Incompressible Pipe Flow Equations

Fully-developed Laminar or Turbulent Flow:
From the momentum equation for fully-developed pipe flow, the wall shear stress $\tau_0 = (\Delta p + \gamma \Delta h)(D/4L)$ where Δp is a pressure *drop*, and Δh is a height *drop* in the flow direction. The wall shear stress is usually expressed by its non-directional equivalent f defined as $\tau_0 / \frac{1}{8}\rho \bar{V}^2$, where $\bar{V}$ is the average velocity across the pipe. f is called the Darcy friction factor.

The form of the energy equation to solve a single-input/single-output pipe flow problem is given by

$$H_P + \alpha_1 \frac{\bar{V}_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = H_T + \alpha_2 \frac{\bar{V}_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L,$$

where all terms are explained below.

The head loss h_L is composed in general of frictional and minor losses $h_L = h_f + h_m$

$$h_f = f\left(\frac{L}{D}\right)\frac{\bar{V}^2}{2g} = f\left(\frac{L}{D}\right)\frac{8Q^2}{\pi^2 D^4 g} = f\left(\frac{L}{D}\right)\frac{8\dot{m}^2}{\pi^2 D^4 \rho^2 g} = f\left(\frac{L}{D}\right)\frac{\nu^2 \mathrm{Re}_D^2}{2D^2 g},$$

where the exact form of f is given later for both laminar and turbulent flow cases. The alternate formulas involving Q, $\dot{m}$ and Re are for circular pipes. Each minor loss h_m is expressed in terms of a loss coefficient K or an effective length of pipe L_{eff}

$$h_m = K\frac{\bar{V}^2}{2g} = f\left(\frac{L_{\mathrm{eff}}}{D}\right)\frac{\bar{V}^2}{2g}.$$

Note as above for h_f the velocity term can be replaced by a term involving Q, $\dot{m}$ or Re_D. Various useful forms of the Reynolds number are

$$\mathrm{Re}_D = \frac{\bar{V}D}{\nu} = \frac{4Q}{\pi D \nu} = \frac{4\dot{m}}{\pi D \mu}.$$

H_P is the pump head delivered to the fluid, H_T is the turbine head extracted from the fluid. The actual work rate put into and obtained from the pump and turbine are given by

$$\dot{W}_P = \frac{\gamma H_P Q}{\eta_P} \quad \dot{W}_T = \gamma H_T Q \eta_T \quad 0 < \eta_P, \eta_T < 1,$$

where the η's are efficiency factors.

The kinetic energy correction factor α for circular pipes is given by

$$\alpha = \begin{cases} 1, & \text{uniform flow} \\ 2, & \text{laminar parabolic flow: } V(r) = V_0(1 - r^2/R^2) \\ \frac{(m+1)^3(m+2)^3}{4(3m+1)(3m+2)}, & \text{turbulence: } V(r) = V_0(1-r/R)^m;\ m = \frac{1}{5}\dots\frac{1}{9};\ \alpha = 1.106, 1.077, 1.058, 1.046, 1.037 \end{cases}$$

The centerline velocity is related to the average velocity by

$$\frac{\bar{V}}{V_0} = \begin{cases} 1, & \text{uniform flow} \\ \frac{1}{2}, & \text{laminar parabolic flow} \\ \frac{2}{(1+m)(2+m)}, & \text{turbulence: } m = \frac{1}{5}\dots\frac{1}{9};\ \bar{V}/V_0 = 0.758, 0.791, 0.817, 0.837, 0.853 \end{cases}$$

Lastly, the entry length L_e required to attain fully-developed flow is given by

$$\frac{L_e}{D} = \begin{cases} 0.06\mathrm{Re}, & \text{laminar flow;} \\ 4.4\mathrm{Re}^{1/6}, & \text{turbulent flow.} \end{cases}$$

Fully-developed Laminar Flow:
$\mathrm{Re}_D < 2300$, wall roughness not a factor, $f = 64/\mathrm{Re}_D$, and thus

$$h_f = \frac{32\nu \bar{V} L}{g D^2} = \frac{128 \nu L Q}{g \pi D^4} = \frac{128 \nu L \dot{m}}{\rho g \pi D^4} = \frac{32 \nu^2 L \mathrm{Re}_D}{g D^3}.$$

Fully-developed Turbulent Flow:
$\mathrm{Re}_D > 2300$, and $f = f(\mathrm{Re}_D, \epsilon/D)$, where ϵ is the wall roughness, is determined by the Moody diagram or equivalently by the *implicit* Colebrook formula

$$\frac{1}{\sqrt{f}} = -2.0 \log \left[\frac{2.51}{\mathrm{Re}_D \sqrt{f}} + \frac{\epsilon/D}{3.7} \right].$$

Another reasonably accurate representation is the *explicit* Haaland formula

$$\frac{1}{\sqrt{f}} = -1.81 \log \left[\frac{6.9}{\mathrm{Re}_D} + \left(\frac{\epsilon/D}{3.7} \right)^{1.11} \right] \Longrightarrow f = \left\{ 1.81 \log \left[\frac{6.9}{\mathrm{Re}_D} + \left(\frac{\epsilon/D}{3.7} \right)^{1.11} \right] \right\}^{-2}.$$

There are basically three types of problems to solve given a particular fluid that use the Moody diagram along with the expression above for h_f (i.e. the energy equation):

1. Find h_f knowing Q and the pipe geometry. Calculate Re_D, from Moody diagram calculate f. Then calculate h_f from the energy equation.

2. Find Q knowing h_f and the pipe geometry. Guess a value of f, use the energy equation to calculate Re_D, and use the Moody diagram to get the next value of f... After convergence, calculate Q from Re_D.

3. Find D knowing h_f and Q. Guess f, use the energy equation to calculate a value of D, and subsequently Re_D. Use the Moody diagram with known value of e to determine the next value of f...

Starting the iteration methods with a guess of f is nice, because the range of f is fairly small. Choose an initial value that is at least as large as the fully-rough value for the pipe under consideration. A value of 0.02 should always work for a smooth pipe. Alternatively, you could start by guessing a value for the quantity desired.

As an alternative to iteration, one can use the explicit equations by Swamee and Jain for h_f, Q and D, given as Eqs. 7.6.31–33 in the textbook *Mechanics of Fluids* by Potter and Wiggert:

$$h_f = 1.07 \frac{Q^2 L}{g D^5} \left\{ \ln \left[\frac{\epsilon}{3.7D} + 4.62 \left(\frac{\nu D}{Q} \right)^{0.9} \right] \right\}^{-2} \qquad \begin{array}{c} 10^{-6} < \epsilon/D < 10^{-2} \\ 3000 < \mathrm{Re} < 3 \times 10^8 \end{array}$$

$$Q = -0.965 \left(\frac{g D^5 h_f}{L} \right)^{0.5} \ln \left[\frac{\epsilon}{3.7D} + \left(\frac{3.17 \nu^2 L}{g D^3 h_f} \right)^{0.5} \right] \qquad \mathrm{Re} > 2300$$

$$D = 0.66 \left[\epsilon^{1.25} \left(\frac{L Q^2}{g h_f} \right)^{4.75} + \nu Q^{9.4} \left(\frac{L}{g h_f} \right)^{5.2} \right]^{0.04} \qquad \begin{array}{c} 10^{-6} < \epsilon/D < 10^{-2} \\ 5000 < \mathrm{Re} < 3 \times 10^8 \end{array}$$

Be careful in using the above equations that you stay within the ranges of validity shown to the right of each equation.

Lastly let's summarize the fluid properties of water and air, so we will have just about everything we need in one place. Note it is my preference to work in SI units, so any other variables in a problem would be changed into SI units before working the problem with the following properties:

$$\rho_w = 10^3 \,\mathrm{kg/m^3} \quad \mu_w = 10^{-3} \,\mathrm{kg/ms} \quad \nu_w = 10^{-6} \,\mathrm{m^2/s}$$

$$\rho_a = 1.2 \,\mathrm{kg/m^3} \quad \mu_a = 1.8 \times 10^{-5} \,\mathrm{kg/ms} \quad \nu_a = 1.5 \times 10^{-5} \,\mathrm{m^2/s}$$

If you do choose to work in British gravitation units the conversion of the values given above are:

$$\rho_w = 1.94 \,\mathrm{slug/ft^3} \quad \mu_w = 2.09 \times 10^{-5} \,\mathrm{slug/ft\,s} \quad \nu_w = 1.08 \times 10^{-5} \,\mathrm{ft^2/s}$$

$$\rho_a = 2.33 \,\mathrm{slug/ft^3} \quad \mu_a = 3.76 \times 10^{-7} \,\mathrm{slug/ft\,s} \quad \nu_a = 1.61 \times 10^{-4} \,\mathrm{ft^2/s}$$