Class Notes 1-30-2019

- **Example: Cards**
  We draw a card at random from a standard deck of 52 cards. The event A that the card is an ace is independent of the event C that the card is a club. However, this breaks down as soon as the king of diamonds is missing from the deck, but not when all kings are missing.

- **Theorem:** Independence of E, F implies independence of E, F^c, of E^c, F and of E^c, F^c.
  Proof (partial):

  $$P(EF^c)+P(EF) = P(E) \Rightarrow P(EF^c) = P(E)-P(E)P(F) = P(E)(1-P(F)) = P(E)P(F^c)$$

- Assume A and B are independent. Find an expression, in terms of only $P(A)$ and $P(B)$, for the probability of the event C that only exactly one of A and B occur.
  Solution: $C = AB^c \cup A^cB$ is a disjoint union, thus

  $$P(C) = P(AB^c)+P(A^cB) = P(A)P(B^c)+P(A^c)P(B) = P(A)(1-P(B))+(1-P(A))P(B)$$

  where the second = uses the previous theorem.

- **Mutually Independent:** Events $A_1, \ldots, A_n$ are called *mutually independent* if for every collection $A_{i_1}, \ldots, A_{i_k}$ with $2 \le k \le n$ and $1 \le i_1 < i_2 < \ldots < i_k \le n$ we have

  $$P(A_{i_1}A_{i_2}\cdots A_{i_k}) = P(A_{i_1})P(A_{i_2})\cdots P(A_{i_k})$$

- **Mutual Independence of 3 Events** A, B, C
  Need to check the truth of

  $$P(AB) = P(A)P(B),\ P(AC) = P(A)P(C),\ P(BC) = P(B)P(C),\ P(ABC) = P(A)P(B)P(C)$$

- **Mutual Independence Carries over to Complements:**
  If events $A_1, \ldots, A_n$ are *mutually independent* so are $A_1^*, \ldots, A_n^*$, where A_i^* is either A_i or A_i^c.

- **Example of Events not Mutually Independent:**
  Choose a point randomly on the interval $\Omega = [0, 1]$ and consider the events

  $$A = \left[\frac{1}{2}, 1\right], \quad B = \left[\frac{1}{2}, \frac{3}{4}\right] \quad \text{and} \quad C = \left[\frac{1}{16}, \frac{9}{16}\right] \quad \text{then} \quad ABC = \left[\frac{1}{2}, \frac{9}{16}\right]$$

  and $P(ABC) = \frac{1}{16} = \frac{1}{2}\frac{1}{4}\frac{1}{2} = P(A)P(B)P(C)$ but $P(AB) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)$.

- **Pairwise Independence:**
  Events $A_1, \ldots, A_n$ are pairwise independent if any two of its events are independent, i.e., $P(A_iA_j) = P(A_i)P(A_j)$ for any $i \neq j$.
  This is a weaker form of independence than mutual independence.

- **Example of Pairwise but not Mutual Independence:**
  Flip 3 fair coins and consider the following events. A is the event of exactly one tails in the

first two flips, B is the event of exactly one tails in the last two flips, C is the event of exactly one tails in the first and last flip.

$$\begin{aligned} A &= \{(\mathtt{T},\mathtt{H},\mathtt{H}),(\mathtt{T},\mathtt{H},\mathtt{T}),(\mathtt{H},\mathtt{T},\mathtt{H}),(\mathtt{H},\mathtt{T},\mathtt{T})\} \quad P(A)=\frac{1}{2} \\ B &= \{(\mathtt{H},\mathtt{T},\mathtt{H}),(\mathtt{T},\mathtt{T},\mathtt{H}),(\mathtt{H},\mathtt{H},\mathtt{T}),(\mathtt{T},\mathtt{H},\mathtt{T})\} \quad P(B)=\frac{1}{2} \\ C &= \{(\mathtt{T},\mathtt{H},\mathtt{H}),(\mathtt{T},\mathtt{T},\mathtt{H}),(\mathtt{H},\mathtt{H},\mathtt{T}),(\mathtt{H},\mathtt{T},\mathtt{T})\} \quad P(C)=\frac{1}{2} \end{aligned}$$

$$AB=\{(\mathtt{T},\mathtt{H},\mathtt{T}),(\mathtt{H},\mathtt{T},\mathtt{H})\},\quad AC=\{(\mathtt{T},\mathtt{H},\mathtt{H}),(\mathtt{H},\mathtt{T},\mathtt{T})\},\quad BC=\{(\mathtt{T},\mathtt{T},\mathtt{H}),(\mathtt{H},\mathtt{H},\mathtt{T})\},\quad ABC=\emptyset$$

$$P(AB)=\frac{1}{4}=P(A)P(B),\quad P(AC)=\frac{1}{4}=P(A)P(C),\quad P(BC)=\frac{1}{4}=P(B)P(C)$$

$$P(ABC)=0\neq P(A)P(B)P(C)=\frac{1}{8}$$

- **A Reliability Example:**
  A system functions, event D, as long as one of two subsystems C_1 or C_2 functions. Subsystem C_1 functions as long both its components function, denoted as events S_1 and S_2. C_2 functions as long as its one component functions, event S_3. Assume that all components function or fail independently, with $P(S_i)=p_i$

$$D=C_1\cup C_2=(S_1S_2)\cup S_3$$

$$\begin{aligned} P(D) &= P(C_1\cup C_2)=P(C_1)+P(C_2)-P(C_1C_2)=P(S_1S_2)+P(S_3)-P(S_1S_2S_3) \\ &= p_1p_2+p_3-p_1p_2p_3 \end{aligned}$$

  Boeing's Scientific Research Laboratory (BSRL) played a big role in developing the field of *Reliability Theory.* Z.W. Birnbaum, founding father of statistics at the UW, was an active contributor to BSRL, as were Barlow, Proschan, Saunders and Pyke.

- **Independence of Random Variables**:
  Random variables $X_1,\ldots,X_n$, defined on the same probability space Ω, are called independent if

$$P(X_1\in B_1,\ldots,X_n\in B_n)=\prod_{i=1}^{n}P(X_i\in B_i)$$

  for all reasonable subsets $B_1,\ldots,B_n$ on the real line $\mathbb{R}$.

  This is not very practical to check, because of the innumeracy of such subsets. For discrete random variables we have the following equivalence.