• Example: Cards
We draw a card at random from a standard deck of 52 cards. The event $A$ that the card is an ace is independent of the event $C$ that the card is a club. However, this breaks down as soon as the king of diamonds is missing from the deck, but not when all kings are missing.

• Theorem: Independence of $E,F$ implies independence of $E,F^c$, of $E^c,F$ and of $E^c,F^c$.
Proof (partial):
\[ P(EF^c) + P(EF) = P(E) \Rightarrow P(EF^c) = P(E) - P(E)P(F) = P(E)(1-P(F)) = P(E)P(F^c) \]

• Assume $A$ and $B$ are independent. Find an expression, in terms of only $P(A)$ and $P(B)$, for the probability of the event $C$ that only exactly one of $A$ and $B$ occur.
Solution: $C = AB^c \cup A^cB$ is a disjoint union, thus
\[ P(C) = P(AB^c) + P(A^cB) = P(A)P(B^c) + P(A^c)P(B) = P(A)(1-P(B)) + (1-P(A))P(B) \]
where the second $=$ uses the previous theorem.

• Mutually Independent: Events $A_1, \ldots, A_n$ are called mutually independent if for every collection $A_{i_1}, \ldots, A_{i_k}$ with $2 \leq k \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ we have
\[ P(A_{i_1}A_{i_2}\cdots A_{i_k}) = P(A_{i_1})P(A_{i_2})\cdots P(A_{i_k}) \]

• Mutual Independence of 3 Events $A, B, C$
Need to check the truth of
\[ P(AB) = P(A)P(B), \ P(AC) = P(A)P(C), \ P(BC) = P(B)P(C), \ P(ABC) = P(A)P(B)P(C) \]

• Mutual Independence Carries over to Complements:
If events $A_1, \ldots, A_n$ are mutually independent so are $A_1^*, \ldots, A_n^*$, where $A_i^*$ is either $A_i$ or $A_i^c$.

• Example of Events not Mutually Independent:
Choose a point randomly on the interval $\Omega = [0,1]$ and consider the events
\[ A = \left[ \frac{1}{2}, 1 \right], \ B = \left[ \frac{1}{2}, \frac{3}{4} \right] \ \text{and} \ \ C = \left[ \frac{1}{16}, \frac{9}{16} \right] \ \text{then} \ ABC = \left[ \frac{1}{2}, \frac{9}{16} \right] \]
and $P(ABC) = \frac{1}{16} = \frac{111}{2^{12}} = P(A)P(B)P(C)$ but $P(AB) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)$.

• Pairwise Independence:
Events $A_1, \ldots, A_n$ are pairwise independent if any two of its events are independent, i.e., $P(A_iA_j) = P(A_i)P(A_j)$ for any $i \neq j$.
This is a weaker form of independence than mutual independence.

• Example of Pairwise but not Mutual Independence:
Flip 3 fair coins and consider the following events. $A$ is the event of exactly one tails in the
first two flips, \( B \) is the event of exactly one tails in the last two flips, \( C \) is the event of exactly one tails in the first and last flip.

\[
A = \{(T, H, H), (T, H, T), (H, T, H), (H, T, T)\} \quad P(A) = \frac{1}{2}
\]

\[
B = \{(H, T, H), (T, T, H), (H, H, T), (T, H, T)\} \quad P(B) = \frac{1}{2}
\]

\[
C = \{(T, H, H), (T, T, H), (H, H, T), (H, T, T)\} \quad P(C) = \frac{1}{2}
\]

\[
AB = \{(T, H, T), (H, T, H)\}, \quad AC = \{(T, H, H), (H, T, T)\}, \quad BC = \{(T, T, H), (H, H, T)\}, \quad ABC = \emptyset
\]

\[
P(AB) = \frac{1}{4} = P(A)P(B), \quad P(AC) = \frac{1}{4} = P(A)P(C), \quad P(BC) = \frac{1}{4} = P(B)P(C)
\]

\[
P(ABC) = 0 \neq P(A)P(B)P(C) = \frac{1}{8}
\]

- **A Reliability Example:**
  A system functions, event \( D \), as long as one of two subsystems \( C_1 \) or \( C_2 \) functions. Subsystem \( C_1 \) functions as long both its components function, denoted as events \( S_1 \) and \( S_2 \). \( C_2 \) functions as long as its one component functions, event \( S_3 \). Assume that all components function or fail independently, with \( P(S_i) = p_i \)

\[
D = C_1 \cup C_2 = (S_1S_2) \cup S_3
\]

\[
P(D) = P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1C_2) = P(S_1S_2) + P(S_3) - P(S_1S_2S_3)
\]

\[
= p_1p_2 + p_3 - p_1p_2p_3
\]

Boeing’s Scientific Research Laboratory (BSRL) played a big role in developing the field of **Reliability Theory**. Z.W. Birnbaum, founding father of statistics at the UW, was an active contributor to BSRL, as were Barlow, Proschan, Saunders and Pyke.

- **Independence of Random Variables:**
  Random variables \( X_1, \ldots, X_n \), defined on the same probability space \( \Omega \), are called independent if

\[
P(X_1 \in B_1, \ldots, X_n \in B_n) = \prod_{i=1}^{n} P(X_i \in B_i)
\]

for all reasonable subsets \( B_1, \ldots, B_n \) on the real line \( \mathbb{R} \).

This is not very practical to check, because of the innumercy of such subsets. For discrete random variables we have the following equivalence.