An Interesting Example

Math& 148

Consider the function

$$f(x) = e^{-\frac{1}{x^2}}$$

(a variation is $e^{-\frac{1}{|x|}}$). The domain is given by $\mathbb{R} \setminus \{0\}$, but we can extend the function by continuity, since, clearly

$$\lim_{x \to 0} f(x) = 0$$

to

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The interesting feature about this function is that *all* derivatives are equal to 0 at $x = 0$, but the function is not constant at all.

Remark The fact that this is "unexpected" may not be obvious right now. Fact is that the following theorem holds (you will probably see it in your third - or higher - calculus class)

Suppose f has n continuous derivatives at $x = x_0$. Then (*Taylor's Theorem*)

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}}{k!}(x_0)(x - x_0)^k + R(x, x_0)$$

where

$$\lim_{x \to x_0} \frac{R(x, x_0)}{(x - x_0)^n} = 0$$

This suggests a tantalizing possibility: maybe some/most/nice functions (more precisely, functions with derivatives of all orders) can be thought of as "sum of an infinite polynomial", $\sum_{k=0}^{\infty} a_k (x - x_0)^k$. It turns out that only "very nice" functions can be represented as that (these are now known as *real analytic* functions), but that "most" (that is something that can be made more precise, but is clearly fuzzy right now) functions with infinitely many derivatives cannot be represented as sums of an infinite sum (which is now known as a *series*). Of course, if our function fell in this category, it should be constantly equal to 0, by choosing $x_0 = 0$, but that's obviously not the case.

Proof

This is a very interesting example, especially in view of future issues concerning the possibility of expressing complicated functions as "polynomials of infinite degree" (or, more scientifically, as "power series"). Aside from future implications, it is a very interesting example of a function having all derivatives, all derivatives being equal to 0 at a given point, but not being constant. To show that, indeed, all derivatives are 0 at x =0, we start by computing the first derivative (since the function is piecewise defined, it would not be precise to compute the derivative of $e^{-\frac{1}{x^2}}$ for $x \neq 0$, and then see how it behaves as $x \to 0$ – or, better said, to show that this would provide the correct answer too we would need a theorem that we have not stated – however, going by the book ends up with the same result as we go on to show). The ratio for $x = 0$ is

$$\frac{e^{-\frac{1}{x^2}}}{x}$$

Let's rewrite the problem for better visibility. Set $t = \frac{1}{x^2}$, so that we are now checking on

$$\lim_{t\to\infty} t^{\frac{1}{2}} e^{-t}$$

Writing this out as

$$\lim_{t\to\infty} \frac{t^{\frac{1}{2}}}{e^t}$$

and applying L'Hospital's Rule, we see that this limit is equal to

$$\lim_{t\to\infty} \frac{1}{2t^{\frac{1}{2}} e^t} = 0$$

since both terms in the denominator are now going to infinity. Now, the first derivative will be

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Computing $f''(0)$, we need to evaluate

$$\lim_{x\to 0} \frac{\frac{2}{x^3} e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{2e^{-\frac{1}{x^2}}}{x^4}$$

Using, for simplicity, the substitution$t = \frac{1}{x^2}$, this translates into

$$\lim_{t\to\infty} t^2 e^{-t} = \lim_{t\to\infty} \frac{t^2}{e^t}$$

and applying L'Hospital's Rule twice shows that this limit is also 0. You can see that a pattern is developing: as we compute higher and higher derivatives, we end up evaluating limits like

$$\lim_{t\to\infty} \frac{t^\alpha}{e^t}$$

for higher and higher $\alpha > 0$. By applying L'Hospital's Rule enough times, we end up with a limit that is obviously equal to 0. Hence, the observation that for any n, $f^{(n)}(0) = 0$