Class Notes 3-4-2019

- **The Law of Large Numbers (LLN):**
Although the LLN can/will be proved without using the CLT we give here a first look at it. The LLN states that for any $p \in [0,1]$ that for any $\epsilon > 0$ we have

$$P\left(\left|\frac{S_n}{n} - p\right| < \epsilon\right) \to 1 \qquad \text{as } n \to \infty$$

For $p = 0$ and $p = 1$ this is trivially true since $P(S_n/n = 0) = 1$ for $p = 0$ and $P(S_n/n = 1) = 1$ for $p = 1$. For $0 < p < 1$ we have from the CLT

$$\begin{aligned} P\left(\left|\frac{S_n}{n} - p\right| < \epsilon\right) &= P\left(\left|\frac{\frac{S_n}{n} - p}{\sqrt{p(1-p)/n}}\right| < \frac{\epsilon}{\sqrt{p(1-p)/n}}\right) \approx P\left(|Z| < \frac{\sqrt{n}\epsilon}{\sqrt{p(1-p)}}\right) \\ &= 2\Phi\left(\frac{\sqrt{n}\epsilon}{\sqrt{p(1-p)}}\right) - 1 \to 1 \quad \text{as } n \to \infty \end{aligned}$$

HW 7, Problem 2 lets you work out a specific example case.

- **Sampling with and without Replacement:**
Suppose we have a large population of size N from which we sample at random n items without replacement, either one by one or as a single unordered grab of n. Then the outcome ω is either an ordered n-tuple, denoted by ω, each with probability

$$P(\omega) = \frac{1}{N \cdot (N-1) \cdot (N-2) \cdots (N-n+1)} = \frac{1}{(N)_n}$$

or an unordered set of n items representing all $n!$ permutations of the previous ω. We will denote this set by $\{\omega\}$ and it has probability

$$P^*(\{\omega\}) = \frac{n!}{N \cdot (N-1) \cdot (N-2) \cdots (N-n+1)} = \frac{1}{\binom{N}{n}}$$

Suppose now that our population consists of two types of items, say M red balls and $N - M$ blue balls. We are interested in the random variable X that counts the number of red balls in ω or in $\{\omega\}$. We can find $P(X = k) = P(\omega : X(\omega) = k)$ or $P^*(X = k) = P^*(\{\omega\} : X(\{\omega\}) = k)$. Any n-tuple ω with k red balls in specified position and with blue balls in the other positions can be obtained in

$$M \cdot (M-1) \cdot \ldots \cdot (M-k+1) \cdot (N-M) \cdot (N-M-1) \cdot \ldots \cdot (N-M-(n-k-1)) = (M)_k (N-M)_{n-k}$$

ways. As far as $P(X = k) = P(\omega : X(\omega) = k)$ is concerned there are $\binom{n}{k}$ n-tuples ω which have k red balls and $n - k$ blues balls in one of the chosen $\binom{n}{k}$ positions of the n-tuple. Thus

$$P(X = k) = \frac{\binom{n}{k}(M)_k (N-M)_{n-k}}{(N)_n} = \frac{\binom{M}{k} \cdot \binom{N-M}{n-k}}{\binom{N}{n}} = P^*(X = k)$$

where the $= P^*$ derives from the fact that in order to get a set $\{\omega\}$ with exactly k red balls I have to grab k from M red balls in one of $\binom{M}{k}$ ways and combine that with a grab of $n - k$

blue balls from $N-M$ in $\binom{N-M}{n-k}$ possible ways. This equality $P(X=k)=P^*(X=k)$ should have been made clear in the beginning of this class and it is predicated on the fact that the event $X=k$ does not involve any order. This common distribution of X is referred to as the hypergeometric distribution and we write $X \sim \text{Hypergeom}(N,M,n)$. See HW 4, Problem 4.

Let us next consider sampling n items with replacement from the same population and let X again denote the number of red balls. Then we are dealing with n independent trials and with success (red ball chosen in a given trial) probability $p = M/N$ and we have $X \sim \text{Bin}(n,p)$. We write

$$\tilde{P}(X=k) = \binom{n}{k}\left(\frac{M}{N}\right)^k\left(\frac{N-M}{N}\right)^{n-k} = \binom{n}{k}p^k(1-p)^{n-k}$$

Let A denote the set of all n-tuples with distinct elements, assuming all N items are labeled $1, 2\ldots, N$. Then

$$\tilde{P}(A) = \frac{N\cdot(N-1)\cdot(N-2)\cdot\ldots\cdot(N-n+1)}{N^n} \to 1 \quad \text{as } N\to\infty,\, n \text{ staying fixed}$$

and since $\tilde{P}(A^c)\to 0$ as $N\to\infty$ we then have

$$\tilde{P}(X=k) = \tilde{P}(X=k|A)\tilde{P}(A) + \tilde{P}(X=k|A^c)\tilde{P}(A^c) \approx \tilde{P}(X=k|A)$$

For n-tuples $\omega \in A$ we have

$$\tilde{P}(\omega|A) = \frac{\tilde{P}(\omega)}{\tilde{P}(A)} = \frac{\frac{1}{N^n}}{\frac{N\cdot(N-1)\cdot(N-2)\cdot\ldots\cdot(N-n+1)}{N^n}} = \frac{1}{(N)_n} = P(\omega)$$

and thus

$$\tilde{P}(X=k) \approx \tilde{P}(X=k|A) = \tilde{P}(\{\omega : X(\omega)=k\}|A) = P(\omega : X(\omega)=k) = P(X=k) = P^*(X=k)$$

Thus the “two sampling without replacement” schemes give approximately the same distribution to X as the sampling with replacement, provided N is large. In order for the binomial distribution $\text{Bin}(n, p = M/N)$ to stay stable we would also need to assume that $p = M/N$ stay stable and stay away from zero or one. In that case the CLT applied to the binomial distribution ($\tilde{P}$) equally applies to the nonreplacement schemes, where the sequential selections (if we sample one by one) are certainly no longer independent.

- **The Poisson Distribution and Poisson Approximation to the Binomial:**
  If $S_n \sim \text{Bin}(n,p)$ with $p\approx 0$ or $p \approx 1$ presents a problem for the CLT, at least for moderate n. In that case the Poisson distribution provides a very useful alternative for $p\approx 0$ and for $p\approx 1$ we switch to $\tilde{S}_n = n - S_n \sim \text{Bin}(n, 1-p)$, counting the number of failures instead.

  A random variable X has a Poisson distribution with parameter $\lambda > 0$, and we write $X \sim \text{Poisson}(\lambda)$ or $X\sim \text{Pois}(\lambda)$, when

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{for } k = 0, 1, 2, \ldots$$

  Its mean and variance are given by

$$E(X) = \lambda \quad \text{and} \quad \text{var}(X) = \lambda$$

$$E(X)=\sum_{k=0}^{\infty}k\frac{e^{-\lambda}\lambda^k}{k!}=\sum_{k=1}^{\infty}k\frac{e^{-\lambda}\lambda^k}{k!}=\lambda\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}=\lambda\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}=\lambda$$

$$E(X(X-1))=\sum_{k=0}^{\infty}k(k-1)\frac{e^{-\lambda}\lambda^k}{k!}=\sum_{k=2}^{\infty}k(k-1)\frac{e^{-\lambda}\lambda^k}{k!}=\lambda^2\sum_{k=2}^{\infty}\frac{e^{-\lambda}\lambda^{k-2}}{(k-2)!}=\lambda^2\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}=\lambda^2$$

Thus $\quad \text{var}(X)=E(X(X-1))+E(X)-[E(X)]^2=\lambda^2+\lambda-\lambda^2=\lambda$

- **Law of Rare Events Or Law of Small Numbers**
  Let $\lambda>0$ and consider positive integer n with $\lambda/n<1$. Let $S_n\sim \text{Bin}(n,\lambda/n)$. Then

$$\lim_{n\to\infty}P(S_n=k)=e^{-\lambda}\frac{\lambda^k}{k!}\quad \text{for } k=0,1,2,\ldots$$

Proof:

$$\begin{aligned}P(S_n=k) &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}=\frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}\\ &\longrightarrow \frac{\lambda^k}{k!}e^{-\lambda}\qquad \text{as } n\to\infty \text{ for any fixed } k \qquad \square\end{aligned}$$