Stat 311: HW 7, Chapter 9, Solutions

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Ch. 9, Problem 5. (a) The X_i are Bernoulli random variables and their distribution belongs to the family of Bernoulli distributions, indexed by the success probability p. We have $\mu = EX_i = p$ and $\sigma^2 = \mathrm{var}X_i = p(1-p)$.

(b) The health officals (guarding the health of the public) should take the conservative positition of $H_0 : \mu = p \ge \mu_0 = 0.02$ unless proven otherwise, i.e., when we have evidence to make us reject H_0 in favor of $H_1 : \mu = p < \mu_0 = 0.02$, in which case the health officials cannot destroy all chickens.

(c) The farmers (having an interest in keeping theirs chickens) would want to take the conservative position of $H_0 : \mu = p \le \mu_0 = 0.02$ unless proven otherwise, i.e., when we have evidence to make us reject H_0 in favor of $H_1 : \mu = p > \mu_0 = 0.02$, in which case the chickens should be destroyed.

(d) For $\mu = \mu_0 = 0.02$ we get $\sigma_0^2 = 0.02 \cdot 0.98 = 0.0196 = 0.14^2$, i.e., $\sigma_0 = 0.14$. The observed value of the $Z_n = (\bar{X}_n - \mu_0)/(\sigma_0/\sqrt{1000})$ statistic for $n = 1000$ is $z_n = (40/1000 - 0.02)/(0.14/\sqrt{1000}) = 4.517540$.

(e) From the health officials' perspective we should reject their H_0 whenever $\bar{x}_n$ is too small, or when z_n is too small. Here we have z_n not anywhere near small (compared to the standard normal distribution), in fact we get as significance probability

$$P_{\mu_0}(\bar{X}_n \le 40/1000) = P_{\mu_0}(Z_n \le 4.517540) \approx \Phi(4.517540) = \mathtt{pnorm}(4.517540) = 0.9999969$$

which is not anywhere near being ≤ 0.001. Thus the health officials' conservative $H_0 : \mu \ge \mu_0 = 0.02$ should prevail, i.e., should not be rejected. The chickens should be destroyed.

(f) From the farmers' perspective we should reject their conservative position $H_0 : \mu \le \mu_0 = 0.02$ only when $\bar{x}_n$ is too large. Here we get as significance probability

$$P_{\mu_0}(\bar{X}_n \ge 40/1000) = P_{\mu_0}(Z_n \ge 4.517540) \approx 1 - \Phi(4.517540) = 1 - \mathtt{pnorm}(4.517540) = 3.128111\mathrm{e}-06$$

which is way below $\alpha = 0.10$, i.e., the farmers' H_0 should be rejected and the chickens should be destroyed.

Ch. 9, Problem 6. The consumer organization want to "prove" that the advertised claim is false. Thus it sets up the conservative $H_0 : \mu \ge 800$ in the hope to get sufficient evidence to reject that hypothesis in favor of $H_1 : \mu < 800$. We would reject H_0 for small values of $\bar{x}_n$. Here $\bar{x}_n = 745.1$ with $s_n = 238.0$ and thus $t_n = (745.1 - 800)/(238.0/\sqrt{100}) = -2.306723$ and we get as significance probability

$$P_{\mu_0}(\bar{X}_n \le \bar{x}_n) = P_{\mu_0}(T_n \le t_n) \approx \Phi(-2.306723) = \mathtt{pnorm}(-2.306723) = 0.01053513$$

which is sufficiently smaller than the a priori agreed $\alpha = 0.05$. Thus we should reject H_0 and pursue the claim further.

Ch. 9, Problem 8. Since each student makes 100 tosses of a fair coin, the normal approximation for $Y =$ number of heads or $\bar{X} =$ proportion of heads should be quite good. Thus the coverage probability of any such interval would be about 0.95. We can consider 0.95 as the success probability of a Bernoulli trial, either the interval contains 0.5 (success) or not (failure). The 600 intervals constitute 600 independent Bernoulli trials and if V is the number of successes in these 600 trials (i.e., the number of intervals containing 0.5) then $EV = 600 \cdot 0.95 = 570$. Thus the answer should be: TRUE.

Ch. 9, Problem 12 (a).

$$\begin{aligned}
\text{i)}\quad \mathbf{p} &= P(Y \le 2) = \texttt{pbinom}(2, 89, 0.30) = \texttt{1.240591e} - 11 \\
\text{ii)}\quad \mathbf{p} &= P(Y \le 2) = P(\bar{X}_n \le 2/89) = P\left(Z_n \le \frac{2/89 - 0.30}{\sqrt{0.3 \cdot 0.7/89}}\right) = P(Z_n \le -5.713369) \\
&\approx \Phi(-5.713369) = \texttt{pnorm}(-5.713369) = \texttt{5.538057e} - 09
\end{aligned}$$

In relative terms the approximation is off by a factor of $445 = |\texttt{5.538057e} - 09 - \texttt{1.240591e} - 11|/\texttt{1.240591e} - 11$, but in absolute terms the error is $|\texttt{5.538057e} - 09 - \texttt{1.240591e} - 11| = \texttt{5.525651e} - 09$, i.e., quite small, which should not matter in most hypothesis testing decisions. In both cases, i) and ii), we had plenty of evidence for rejecting H_0.