Homework 6

Due February 27, 2019

Problem 1: Let X be a discrete random variable with values 1,2,3,4,5 and corresponding probabilities $1/7, 1/14, 3/14, 2/7, 2/7$. a) Compute $E(X)$ b) compute $E[|X-2|]$.

Problem 2: Let X be a continuous random variable with pdf $f(x) = 3\exp(-3x)I_{(0,\infty)}(x)$.
a) Find $E(X)$ and compute b) $E(\exp(2X))$.

Problem 3: Let X have pmf $P(X=-1) = 1/2$, $P(X=0) = 1/3$ and $P(X=1) = 1/6$. Calculate $E[|X|]$ using both approaches a) and b) below.
a) First find the pmf of $Y = |X|$ to compute $E(Y) = E[|X|]$
b) Use formula (3.24) with $g(x) = |x|$.

Problem 4: a) Compute the median of the random variables in problems 1 and 2.
b) compute the 0.9-quantile of the random variable in problem 2.

Problem 5: Suppose the random variable X has mean $E(X) = 3$ and variance $\text{var}(X) = 4$. Compute the following quantities: a) $E[3X+2]$, b) $E[X^2]$, c) $E[(2X+3)^2]$, d) $\text{var}(4X-2)$.

Problem 6: Let X be a discrete random variable with possible values $0, 1, 2, \ldots$. Show that an alternate way of computing $E(X)$ is via $\sum_{k=1}^{\infty} P(X \geq k)$. Hint: Write down $P(X \geq k)$ in terms of its pmf terms for $k = 1, 2, 3, \ldots$ in one row below the previous and you should get the idea. Examine how many times you see $P(X=1)$, $P(X=2)$, $P(X=3)$ and so on.

Problem 7: Assume that $E(|X|) < \infty$. By the triangle inequality $|X-a| \leq |X| + |a|$ this implies that $E(|X-a|) < \infty$ as well for any $a \in \mathbb{R}$. Show that $E(|X-a|)$ is minimized over a whenever a is a median of X. Let m be any median median of the distribution of X. I will show that $E(|X-a|) \geq E(|X-m|)$ when $a < m$. You will show the other half, namely $E(|X-a|) \geq E(|X-m|)$ when $a > m$.
Assume $a < m$ and by examination of the cases $X \leq a$, $X \geq m$ and $a < X < m$ write

$$|X-a| - |X-m| = (a-m)I_{(-\infty,a]}(X) + (m-a)I_{[m,\infty)}(X) + (2X-a-m)I_{(a,m)}(X)$$

Taking expectations on both sides and using $E(|X-a|) - E(|X-m|) = E(|X-a| - |X-m|)$ (because $E(g_1(X) \pm g_2(X)) = E(g_1(X)) \pm E(g_2(X))$ for discrete and continuous X) we get

$$\begin{aligned}
E(|X-a|) &- E(|X-m|) \\
&= (m-a)\,[P(X \geq m) - P(X \leq a)] + E[(2X-a-m)\,I_{(a,m)}(X)] \\
&= (m-a)\,[P(X \geq m) - P(X \leq a)] + E[(2X-2a-(m-a))\,I_{(a,m)}(X)] \\
&= (m-a)\,[P(X \geq m) - P(a < X < m) - P(X \leq a)] + 2E[(X-a)\,I_{(a,m)}(X)] \\
&= (m-a)\,[P(X \geq m) - P(X < m)] + 2E[(X-a)\,I_{(a,m)}(X)]
\end{aligned}$$

Here the last term is ≥ 0 because $X - a > 0$ for all $X \in (a,m)$ and the first term is ≥ 0 because $m - a > 0$ and $P(X \geq m) \geq P(X < m) = 1 - P(X \geq m)$ since $P(X \geq m) \geq 1/2$ by definition of the median. Thus $E(|X-a|) \geq E(|X-m|)$, as claimed. Now you show $E(|X-a|) \geq E(|X-m|)$ for $m < a$. (Examine $|X-a| - |X-m|$ for $X \leq m$, $X \geq a$ and $m < X < a$. Or reduce it to what was shown above by looking at $Y = -X$.) From this it follows as a corollary that $|\mu - m| \leq \sigma$ where μ and σ are mean and standard deviation of X and m is its median.
proof:

$$|\mu - m| = |E(X-m)| \leq E|X-m| \leq E|X-\mu| \leq \sqrt{E(|X-\mu|^2)} = \sigma$$

where the first $\leq$ follows from $-|X-m| \leq X-m \leq |X-m|$ and the corresponding inequalities of expectations. The second $\leq$ is the above exercise concerning the minimization property of the median, and the third $\leq$ follows from $E|X-\mu|^2 \geq (E|X-\mu|)^2$ which follows from $\text{var}(|X-\mu|) \geq 0$.