- **In General:**
Let $Y = g(X)$ with X being a discrete random variable. Then the pmf of Y can be expressed as follows

$$p_Y(\ell) = P(Y = \ell) = P(g(X) = \ell) = P\left(\cup_{k:g(k)=\ell}\{X = k\}\right) = \sum_{k:g(k)=\ell} p_X(k)$$