Homework 5

Due February 22, 2019

Problem 1: Let $B \sim \text{Ber}(0.3)$ and $U \sim \text{Unif}[0,1]$ be independent random variables.
a) For any $x \in \mathbb{R}$ find the cdf $F_X(x)$ of $X = B \cdot U$ using the law of total probability

$$F_X(x) = P(X \leq x) = P(X \leq x | B = 0)P(B = 0) + P(X \leq x | B = 1)P(B = 1)$$

and exploiting the independence of U and B.
b) Is X a continuous random variable?
c) Find $F_X(.5)$ and $P(0 < X < .8)$.
d) Find q such that $P(X \leq q) = .8$.

Problem 2: You toss 15 fair coins until the number of heads in each such toss is either divisible by 5 or 4. (0 is divisible by any $x \neq 0$.)
What is the chance that you wind up with a result divisible by 5?

Problem 3: Let $X_1 \sim \text{Ber}(p_1)$ and $X_2 \sim \text{Ber}(p_2)$ be independent random variables and let $Y = X_1 + X_2$.
a) Find the pmf $p_Y(y) = P(Y = y)$.
b) Give a graphical rendition of this pmf when $p_1 = 1/4$ and $p_2 = 1/2$.

Problem 4: Suppose that X is a continuous random variable with cdf $F_X(x)$. Assume that its pdf $f_X(x)$ is continuous except at finitely many points, so that $F'(x) = f_X(x)$, except at those points.
a) For given constants $b > 0$ and a find the cdf and pdf of $Y = a + bX$ in terms of F_X and f_X respectively. Hint: turn $Y \leq y$ into an equivalent statement $X \leq ??$.
b) Apply the previous result to show that $Y \sim \text{Unif}[a, a + b]$ when $X \sim \text{Unif}[0,1]$.
c) How would the answer in a) change if we do this instead for $Y = a - bX$?
d) If $X \sim \text{Unif}[0,1]$ show that $Y = a - bX$ is uniform over what range. In particular, what is the distribution of $1 - X$.

Problem 5: Let $X_1, \ldots, X_n$ be mutually independent with $X_i \sim \text{Unif}[0,1]$, $i = 1, \ldots, n$. Define the random variable $Y = \min(X_1, \ldots, X_n)$.
a) Find $P(Y > x)$ as a simple function of x.
b) Find the density of Y.

Problem 6: A planned nuclear power plant is at a location in direct line of a known earthquake fault line of length L. For design and risk assessment considerations it is important to know the distribution of the closest distance $\tilde{D}_n$ the earth quake epicenter could have to the power plant during n independent earthquake events above a specified magnitude. Assume that the epicenters occur uniformly along the fault line $[0, L]$ (visualized as an interval on the real line) and that the planned location of the power plant is on the same line at location $x_0 > L$, i.e, to the right of interval $[0, L]$. Find the cdf of the distance D_i for the i^{th} earthquake and then the cdf of $\tilde{D}_n = \min(D_1, \ldots, D_n)$. Hint: you may want to make use of what you learned in problems 4) and 5). You can write $D_i = x_0 - LU_i = x_0 + (1 - U_i)L - L$, where the U_i are mutually independent $\sim \text{Unif}[0,1]$, as are the $V_i = 1 - U_i$ mutually independent $\sim \text{Unif}[0,1]$.