Oscillatory flow in microchannels
Comparison of exact and approximate impedance models with experiments

Christopher J. Morris, Fred K. Forster

Abstract
Commonly used, lumped-parameter expressions for the impedance of an incompressible viscous fluid subjected to harmonic oscillations in a channel were compared with exact expressions, based on solutions of the Navier-Stokes equations for slots and channels of circular and rectangular cross-section, and were found to differ by as much as 30% in amplitude. These differences resulted in predicted discrepancies by as much as 400% in frequency response amplitude for simple second-order systems based on size scales and frequencies encountered in microfluidic devices. These predictions were verified experimentally for rectangular microchannels and indicate that underdamped fluidic systems operating near the corner frequency of any included flow channel should be modeled with exact expressions for impedance to avoid potentially large errors in predicted behavior.

Keywords
Flow in microsystems, Microfluidics, Microchannels, Low-order model, Macromodel, Impedance

List of symbols

\[ A \] Channel cross-sectional area \( (m^2) \)
\[ A_c \] Membrane area \( (m^2) \)
\[ a \] Rectangular duct and slot half-width or radius \( (m) \)
\[ b \] Rectangular duct half-depth and slot depth \( (m) \)
\[ C \] Capacitance \( (m^3/Pa) \)
\[ C^* \] \( C + (A_c \gamma)^2 / k \)
\[ D_h \] Channel hydraulic diameter \( 4A/P_c (m) \)
\[ E \] Voltage \( (V) \)
\[ f \] Darcy friction factor
\[ F \] Force \( (N) \)
\[ I \] Channel inertance \( (Pa \ s^2/m^3) \)
\[ i \] \( \sqrt{-1} \)
\[ \Im \] Imaginary part of a complex number
\[ J_k \] Bessel function of the first kind of order \( k \)
\[ \hat{H} \] System transfer function
\[ K \] Sum of minor loss factors
\[ k \] Membrane stiffness \( (N/m) \)
\[ l \] Channel length \( (m) \)
\[ n \] Outward unit normal vector
\[ P \] Fluid pressure \( (Pa) \)
\[ p_n \] \( \frac{1}{2} (2n + 1) \)
\[ Q \] Volumetric flow rate \( (m^3/s) \)
\[ R \] Channel resistance \( (Pa \ s/m^3) \)
\[ R_I \] Real part of a complex number
\[ R_e \] Reynolds number, \( V D_h / v \)
\[ V \] Velocity \( (m/s) \)
\[ V \] Volume \( (m^3) \)
\[ w \] Axial component of velocity \( (m/s) \)
\[ X_m \] Harmonic amplitude of membrane centerline displacement
\[ Z \] Fluid impedance \( (kg/m^4 \ s) \)
\[ \alpha \] Duct aspect ratio, \( b/a \)
\[ \eta \] Nondimensional frequency parameter, \( \omega a^2 / v \)
\[ \eta_c \] Nondimensional corner frequency, \( \omega C a^2 / v \)
\[ \gamma \] Membrane shape factor
\[ \lambda \] \( C/C^* \)
\[ \mu \] Fluid dynamic viscosity \( (Pa \ s) \)
\[ \nu \] Fluid kinematic viscosity \( (m^2/s) \)
\[ \rho \] Mass density \( (kg/m^3) \)
\[ \omega \] Radian frequency
\[ \omega_c \] Undamped natural frequency
\[ R/I_s \] Cutoff or corner frequency
\[ \Phi \] Channel shape parameter in Eqs. 29 and 30
\[ \zeta \] Damping ratio
\[ ( )_e \] Exact property
\[ ( )_s \] Simplified property
\[ ( ) \] Spatial average
\[ (" ) \] Complex quantity

1 Introduction
Current micromachining technology enables the fabrication of microfluidic systems with many mechanical, electrical, and fluidic components operated in a time-dependent fashion. Because continuum modeling is difficult even for simple systems, accurate low-order modeling is essential for design. One example is the membrane-driven micropump that produces maximum pump output at a resonant frequency near the corner, or cutoff, frequency of the valves (Forster et al. 1995). The valves for this pump are of fixed-geometry, etched in silicon using
the deep reactive ion etching (DRIE) process. The valves behave as fluid diodes, and because they exhibit relatively low diodicity, their dynamic behavior for design of system 
resonance is approximated by zero-mean oscillatory flow 
through straight rigid channels (Morris and Forster 2003).

Fluid channels are prevalent elements in many micro-
and macro-sized fluidic systems. A common, low-order 
model for harmonic incompressible viscous flow in a rigid 
channel is impedance, $Z$, the ratio of complex amplitudes 
of pressure drop to volume flow rate, represented by a 
series combination of resistance, $R$, and inerter $I$,

$$Z = \frac{\Delta P}{Q} = R + i\omega I. \quad (1)$$

Simplified ad hoc approaches are often used to deter-
mine $R$ and $I$ by neglecting fluid mass effects to estimate $R$ 
and neglecting viscous effects to estimate $I$, with neither 
quantity being a function of frequency. This approach is 
typically found in textbooks on system modeling (for 
example, Rowell and Wormley 1997). The same approach 
has also been used to account for the viscous and inertial 
effects with additional ad hoc methods to account for 
compressibility (Rohmann and Grogan 1957). Similar 
analyses in acoustics are based on work by Rayleigh (1945) 
dating back over a century. Based on this earlier work, 
Beranek (1954) gives approximate relations for $R$ and $I$ that 
also are independent of frequency for “very small” circular 
tubes and slits, and for “intermediate-sized” circular tubes 
with $R$ being a linear function of $\sqrt{\omega}$. However, these 
results collectively do not cover all frequencies. In addition, 
the approximations used in these models of acoustical 
elements are only valid for cross-sectional shapes for 
which the radius of curvature is large compared to the 
boundary layer thickness (Ingard 1953), which is not the 
 case for rectangular ducts at any frequency. Nevertheless, 
for very short ducts, i.e., orifices, ad hoc acoustical models 
for $R$ exist based on boundary layer thickness associated 
with flat plates (Morse and Ingard 1968).

Closely associated with impedance modeling are complete 
solutions for the incompressible pressure-velocity field 
for fully developed, pressure-driven harmonic flow in 
channels based on the Navier-Stokes equations. Solutions 
can be found in the literature for various cross-sectional 
shapes, including circular (Uchida 1956; Womersley 1955), 
the two-dimensional slot (Panton 1996), and rectangular 
(Drake 1965; Fan and Chao 1965; O’Brien 1975). These 
solutions can be readily used to obtain exact expressions 
for impedance. An example of such an analysis for the case 
of the circular cross-section that extends Uchida (1956) to 
include compressibility of liquids in the governing equa-
tions and makes comparisons with experimental results is 
presented by D’Souza and Oldenberger (1964).

The above review indicates that the literature con-
cerning low-order modeling of fluidic impedance for 
channels is significantly fragmented in terms of geometry 
and frequency range. In addition, textbooks on low-order 
modeling of fluidic systems, typically, only present highly 
Simplified impedance is composed of resistance and 
inductance values, as shown in Eq. 1, where resistance, $\frac{\Delta P}{Q}$, 
and inductance, $\Delta P/\frac{dQ}{dt}$, are real and independent of fre-
quency. The commonly used expression for simplified 
resistance, denoted in this study by $R_s$, is based on steady, 
viscous, fully developed flow through a straight channel of 
constant cross-section. Examples of different cross-
 sectional shapes are readily found in the literature. For a 
circular cross-section of radius $a$ (White 1991, Eq. 3–36)

$$R_s,\text{circ} = \frac{8\mu L}{\pi a^4}, \quad (2)$$

and for a slot of width $2a$ (White 1991, Eq. 3–45)
\[ R_{s,\text{slot}} = \frac{3\mu L}{2ba^2}, \]  
\[ R_{s,\text{rect}} = \frac{\mu L}{4za^2} \left( 1 - \frac{2x^2}{3(2n + 1)^2} \sum_{n=0}^{\infty} \frac{\tanh (p_n a)}{p_n a^3} \right)^{-1}, \]

where \( b \) is the slot depth considered. For a rectangular channel of width \( 2a \), depth \( 2b \), and aspect ratio \( \alpha = b/a \) (White 1991, Eq. 3–48)

\[ I_{s,\text{rect}} = \frac{\rho L}{4za^2}. \]

and for the rectangular duct,

A general expression for simplified inductance, \( I_{s} \), can be obtained by direct application of the momentum equation to unsteady, inviscid, fully developed axial flow in a straight duct of arbitrary, constant cross-sectional shape. The momentum equation for a control volume, CV, consisting of the fluid in a straight duct between streamwise locations, \( z_1 \) and \( z_2 \), yields the relation between the axial-direction force, \( F_a \), and the rate of change of momentum in the control volume. For fully developed flow, i.e., when axial velocity, \( w \), is not a function of \( z \),

\[
F_a = \frac{d}{dt} \int_{CV} q w(x, y, t) \, dv + \int_{\text{CS}} q w(x, y, t) V(x, y, t) \cdot ndA \n
= \rho \frac{d}{dt} \int_{A} w(x, y, t) \, dv - \int_{A(z_1)} w^2(x, y, t) \, dv + \int_{A(z_2)} w^2(x, y, t) \, dv,
\]

where \( A \) is the portion of the control surface, CS, through which the fluid flows. For a constant cross-sectional area, the integral over \( z \) in the first term can be replaced by the length, \( L \), of the duct considered, and the second and third terms cancel. In addition, for inviscid flow with gravity neglected, the net force is due only to the pressure difference over length, \( L \). Thus,

\[
\Delta P = \frac{\rho L}{A} \int_{A} w(x, y, t) \, dv.
\]

Noting that the integral of the velocity over the cross-sectional area is the volume flow rate, \( Q \),

\[
\Delta P = \frac{\rho L}{A} \frac{dQ}{dt},
\]

or

\[
I_s = \frac{\rho L}{A},
\]

which is independent of the velocity profile. For the circular pipe,

\[ I_{s,\text{circ}} = \frac{\rho L}{\pi a^2}, \]

for the slot,

\[ I_{s,\text{slot}} = \frac{\rho L}{2ab}, \]

and for the rectangular duct,

The equations for \( R_s \) and \( I_s \) constitute simplified expressions commonly found in textbooks and are frequently used in microdevice simulations, as discussed in Sect. 1. However, it is re-emphasized that \( R_s \) is derived for steady, viscous flow, while \( I_s \) is derived from unsteady, inviscid flow. Thus, the combined use to obtain a simplified fluid impedance, \( Z_s \), according to Eq. 1 represents an ad hoc and inconsistent approach.

The amplitude of the simplified transfer function for the volume flow rate through a system consisting of only the channel driven by a harmonic pressure difference is given by

\[
|H_e(j\omega)| = \left| \frac{\dot{Q}}{\Delta P} \right| = \left| \frac{1}{Z_s} \right| = \frac{1}{\sqrt{R_s^2 + j\omega^2 I_s^2}}.
\]

This first-order system has low- and high-frequency asymptotes intersecting at a corner frequency, \( \omega_c = R_s/I_s \). It follows that by defining the nondimensional corner frequency, \( \eta_c^2 \), such that

\[ \eta_c^2 = \frac{\omega_c a^2}{v}, \]

the corner frequency can be represented for any cross-sectional shape considered through the corresponding shape-dependent frequency, \( \eta_c \), given in Table 1. Table 2 lists numerical values of \( \eta_c^2 \) for the rectangular duct.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>( \eta_c^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular</td>
<td>8</td>
</tr>
<tr>
<td>Slot</td>
<td>3</td>
</tr>
<tr>
<td>Rectangular</td>
<td>\left[ \frac{1}{3} - 2x^4 \sum_{n=0}^{\infty} \frac{\tanh (p_n a)}{p_n a^3} \right]^{-1}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \eta_c^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.1135</td>
</tr>
<tr>
<td>2</td>
<td>4.3729</td>
</tr>
<tr>
<td>4</td>
<td>3.5611</td>
</tr>
<tr>
<td>8</td>
<td>3.2566</td>
</tr>
<tr>
<td>16</td>
<td>3.1230</td>
</tr>
<tr>
<td>128</td>
<td>3.0148</td>
</tr>
<tr>
<td>1024</td>
<td>3.0018</td>
</tr>
</tbody>
</table>
cross-section driven by the pressure difference $\Delta P_{(\text{int})}$, and where $Q$ is the integral of $\dot{w}(x,y)$ over the cross-section. The actual velocity field,
\[
\dot{w}(x,y,t) = \Re \left[ \dot{w}(x,y)e^{(i\omega t)} \right],
\]
(15)
corresponds to the actual driving pressure,
\[
\Delta P = \Re \left[ \Delta P_{(\text{int})} \right].
\]
(16)

For a circular pipe (White 1991, Eq. 3–98)
\[
\dot{w}(r) = \frac{\Delta P}{\mu \eta^2} \left[ 1 - \frac{J_0 \left( \frac{\eta}{\sqrt{\eta^2}} \right)}{J_0 \left( \sqrt{\eta^2} \right)} \right],
\]
(17)
where $\eta^2$ is the nondimensional frequency parameter
\[
\eta^2 = \frac{\omega a^2}{v}.
\]
(18)
The parameter, $\eta$, is also referred to as the Womersley number (Womersley 1955). The complex amplitude of the volume flow rate is
\[
\dot{Q} = \frac{\Delta P \pi a^4}{\mu \eta^2} \left[ 1 - \frac{2 \frac{J_1 \left( \eta \right)}{\sqrt{\eta^2}}}{J_0 \left( \sqrt{\eta^2} \right)} \right],
\]
(19)
also readily obtained from Iberall (1950), and it follows that the impedance can be expressed as
\[
\dot{Z}_e = \frac{\rho \nu a \left( 1 - \frac{2 \frac{J_1 \left( \eta \right)}{\sqrt{\eta^2}}}{J_0 \left( \sqrt{\eta^2} \right)} \right)}{1 - \frac{2 \frac{J_1 \left( \eta \right)}{\sqrt{\eta^2}}}{J_0 \left( \sqrt{\eta^2} \right)}},
\]
(20)

For a slot (Panton 1996, Eq. 11.5.13), the velocity field is
\[
\dot{w}(y) = \frac{\Delta P a^2}{\mu a^2} \left[ 1 - \frac{\cosh \left( \frac{y}{a} \right)}{\cosh \left( \frac{\eta^2}{a^2} \right)} \right],
\]
(21)
the volume flow rate is
\[
\dot{Q} = \frac{\Delta P 2 \pi a^3}{\mu \eta^2} \left[ 1 - \frac{\tanh \left( \frac{\eta^2}{a^2} \right)}{\sqrt{\eta^2}} \right],
\]
(22)
and the impedance can be expressed as
\[
\dot{Z}_e = \frac{\rho \nu a \left( 1 - \frac{\tanh \left( \frac{\eta^2}{a^2} \right)}{\sqrt{\eta^2}} \right)}{1 - \frac{\tanh \left( \frac{\eta^2}{a^2} \right)}{\sqrt{\eta^2}}},
\]
(23)

For the rectangle (see Appendix),
\[
\dot{w}(x,y) = \frac{\Delta P a^2}{\mu \eta^2} \left[ 1 - 2 \sum_{n=0}^{\infty} \frac{\sqrt{1+\xi}}{p_n} \right]
\times \left[ \cosh \left( \frac{\eta \left( x/a \right)}{\cosh \left( \frac{\eta \left( y/b \right)}{p_n} \right)} \right. \cos \left( p_n \left( y/b \right) \right) \right]
\times \left. \cosh \left( \frac{\eta \left( y/b \right)}{\cosh \left( \frac{\eta \left( x/a \right)}{p_n} \right)} \right) \cos \left( p_n \left( x/a \right) \right) \right],
\]
(24)
where
\[
r_n = \sqrt{\eta^2 + \frac{p_n^2}{4}} \quad \text{and} \quad s_n = \sqrt{\eta^2 x^2 + \frac{p_n^2}{4}}.
\]
(25)
The volume flow rate is
\[
\dot{Q} = \frac{\Delta P 2 \pi a^3}{\mu \eta^2} \left[ 1 - \frac{2 \frac{J_1 \left( \eta \right)}{\sqrt{\eta^2}}}{J_0 \left( \sqrt{\eta^2} \right)} \right]
\times \left[ \frac{\tanh \left( \frac{\eta \left( x/a \right)}{s_n} \right)}{\frac{\eta \left( x/a \right)}{s_n}} + \frac{\tanh \left( \frac{\eta \left( y/b \right)}{s_n} \right)}{\frac{\eta \left( y/b \right)}{s_n}} \right],
\]
(26)
and the impedance can be expressed as
\[
\dot{Z}_e = \frac{\rho \nu a \left( 1 - \frac{2 \frac{J_1 \left( \eta \right)}{\sqrt{\eta^2}}}{J_0 \left( \sqrt{\eta^2} \right)} \right)}{1 - \frac{2 \frac{J_1 \left( \eta \right)}{\sqrt{\eta^2}}}{J_0 \left( \sqrt{\eta^2} \right)}},
\]
(27)

Expressions for the exact resistance and inductance for each of the geometries considered above, according to Eq. 1, are given by
\[
R_e = \Re \left[ \dot{Z}_e \right] \quad I_e = \frac{1}{\omega \Re} \Im \left[ \dot{Z}_e \right],
\]
(28)
where $R_e$ and $I_e$ are functions of $\omega$ or, equivalently, $\eta^2$. Generalized expressions can also be formed that relate the expressions for exact resistance and impedance to the simplified expressions, namely
\[
\frac{L_e(\eta^2)}{I_e} = \Im \left[ \Phi \right],
\]
(29)
and with the use of Eqs. 14 and 18,
\[
\frac{R_e(\eta^2)}{R_e} = \frac{I_e}{R_e} = \frac{L_e(\eta^2)}{I_e},
\]
(30)
where expressions for $\Phi$ for the cross-sectional shapes considered are listed in Table 3. Note that $\eta^2/\eta_c^2$ is simply $\omega/\omega_c$, i.e., the excitation frequency normalized by the corner frequency of the channel.

### 2.3 System modeling

Two systems were considered for the purpose of investigating the differences resulting from the choice of simplified or exact expressions for impedance of a fluid channel. The first system is the most basic fluid system that could be considered. The second system is a basic model of a micropump, or a pump-like device with straight channels that replace the valves, as shown in

<table>
<thead>
<tr>
<th>Geometry</th>
<th>$\Phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular</td>
<td>$\eta^2/\eta_c^2$</td>
</tr>
<tr>
<td>Slot</td>
<td>$\eta^2/\eta_c^2$</td>
</tr>
<tr>
<td>Rectangular</td>
<td>$\frac{1}{\omega} \frac{1}{\sqrt{\eta^2}}$</td>
</tr>
</tbody>
</table>

| Table 3. Shape parameter, $\Phi$, in Eqs. 29 and 30 |
Fig. 1. The latter model was necessary to compare theoretical calculations to actual measurements and represented the experimental apparatus used.

2.3.1 Series RIC circuit

A series combination consisting of a pressure source, \( P \), resistor, \( R \), inductor, \( L \), and capacitor, \( C \), yields a second-order system. Utilizing the simplified expressions for \( R \) and \( L \) from Sect. 2.1, the magnitude of the corresponding transfer function, \( H_s \), relating the pressure across the capacitor, \( P_C \), to the source pressure is given by

\[
|H_s(i\omega)| = \frac{|P_C(i\omega)|}{|P(i\omega)|} = \frac{1}{\sqrt{(1 - \zeta_s^2)^2 + (2\zeta_s\omega_s)^2}},
\]  

(31)

where \( \omega_s = \omega / \omega_m \), and where \( \omega_m = 1 / \sqrt{L/C} \) and \( \zeta_s = (R_s/2)\sqrt{L/C} \) are the undamped natural frequency and the damping ratio, respectively, for the simplified impedance case.

An expression for the exact transfer function, \( H_e \), can also be expressed in terms of \( \omega_s \) by noting that \( \omega_e = \omega_s \sqrt{L_s/I_s} \) and \( \zeta_e = \zeta_s (R_e/R_s) \sqrt{I_s/I_e} \). The result is given by

\[
|H_e(i\omega)| = \frac{1}{\sqrt{(1 - \zeta_e^2)^2 + (2\zeta_e\omega_s)^2}},
\]  

(32)

In the above equation, \( I_e/I_s \) and \( R_e/R_s \) can be expressed in terms of \( \Phi \) with the use of Eqs. 29 and 30, and with Eqs. 14 and 18 it can be shown that for this particular system

\[
\eta_e^2 = \frac{o_s}{2\zeta_s}. \tag{33}
\]

The above equation relates the two frequency scales, \( \eta_e^2 \), representing just the channel, and \( \omega_s \), a characteristic frequency of the entire system. Using these relations, Eq. 32 can be written as

\[
|H_e(i\omega)| = \frac{1}{\sqrt{(1 - \omega_e^2)^2 + (2\omega_e\omega_s)^2}}. \tag{34}
\]

The transfer function amplitude for the exact case in this form can be readily compared to the simplified form given by Eq. 31 as both are expressed as functions of \( \omega_s \).

2.3.2 Pump model

The system used to model the device shown in Fig. 1 consisted of a voltage source, a piezoelectric bimorph element defined by a centerline displacement per volt, \( f_o \), and stiffness, \( K \), swept volume per centerline displacement, \( L/2 \), combined chamber and fluid compliance, \( C \), and identical inlet and outlet channels each having resistance and inertance, \( R \) and \( I \), open to atmospheric pressure at the inlet and outlet. The resulting system is similar to the simple second-order system equations described previously, but with \( C \) being replaced by \( C = C + (A/2)^3/k \). For further information on these parameters, refer to Morris and Forster (2003). The magnitude of the transfer function that relates the membrane centerline displacement to the input voltage for this case is

\[
|H_e(i\omega)| = \frac{|X_m(i\omega)|}{|E(i\omega)|} = \frac{1 - (2\zeta_s\omega_s)^2 + (2\zeta_s\omega_s)^2}{\sqrt{(1 - \omega_e^2)^2 + (2\omega_e\omega_s)^2}}, \tag{35}
\]

where \( \omega_s = \omega \sqrt{C_s^3/2}, \zeta_s = (R_s/4)\sqrt{2C_s/I_s} \), and \( \lambda = C/C_s \). The factor of 2 appears because the inlet and outlet channels act in parallel, and are modeled as a single channel having half the resistance and inertance of each individual channel. Similar to Eqs. 32 and 34, the exact transfer function amplitude can be written as

\[
|H_e(i\omega)| = \frac{1 - (2\zeta_s\omega_s)^2 + (2\zeta_s\omega_s)^2}{\sqrt{(1 - \omega_e^2)^2 + (2\omega_e\omega_s)^2}}f_o, \tag{36}
\]

or

\[
|H_e(i\omega)| = \frac{1 - (2\zeta_s\omega_s)^2 + (2\zeta_s\omega_s)^2}{\sqrt{(1 - \omega_e^2)^2 + (2\omega_e\omega_s)^2}}f_e. \tag{37}
\]

2.4 Fully developed flow considerations

To investigate the accuracy of the simplified and exact models of fluid impedance, both of which are based on fully developed flow, the experimental measurements were performed to reflect that condition by minimizing additional pressure losses from entry and exit effects. The
steady flow pressure drop in a channel is represented by White (1991)

\[ \Delta P = \left[ f \left( \frac{L}{D_h} \right) + K \right] \frac{1}{2} \rho V^2. \]  

(38)

When \( f \left( L/D_h \right) \) is large compared to the total minor loss factor, \( K \), entry and exit losses are negligible, which is true at low Re since the Darcy friction factor, \( f \), is proportional to \( Re^{-1} \) for laminar flow. An upper bound for \( Re \) for all experiments performed was determined with \( K=2.3 \). Entry losses were represented by \( K_{entry}=1.3 \) for a rectangular duct with \( x=1 \) (Shah and London 1978), and for exit losses, \( K_{exit}=1 \) (White 1994).

2.5 Experiments and simulations

Six devices like the one shown in Fig.1 were used to investigate the effects of using \( Z_e \) and \( Z_s \), and, therefore, \( H_s \) and \( H_e \), with model predictions. Each device was similar to a micropump, consisting of a 3, 6, or 10 mm-diameter chamber, but with straight, rectangular channels located on opposite sides of the chamber that replaced the valves. Features were either etched in silicon using DRIE techniques or machined in plastic using computer numerical-control (CNC) machining.

Two 3 mm and three 6 mm devices were fabricated in silicon with 120 μm-wide channels and with the valve depth equal to the chamber depth. Chamber depth was measured using an optical interference measurement system (Fotonic MTI-2000, MTI Instruments, Albany, New York). For the 3 mm-diameter devices, 250 μm-thick Pyrex covers were anodically bonded, and a 2.5 mm-diameter by 190 μm-thick piezoelectric actuator was glued onto the Pyrex using electrically-conductive epoxy. Holes were drilled through the silicon to allow inlet and outlet connections. The same fabrication procedure was used for the 6 mm-diameter devices, but with 500 μm-thick Pyrex covers and 5 mm-diameter piezoelectric actuators. For additional details see Morris and Forster (2003).

One 10 mm-diameter device was fabricated in plastic using CNC machining. Using a cyanoacrylate adhesive, a polycarbonate membrane was bonded to an acrylic pump housing containing machined pump features including 135 μm-wide channels. A piezoelectric actuator 9 mm in diameter and 127 μm-thick was bonded to the membrane as described above. Electrical connections were made to a 1.5 mm-wide “tab” on the piezoelectric driver that extended 2.5 mm beyond the pump chamber. A small pocket was cut through the membrane and into the pump housing below this tab to allow for electrical lead clearance. For additional fabrication details, see Gamboa et al. (2003). Table 4 summarizes the parameters for the six pump-like devices constructed and tested in this study.

Table 4. Parameters of the pump-like devices constructed and tested in this study

<table>
<thead>
<tr>
<th>Device ID</th>
<th>Chamber diameter (mm)</th>
<th>( \omega_n )</th>
<th>( \zeta_s )</th>
<th>( \chi )</th>
<th>( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>p33</td>
<td>10</td>
<td>2.09</td>
<td>0.183</td>
<td>0.010</td>
<td>2.78</td>
</tr>
<tr>
<td>76</td>
<td>6</td>
<td>21.88</td>
<td>0.042</td>
<td>0.275</td>
<td>0.96</td>
</tr>
<tr>
<td>77</td>
<td>6</td>
<td>21.88</td>
<td>0.042</td>
<td>0.275</td>
<td>0.96</td>
</tr>
<tr>
<td>78</td>
<td>6</td>
<td>24.08</td>
<td>0.031</td>
<td>0.318</td>
<td>1.23</td>
</tr>
<tr>
<td>79</td>
<td>3</td>
<td>93.41</td>
<td>0.008</td>
<td>0.647</td>
<td>1.27</td>
</tr>
<tr>
<td>80</td>
<td>3</td>
<td>87.26</td>
<td>0.011</td>
<td>0.574</td>
<td>0.92</td>
</tr>
</tbody>
</table>

To validate Eq. 38 and use it to determine the maximum Re that ensured fully developed channel flow, a steady flow test was performed using two of the 6 mm straight-channel devices and a syringe pump (Model 200, KD Scientific, Boston, Massachusetts). The pressure drop through each device was measured with a water column for pressures below 10 kPa, and with a pressure transducer (EPI-127, Entran Sensors & Electronics, Fairfield, New Jersey) for pressures above 10 kPa. The resulting pressure drop was divided by two to arrive at \( \Delta P \) across a single channel, assuming each channel experienced the same entry and exit effects and negligible pressure drop through the pump chamber.

To investigate differences between the simplified and exact relationships derived in this study, the transfer functions for the system models were evaluated numerically using Matlab. Finite element analysis was used to obtain mechanical parameters values (Morris and Forster 2000; 2003). Impedance parameters were calculated using Eqs. 4 and 12 for \( Z_s \), and Eq. 27 for \( Z_e \), for the viscosity and density of water at 25. Fluid capacitance was calculated based on chamber volume using the volumetric fraction 1.7151x10^3 Pa^-1. This value depended on the condition of the working fluid, which was partially degassed water, and was determined empirically (Morris and Forster 2003). This fluid capacitance was added to the housing capacitance determined by FEA to obtain a value for the combined fluid and housing capacitance. C. Frequency response data were gathered for the membrane centerline velocity for each chamber/straight-channel device using a laser vibrometer (Model OVF 302, Polytec, Waldbronn Germany), while the device was driven such that the Reynolds number was low enough to neglect additional entry/exit losses. These data were compared to the system model predictions using \( Z_s \) and \( Z_e \).

3 Results and discussion

3.1 Simplified vs exact impedance

Figure 2 is a comparison of simplified and exact impedance for a rectangular duct of aspect ratio \( a=1 \). The same information is plotted in Fig. 3, along with results for the slot and circular channel. In Fig. 3, the difference between \( Z_e \) and \( Z_s \) is more quantitatively evident; the maximum deviation for the rectangular duct is approximately 30%, and is the largest of the three geometries. This maximum deviation occurs in the region of the corner frequency \( \ell^2_c = 7.1 \) according to Eq. 14 and Table 2. Figure 4 shows similar behavior in terms of normalized resistance and inertance ratios based on Eqs. 29 and 30, again for a rectangular duct with \( a=1 \). At low frequency, \( R \) dominates and \( R_e/R_s = 1 \), at high frequency, \( I \) dominates and \( I_e/I_s \approx 1 \), but at intermediate
frequency, both \( R_s \) and \( I_s \) deviate significantly from \( R_e \) and \( I_e \).

### 3.2 Series RIC circuit

Figure 5 shows the system response for the basic second-order system, the RIC series circuit given by Eqs. 31 and 34 for a square duct. The response curves corresponding to the exact expression for impedance are shifted to a lower frequency and lower amplitude relative to the response curves based on simplified impedance. This trend is similar to the behavior expected for a second-order system with added inertia and resistance. Indeed, Fig. 4 shows that both \( I_e \) and \( R_e \) are greater than the corresponding simplified expressions for the entire frequency range considered. Also shown in Fig. 5 is the increasing discrepancy between \( |H_s| \) and \( |H_e| \) with decreasing damping ratio \( \zeta \). For \( \zeta = 0.01 \), a value not uncommon for actual microsystems, the error in peak amplitude is approximately 400%. Figure 6 shows the predicted error in both the amplitude response and the resonant frequency over a wide range of underdamped oscillation, \( \zeta < 1/\sqrt{2} \). As \( \zeta \) decreases, the peak amplitude error of \( |H_s| \) increases monotonically, while the error in resonant frequency diminishes above and below \( \zeta \approx 0.23 \), where the error is approximately 18%.

To understand the physical basis for the difference between the simplified impedance model, which assumes a parabolic velocity profile for resistance and no profile dependence for inertere, and the exact model, velocity profiles for the exact velocity field are shown in Fig. 7 for \( \eta^2 \) from 5 to 300. As \( \eta^2 \) increases, the deviation from the parabolic shape is evident, which is in agreement with the increasing deviation in resistance between the two models.
For the RIC circuit, whose response is shown in Fig. 5, the values $g_2=30$ and 300 (see Fig. 7) correspond to the resonant frequency of the exact curves for $f_s=0.1$ and 0.01 from Eq. 33. It is clear from Fig. 3 that these values of $g^2$ are in the range where the deviation between the models for impedance differ significantly. Equation 33 is specific to the RIC circuit, but regardless of the specific relation between $g^2$ and $x$ for a given system, a key finding of this study is that model predictions based on simplified impedance may produce unacceptable errors in any underdamped fluid system when $g^2$ in any flow channel is in the range $1 \leq g^2 \leq 1000$.

### 3.3 Measurements compared to pump model using exact and simplified expressions

Measurements on two test devices similar to that shown in Fig. 1 were first performed under steady flow conditions to determine the maximum Reynolds number for which entry and exit losses were negligible. Pressure drop vs volume flow rate for two of the test devices is shown in Fig. 8, along with predictions using Eq. 38. The departure from the fully developed flow assumption ($K=0$) was at roughly $Re > 50$ and was in good agreement with the predictions.

Figures 9 and 10 show the unsteady flow experimental results for the devices used in this study. For these results, the Reynolds number in the channels of each device at resonance was less than 50, based on RMS flow velocity calculated from the measured membrane centerline velocity. Typical membrane centerline velocity amplitude response is shown in Fig. 9 for one of the 6 mm-chamber diameter experimental devices and for the pump model based on simplified and exact channel impedance, Eqs. 35 and 37. The over-prediction of both amplitude and resonant frequency for the simplified impedance model is evident.

Figure 10 shows the difference in predicted peak amplitude and frequency for the pump model based on simplified channel impedance, Eq. 35, and measurements on all devices listed in Table 4 to show the effect of all three dimensionless model parameters, $a$, $\lambda$, and $\zeta_\infty$ that determine any differences between simplified and exact impedance for the pump model. Also shown is the difference between the pump model results for the simplified and exact cases, as shown for the RIC model in Fig. 6. The agreement between symbols and lines demonstrates good predictive value of the pump model based on exact impedance, and serious inaccuracies using simplified

---

**Fig. 6.** Error in peak amplitude and resonant frequency of the simplified transfer function, $|H_s|$, from Eq. 31 relative to the exact transfer function, $|H_e|$, from Eq. 34 for a rectangular duct of aspect ratio $a=1$. The error in amplitude reaches $-8.8\%$ at $\zeta_s=0.48$ and increases monotonically with decreasing $\zeta_s$. The error in frequency increases to $18\%$ at $\zeta_s=0.23$ and then reaches $-100\%$ at $\zeta_s=1/\sqrt{2}$. Results for other values of $a$ are of similar magnitude.

**Fig. 7.** Velocity profiles for a square duct based on Eq. 24, at a typical phase angle of $\pi/4$ relative to the pressure and for various values of $\eta^2$. For the series RIC system, $\eta^2=5$ corresponds to $\zeta_s = 1/\sqrt{2}$ at $\omega_0 a_0 = 1$. Values of $\eta^2=30$ and 300 correspond to the resonant frequency of $|H_e|$ and the values of $\zeta_s=0.1$ and 0.01 shown in Fig. 5.

**Fig. 8.** Nondimensional pressure drop vs steady flow Reynolds number, $Re$, for two devices like the one shown in Fig. 1, with etch depths of 114 $\mu$m and 115 $\mu$m, plotted on a log scale. Predictions were based on Eq. 38. The figure shows that, in a quasi-steady sense, additional resistance due to entry/exit losses may be neglected at low $Re$. 

---
channel impedance. The results shown are similar to those in Fig. 6 for the RIC system; the error in velocity amplitude based on simplified impedance, $Z_s$, increased with decreasing $\zeta_s$. For example, the device with the smallest value of $\zeta_s$, device 79, had a difference of 245% between the simplified model and experiment for peak amplitude, whilst with respect to the exact model, it was only 20%. The larger differences between measurement and exact pump model corresponded to the smallest size pump (3 mm-diameter chamber), which we attribute to the difficulty of manual fabrication of smaller scale devices (Morris and Forster 2003). However, the overall agreement between experiment and the pump model utilizing exact impedance was far better than that for the simplified case.

Even though the devices tested spanned over an order of magnitude range for $\zeta_s$ and $\lambda$, and a factor of three for $x$, Fig. 10 shows that variations in these parameters do not account for the errors associated with the simplified impedance model. Firstly, Fig. 10 shows that changes in $\zeta_s$ cannot simultaneously reduce errors in resonant frequency and peak amplitude. Secondly, for the nondimensional parameter, $\lambda$, which has a physically possible range of $0<\lambda<1$, Fig. 10 shows no effect on error for the range $0.01<\lambda<0.65$. Thirdly, even a variation in aspect ratio, $x_s$, well beyond the range of the tested devices is not expected to affect the error since Fig. 3 shows that the difference in $Z_s$ relative to $Z_e$ is significant, even for the slot. One additional parameter that has a significant effect on the predictions of the pump model is fluid capacitance. Except for fluid viscosity and density, fluid capacitance was the only parameter that was not calculated from first principles. The relationship between $C$ to the chamber volume was determined empirically by adjusting $C$ in an independent set of experiments using a number of micropumps until the model using exact impedance agreed with experimental data (Morris and Forster 2003). The resulting value for capacitance varied linearly with chamber volume, as would be expected if due to dissolved gas. This value, reported in Sect. 2, was used in this study. It can be argued that the same procedure could have been used with the simplified relationships based on $Z_s$ to achieve better agreement. However, changing $C$ primarily affects the damping ratio $\zeta_s$, and it can be seen from Fig. 10 that no value of $\zeta_s$ yielded acceptable errors for both peak amplitude and frequency. Any attempts to match the simplified model and experimental data would require a compromise between these errors.

4 Conclusions

Proper handling of resistance and inductance is essential to accurate low-order modeling of unsteady behavior of fluidic devices. While differences between simplified and exact expressions for channel impedance have been long recognized, this study demonstrated with a microfluidic device that significantly larger errors in system response caused by these inaccuracies can occur, even for simple fluidic systems. We conclude that any underdamped fluidic system for which the dimensionless frequency parameter lies in the range $1<\eta^2<1000$ for any contained fluidic channel should be modeled with exact expressions for impedance, as described in this study, otherwise potentially large errors in the predicted behavior can result.

5 Harmonic pressure-velocity field for a rectangular duct

The equations for oscillating flow in the rectangular duct were derived from the one-dimensional Navier-Stokes equation for fully developed flow,

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

(39)

for the quadrant, $0<x<a$, $0<y<b$, subject to the zero velocity boundary condition on $x=a$, $y=b$, and the sym-
metry conditions, \( \partial \tilde{w}/\partial x|_{x=0} = 0 \) and \( \partial \tilde{w}/\partial y|_{y=0} = 0 \). Complex spatial solutions, \( \tilde{w}(x, y) \), were sought for a complex pressure gradient, \( K e^{(i \omega t)} \), where

\[
K = -\frac{1}{\rho} \frac{\partial p}{\partial z}.
\]

(40)

The function

\[
w(x, y, t) = \Re \left[ \tilde{w}(x, y)e^{(i \omega t)} \right]
\]

represents the actual velocity for the actual pressure gradient

\[
-\frac{1}{\rho} \frac{\partial p}{\partial z} \cos (\omega t).
\]

(42)

It follows that \( \tilde{w} \) satisfies the equation

\[
\frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{\partial^2 \tilde{w}}{\partial y^2} - \frac{i \omega}{\nu} \tilde{w} = -\frac{K}{\nu}.
\]

(43)

By defining

\[
\hat{g}(x, y) = \tilde{w}(x, y) - \frac{K}{i \omega},
\]

the problem reduces to finding the solution to the homogeneous equation

\[
\frac{\partial^2 \hat{g}}{\partial x^2} + \frac{\partial^2 \hat{g}}{\partial y^2} - \frac{i \omega}{\nu} \hat{g} = 0
\]

(45)

subject to symmetry and \( \hat{g} = -K/i \omega \) on \( x=a, y=b \). A final decomposition into \( \hat{g} = \hat{g}_1 + \hat{g}_2 \), where \( \hat{g}_1|_{y=b} = -K/i \omega \) and \( \hat{g}_2|_{x=a} = -K/i \omega \) are the only nonhomogeneous boundary conditions yields explicit solutions with the use of separation of variables. These are

\[
\hat{g}_1 = -\frac{2K}{i \omega} \sum_{n=0,1,2} \left( (-1)^n \cosh(r_n(y/b)) \right) \cos(p_n(x/a))
\]

(46)

and

\[
\hat{g}_2 = -\frac{2K}{i \omega} \sum_{n=0,1,2} \left( (-1)^n \cosh(s_n(x/b)) \right) \cos(p_n(y/b)).
\]

(47)

The above equations form the basis for the velocity given by Eq. 24.

At least three known solutions to this problem exist: Drake (1965), Fan and Chao (1965), and O’Brien (1975). Our expression is equivalent to that by O’Brien (1975, Eq. 11) after corrections \( W_0 = P^2/2 \) should read \( W_0 = P^2 \), Eq. 7 should read \( \nabla^2 \tilde{W} = \nabla^2 \tilde{W} \) and Eq. 8 should read \( W_0 = -1/(2(i \omega)^2) \). Calculated velocity fields based on the above expressions and Drake (1965), Fan and Chao (1965), and O’Brien (1975) were found to be equivalent, based on numerical evaluation over a wide range of \( \eta^2 \) using Matlab (v6.5, The Mathworks Inc, Natick, MA), although one expression was less computationally efficient due to a double summation (Fan and Chao 1965).

References


Fan C, Chao BT (1965) Unsteady, laminar, incompressible flow through rectangular ducts. ZAMP 16:351–60


