Homework 8
Due March 13, 2019

Problem 1: The sum of independent Poisson random variables $X_i \sim \text{Pois}(\lambda_i), i = 1, 2, \ldots, n$ is again a Poisson random variable $Y \sim \text{Pois}(\lambda = \lambda_1 + \ldots + \lambda_n)$. This is shown right a the start of
http://faculty.washington.edu/fscholz/DATAFILES394_2019/POIBIN.pdf
Conversely, any Poisson random variable can be viewed as a sum of such independent random variables. Hence one would expect that the CLT applies for sufficiently large λ. Use the same rationale as in the development of the rule of thumb for the validity of the CLT in the binomial case to get a rule of thumb for the validity of the CLT in approximating the Poisson distribution by a normal approximation $\mathcal{N}(\lambda, \lambda)$ when λ is sufficiently large. Assuming $Y \sim \text{Pois}(\lambda)$ apply such a normal approximation to $P(Y \geq 10)$ when $\lambda = 5, 10, 15$. I give you the exact values for comparison: 0.03182806, 0.5420703, 0.9301463. Make the comparison with and without continuity correction, as applied in the binomial case and check your rule of thumb. Comment on the approximation quality.

Problem 2: A random variable W has a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$, and we write $W \sim \text{Weib}(\alpha, \beta)$, if its cdf is

$$F_W(x) = P(W \leq x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta} \quad \text{if } x \geq 0, \text{ and } F_W(x) = 0 \text{ for } x < 0$$

a) Argue that $F_W(x)$ is a genuine cdf and find the density $f_W(x)$ of W.
b) Let $W_i \sim \text{Weib}(\alpha, \beta)$, $i = 1, 2, \ldots n$ be mutually independent random variables. Show that $Y = \min(W_1, \ldots, W_n)$ is again a Weibull random variable with which parameters. (Hint: look $P(Y > y)$ and recall what we did in HW5 Problem 5.)
c) Is $Y = \min(W_1, \ldots, W_n)$ still a Weibull random variable when $W_i \sim \text{Weib}(\alpha_i, \beta)$, $i = 1, 2, \ldots n$ are mutually independent random variables?

Comment: Before joining Boeing I hardly knew anything about the Weibull distribution. Basically I had heard just its name. The Weibull distribution is very popular with engineers who deal with strength of materials. It models very well such strengths by the principle that a material, when viewed as the collection/linkage of many sub-specimens, is as strong as the strength of its weakest link. In fact, there is a limit theorem akin to the CLT that shows that $Y = \min(X_1, \ldots, X_n)$ has an approximate Weibull distribution if the strengths $X_1, \ldots, X_n$ of the links are independent random variables, sufficiently well behaved. What you show in part b) is very similar to the situation when summing independent normal random variables, i.e., the sum is again normal. In b) it is the minimum (weakest strength) that is again Weibull if the W_i are all Weibull.

If you ever do statistics using a Weibull model for your data you may find the following useful.
http://www.tqmp.org/RegularArticles/vol11-3/p148/p148.pdf
It includes R code for Weibull analysis.

Problem 3: On the first 300 pages of a book, you notice that there are, on average, 2 typos per page. What is the probability that there will be at least 4 typos on page 301? State clearly the assumptions you are making.

Problem 4: A hockey player scores at least one goal in roughly half of his games. How would you estimate the percentage of games where he scores a hat-trick (3 goals)?

Problem 5: Let $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ be independent Poisson random variables. Exploiting the Poisson approximation to the binomial distribution, namely $\text{Pois}(\lambda) \approx \text{Bin}(n, \lambda/n)$ and $\text{Pois}(\mu) \approx \text{Bin}(m, \mu/n)$, argue that $X + Y$ has a Poisson distribution with parameter $\lambda + \mu$.
Hint: Use an approximation argument without taking it to the full limit, i.e., no need to make it

air tight. View $X' \sim \text{Bin}(n, \lambda/n) \approx \text{Pois}(\lambda) \sim X$ as a sum of n independent Bernoulli random variables and similarly $Y' \sim \text{Bin}(m, \mu/m) \approx \text{Pois}(\mu) \sim Y$ and choosing m and n large so that λ/n and μ/m are about equal so that $X' + Y'$ is a sum of $m + n$ independent, almost identically distributed Bernoulli random variables, etc.

Problem 6: If $X \sim \text{Bin}(n, p)$ then the pmf $P(X = k)$ usually first increases with k and then decreases. Show this by looking at $P(X = k+1)/P(X = k)$ and examining it for being > 1 or < 1. Characterize the situations in terms of p where $P(X = k)$ is monotone in only one direction, i.e., only increasing or only decreasing in k.
Make a similar examination for the pmf $P(Y = k)$ where $Y \sim \text{Pois}(\lambda)$.

Problem 7: Based on past history a famous basketball player makes about 93% of his free throws. What is the approximate chance that he will make less than 80% of his 30 free throws in today's game? Compare that with the exact probability.