Homework 4

Due February 6, 2019

Problem 1: You ask a friend to pick two integers $x < y$ from $\{1, 2, \ldots, 10\}$. This friend is to write these 2 numbers on 2 slips of paper, unseen by you. They are placed face down in front of you and you randomly pick one and turn it over. You are supposed to say whether this upturned number U is y or not. Your sight of the upturned number U provides some information. It is not as though you had to say which of the down turned slips has the larger number. Your chance of making the correct guess will be better than 1/2 if you proceed as follows. You pick a number Z at random from $\{1, 2, \ldots, 10\}$ and you decide $U = x$ if $Z \geq U$ and decide $U = y$ if $Z < U$. Show by how much your chance of a correct decision exceeds 1/2. The answer depends on x and y.
Hint: U and Z are random and independent and x, y are not random, but unknown to you. For example, one of the two ways of making a correct decision can be expressed as $\{U = x\} \cap \{Z \geq U\}$ which is the same as $\{U = x\} \cap \{Z \geq x\}$ and use independence of U and Z.

Problem 2: A test is 95% effective on persons with the disease and has a 1% false alarm rate. Suppose that the prevalence of the disease among patients that present a certain set of symptoms is 40%. What is the chance that such a tested symptomatic patient actually has the disease (event D), given that the test is positive (event E)? In presenting the positive result to the patient how would the doctor proceed? Use similar reasoning as presented in class.

Problem 3: Assume A, B, C are mutually independent, show that $A \cup B$ and C are independent.

Problem 4: You perform a sequence of $m+n$ independent Bernoulli trials with success probability $p \in (0, 1)$. Let X denote the number of successes in the first m trials and Y be the number of successes in the last n trials. Find $f(x|z) = P(X = x|X + Y = z)$. Show that this function of x, which will not depend on p, is a pmf in x with integer values in $[\max(0, z - n), \min(z, m)]$.
Hint: Note that, as in problem 1, the intersection of seemingly dependent events $\{X = x\}$ and $\{X + Y = z\}$ can be written as the intersection of independent events $\{X = x\}$ and $\{Y = z - x\}$.

Problem 5: An airplane engine after maintenance has functioned properly for 2 takeoffs but failed during the 3rd takeoff. From past experience it is known that an engine's failure probability per takeoff is .7 whenever a certain seal is missing, but it is only .001, whenever the seal is present. Also, from experience we can assume the chance for noninstallation of this seal during the maintenance to be 0.2%.

a) What are the two chances of the above history $F_1^c F_2^c F_3$, i.e., (funct.,funct., failed), under either of the two seal conditions ("seal absent" = event A and "seal present"= event A^c), i.e., find $P(F_1^c F_2^c F_3|A)$ and $P(F_1^c F_2^c F_3|A^c)$. Assume independent trials for the takeoffs in either case?

b) What is the conditional chance for the seal to be missing given the observed history $F_1^c F_2^c F_3$?

Problem 6: Let $X \sim \text{Geom}(p)$. a) Show that $P(X \geq k) = (1 - p)^{k-1}$ for $k = 1, 2, \ldots$.
b) Show that any discrete random variable Y with values $1, 2, \ldots$ and with the property $P(Y \geq k) = (1 - p)^{k-1}$ for $k = 1, 2, \ldots$ must have a geometric distribution with parameter p, i.e., $Y \sim \text{Geom}(p)$.
c) Find $P(X \geq k + h - 1|X \geq k) = P(X - k + 1 \geq h|X \geq k)$ for any integers k and $h \geq 1$. What can you conclude about the conditional distribution of $X - k + 1$ given that $X \geq k$?
Why does this make sense intuitively.

Problem 7: Suppose the events A, B and C are mutually independent with $P(A) = 1/2$, $P(B) = 1/3$ and $P(C) = 1/4$. Compute $P(AB \cup C^c)$.