Problem 1: Let $X$ be a discrete random variable with values $1, 2, 3, 4, 5$ and corresponding probabilities $1/7, 1/14, 3/14, 2/7, 2/7$. a) Compute $E(X)$ b) compute $E[|X - 2|]$.

Problem 2: Let $X$ be a continuous random variable with pdf $f(x) = 3 \exp(-3x)I_{(0,\infty)}(x)$. a) Find $E(X)$ and compute $E[\exp(2X)]$.

Problem 3: Let $X$ be a discrete random variable with values $1, 2, 3, 4, 5$ and corresponding probabilities $1/7, 1/14, 3/14, 2/7, 2/7$. a) First find the pmf of $Y = |X|$ to compute $E(Y) = E[|X|]$ b) Use formula (3.24) with $g(x) = |x|$.

Problem 4: a) Compute the median of the random variables in problems 1 and 2. b) Compute the 0.9-quantile of the random variable in problem 2.

Problem 5: Suppose the random variable $X$ has mean $E(X) = 3$ and variance $\text{var}(X) = 4$. Compute the following quantities: a) $E(3X + 2)$, b) $E(X^2)$, c) $E((2X + 3)^2)$, d) $\text{var}(4X - 2)$.

Problem 6: Let $X$ be a discrete random variable with possible values $0, 1, 2, \ldots$. Show that an alternate way of computing $E(X)$ is via $\sum_{k=1}^{\infty} P(X \geq k)$. Hint: Write down $P(X \geq k)$ in terms of its pmf terms for $k = 1, 2, 3, \ldots$ in one row below the previous and you should see the idea. Examine how many times you see $P(X = 1)$, $P(X = 2)$, $P(X = 3)$ and so on.

Problem 7: Assume that $E(|X|) < \infty$. By the triangle inequality $|X - a| \leq |X| + |a|$ this implies that $E(|X - a|) < \infty$ as well for any $a \in \mathbb{R}$. Show that $E(|X - a|)$ is minimized over $a$ whenever $a$ is a median of $X$. Let $m$ be any median median of the distribution of $X$. I will show that $E(|X - a|) \geq E(|X - m|)$ when $a < m$. You will show the other half, namely $E(|X - a|) \geq E(|X - m|)$ when $a > m$.

Assume $a < m$ and by examination of the cases $X \leq a$, $X \geq m$ and $a < X < m$ write

$$
|X - a| - |X - m| = (a - m)I_{(-\infty,a]}(X) + (m - a)I_{[m,\infty)}(X) + (2X - a - m)I_{(a,m)}(X)
$$

Taking expectations on both sides and using $E(|X - a|) - E(|X - m|) = E(|X - a| - |X - m|)$ (because $E(g_1(X) + g_2(X)) = E(g_1(X)) + E(g_2(X))$ for discrete and continuous $X$) we get

$$
E(|X - a|) - E(|X - m|) = (m - a)E[(X \geq m) - P(X \leq a)] + E[(2X - a - m)I_{(a,m)}(X)]
$$

Here the last term is $\geq 0$ because $X - a > 0$ for all $X \in (a, m)$ and the first term is $\geq 0$ because $m - a > 0$ and $P(X \geq m) \geq P(X \leq a)$ since $P(X \geq m) \geq 1/2$ by definition of the median. Thus $E(|X - a|) \geq E(|X - m|)$, as claimed. Now you show $E(|X - a|) \geq E(|X - m|)$ for $m < a$. (Examine $|X - a| - |X - m|$ for $X \leq m$, $X \geq a$ and $m < X < a$. Or reduce it to what was shown above by looking at $Y = -X$.) From this it follows as a corollary that $|\mu - m| \leq \sigma$ where $\mu$ and $\sigma$ are mean and standard deviation of $X$ and $m$ is its median.

proof:

$$
|\mu - m| = |E(X - m)| \leq E|X - m| \leq E|X - \mu| \leq \sqrt{\text{var}(|X - \mu|)} = \sigma
$$

where the first follows from $-|X - m| \leq X - m \leq |X - m|$ and the corresponding inequalities of expectations. The second $\leq$ is the above exercise concerning the minimization property of the median, and the third $\leq$ follows from $E|X - \mu|^2 \geq (E|X - \mu|)^2$ which follows from $\text{var}(|X - \mu|) \geq 0$. 

Homework 6
Due February 27, 2019