Trigonometric Identities

Part I. Connecting Angles

Using the unit circle representation, it is not too hard to verify a number of relations between trig functions of different angles. Most of these relations can also be proved by using the addition/subtraction formulas that we will meet very soon.

Of course, we also need to recall that

1. $\sin(-x) = \sin x$
2. $\cos(-x) = \cos x$
3. $\tan(-x) = -\tan x$

We also noted that

1. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
2. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
3. $\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan x}$
4. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$
5. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

A look at the unit circle (or reading backwards numbers 4, and 5) will convince us also that

1. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$
2. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$
   and also that
3. $\sin(\pi - x) = \sin x$
4. $\cos(\pi - x) = -\cos x$
   hence
5. $\sin(x - \pi) = -\sin x$

6. $\cos(x - \pi) = -\cos x$
   We can also notice that

7. $\sin(\pi + x) = -\sin x$

8. $\cos(\pi + x) = -\cos x$

There is no point in memorizing relations like these: a look at the circle will tell you what they are. Also, the upcoming addition/subtraction formulas (that *are* useful to remember) provide these (and more) for free

Part II. Connecting Different Trig Functions

1 Basic Identities

All trigonometric identities are manipulations of the basic ones:

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan x = \frac{\sin x}{\cos x}$

Note that given the value of $\sin x$ or of $\cos x$ x is not determined, even if we restrict $0 \leq x < 2\pi$. We need the *sign* of the other to determine the quadrant where x lies. Note that the sign of $\tan x$ is also good enough, as it tells us whether the sine and the cosine function have the same or opposite sign.

Even if not really necessary, one can also refer to the definitions

1. $\sec x = \frac{1}{\cos x}$
2. $\csc x = \frac{1}{\sin x}$
3. $\cot x = \frac{1}{\tan x}$

Probably the most commonly used of these last three auxiliary functions is the third, but it is also clear how they are somewhat redundant.

2 Derived Identities

While we could do with the first two identities alone, a number of easy consequences are helpful to shortcut the work with complicated trigonometric expressions.

For example:

1. $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1 \Rightarrow 1 + \tan^2 x = \frac{1}{\cos^2 x}, \cos^2 x = \frac{1}{1+\tan^2 x}$

2. $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\sin^2 x}{1-\sin^2 x} = \frac{1}{\frac{1}{\sin^2 x}-1} \Rightarrow \frac{1}{\tan^2 x} = \frac{1}{\sin^2 x} - 1 \Rightarrow 1 + \frac{1}{\tan^2 x} = \frac{\tan^2 x+1}{\tan^2 x} = \frac{1}{\sin^2 x}, \sin^2 x = \frac{\tan^2 x}{1+\tan^2 x}$

As a consequence, we also have that (the choice of sign depends on the quadrant where x lies) $\cos x = \pm\frac{1}{\sqrt{1+\tan^2 x}}, \sin x = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}}$.

While the previous two identities are very useful, you can create as many as you wish. For example, pretty much at random,

1. $1 - \cos x = \frac{1-\cos^2 x}{1+\cos x} = \frac{\sin^2 x}{1+\cos x} \Rightarrow \frac{1+\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} + \frac{\cot x}{\sin x} = \frac{1+\sin x \cdot \cot x}{\sin^2 x} = \frac{1}{1-\cos x}$

2. $1 + \cos x = \frac{1-\cos^2 x}{1-\cos} x = \frac{\sin^2 x}{1-\cos x}$, etc., following the previous example

3. $1 - \sin x = \frac{1-\sin^2 x}{1+\cos x} = \frac{\cos^2 x}{1+\cos x} \Rightarrow \frac{1}{1-\sin x} = \frac{1+\cos x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{1}{\cos x}$

4. $1 + \cos^2 x = 1 + 1 - \sin^2 x = 2 - \sin^2 x = \sin^2 x + 2\cos^2 x$

5. $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

6. $\cos x + \sin x = \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \frac{1-2\sin^2 x}{\cos x - \sin x}$

7. $\cos x - \sin x = \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}$, etc.

Of these, only #5 is of some real interest, but you never know when some unusual identity might turn useful.

3 Involving Inverse Functions

3.1 Combining a trigonometric function with an inverse trigonometric

The handling of inverse and trig functions together can be really useful. Bear in mind that $y = \sin^{-1} x$, means that $x = \sin y$. Hence, for example,

1. $\sin\left(\tan^{-1} x\right)$, setting $y = \tan^{-1} x$, is equal to $\sin y = \frac{\tan y}{\sqrt{1+\tan^2 y}}$ (we choose the "+" sign, since $\tan^{-1} x$ is between $-\frac{\pi}{2}$, and $\frac{\pi}{2}$, and hence has the same sign as $\sin y$). Since $\tan\left(\tan^{-1} x\right) = x$,
$$\sin\left(\tan^{-1} x\right) = \frac{x}{\sqrt{1+x^2}}$$
2. $\sin\left(\cos^{-1} x\right)$, setting $y = \cos^{-1} x$, is $\sin y = \sqrt{1-\cos^2 y}$, where y lies between 0 and π, so that $\sin y > 0$. Consequently,
$$\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}$$
3. $\cos\left(\sin^{-1} x\right)$, setting $y = \sin^{-1} x$, $\cos y = \sqrt{1-\sin^2 y}$, where y lies between $-\frac{\pi}{2}$, and $\frac{\pi}{2}$, so that $\cos y > 0$. Consequently
$$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$$
as well.
4. $\tan\left(\sin^{-1} x\right)$, $y = \sin^{-1} x$, $\tan(y) = \frac{\sin y}{\sqrt{1-\sin^2 y}}$ (tangent and sign have the same sign between $-\frac{\pi}{2}$, and $\frac{\pi}{2}$), and
$$\tan\left(\sin^{-1} x\right) = \frac{x}{\sqrt{1-x^2}}$$

Formulas like these (you can create as many as you wish) turn out to be very useful in extricating complicated formulas that come up in calculus.

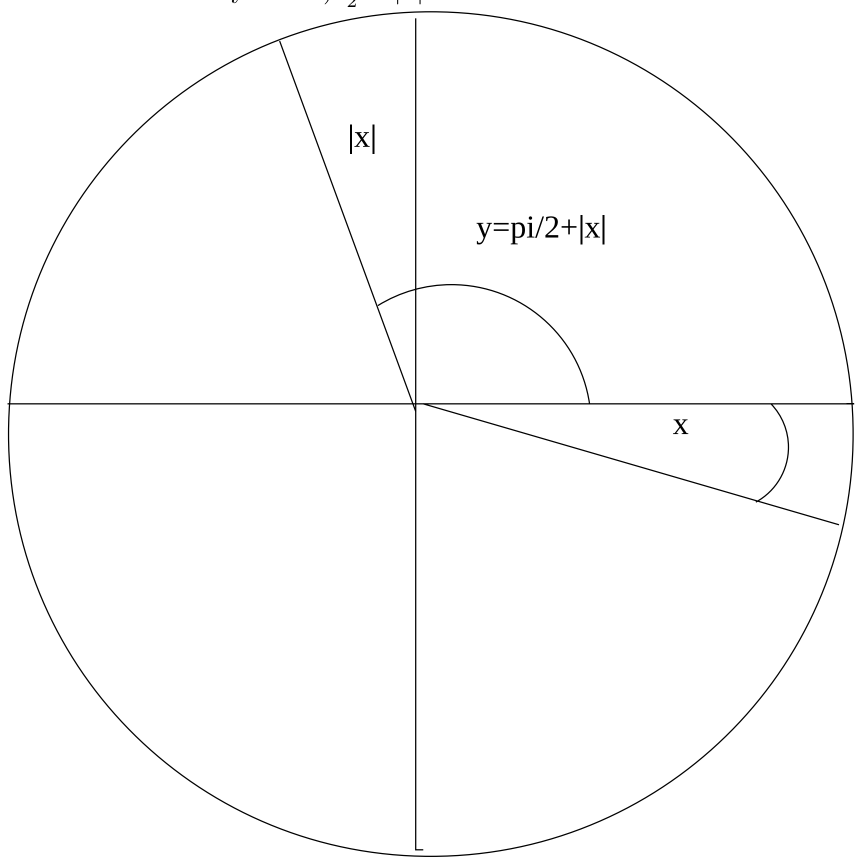
3.2 The inverse combination

Care must be exercised when the inverse function is the "outer" one. That's because the range of $\sin^{-1}$, and $\cos^{-1}$ is very specific (at least as we agree to consider the *principal determinations*, but, in any case, we have to choose *one*, excluding all the others). Thus, it is, in general, *not true that* $\sin^{-1}(\sin(x)) = x$. That's true only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Similarly for $\cos^{-1}(\cos(x))$ (it's equal to x, only for $0 \le x \le \pi$), and $\tan^{-1}(\tan(x))$ (equal to x only if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$).

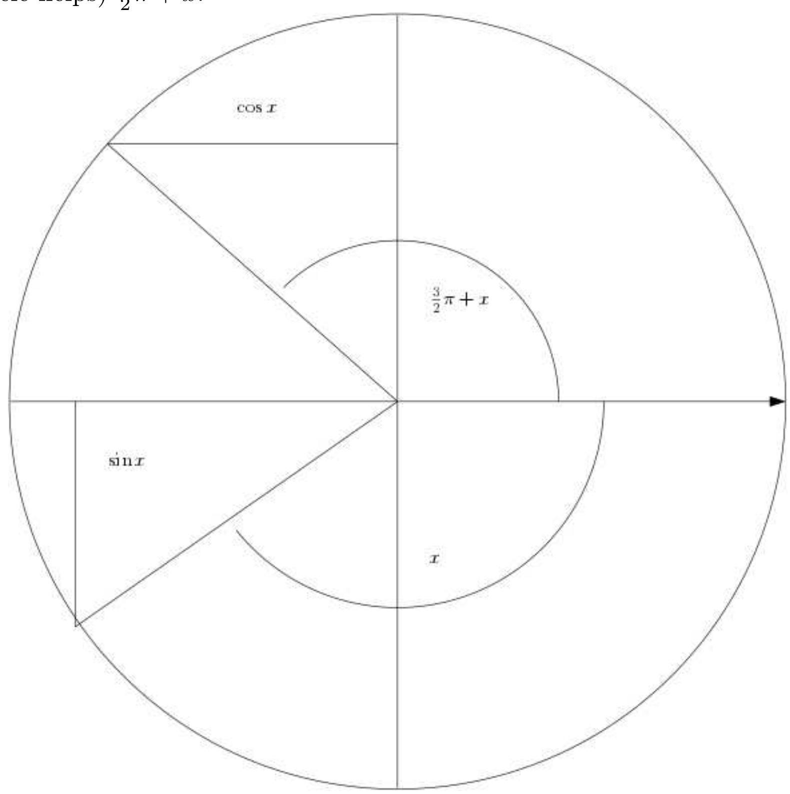
That makes other combinations likewise delicate to handle. For example, consider $\cos^{-1}(\sin(x))$. If we call $y = \sin(x)$, we are looking at $\cos^{-1}(y)$, that is the angle between 0 and π, whose cosine is y. Consider a few cases.

First, suppose $y > 0$ and $0 \leq x \leq \pi$. Then the angle between 0 and π whose cosine is equal to $y = \sin(x)$ is $\frac{\pi}{2} - x$ if $x < \frac{\pi}{2}$, but is $x - \frac{\pi}{2}$, if $x > \frac{\pi}{2}$ (compare the identities in Part I).

If $y < 0$, and $-\pi \leq x \leq 0$, we must again distinguish whether $-\frac{\pi}{2} \leq x \leq 0$, or not. If that's the case then the angle with cosine equal to y is (consider the unit circle to convince yourself) $\frac{\pi}{2} + |x|$.



On the other hand, if $-\pi \leq x \leq -\frac{\pi}{2}$, then the result will be (again, the unit circle helps) $\frac{3}{2}\pi + x$.



We could go on, but you realize that there is caution to be exerted in solving these problems. You also realize that it would not be very sensible to try to list all possible situations! The reasonable approach is to look at the specific situation, and use basic identities and pictures to get the right grip on the problem.