Graphing Rational Functions

1 Introduction

A rational function being the quotient of two polynomials, we can apply much of what we know about polynomials to the study of rational functions. Some things do get more complicated - for example, the search for maxima and minima is quite a bit more involved. On the other hand, the search for zeros is mostly the same as the search for zeros of the numerator only (hence, just as easy or difficult as for a polynomial), and the search for asymptotes is also similar, with both giving indications on the behavior of the function.

The only thing one might want to examine closer is the relation between the function and its horizontal or oblique asymptote.

2 Oblique and Horizontal Asymptotes

Suppose a rational function has a horizontal asymptote, say $y = a$. This means that numerator and denominator have the same degree, say n, and the ratio of their highest order coefficients is a. If we look for possible intersection of the graph with this asymptote, we will have to solve an equation like (suppose the numerator is $ax^n + bx^{n-1} + \cdots$, and the denominator $x^n + cx^{n-1} + \cdots$, where the dots refer to lower order terms)

$$\frac{ax^n + bx^{n-1} + \cdots}{x^n + cx^{n-1} + \cdots} = a$$

Clearing the denominator results in

$$ax^n + bx^{n-1} + \cdots = ax^n + acx^{n-1} + \cdots$$

and we see that this is an equation of degree $n - 1$.

The same reduction occurs when dealing with an oblique asymptote. Suppose the numerator is of the form $ax^n + bx^{n-1} + \cdots$, and the denominator of the form $x^{n-1} + cx^{n-2} + \cdots$. Now the intersections would be solutions of

$$\frac{ax^n + bx^{n-1} + \cdots}{x^{n-1} + cx^{n-1} + dx^{n-2} \cdots} = ax + e$$

where the right hand side is the quotient of the two polynomials (the two sides get closer and closer as $|x|$ becomes larger and larger, but the curves may intersect before that).

$$ax^n + bx^{n-1} + \cdots = ax^n + (ec + ad)\, x^{n-1} + \cdots$$

In any case, unless we are dealing with quadratic functions or factored polynomials, we are faced with high degree algebraic equations. However, when these equations are tractable, we can say a lot of things... As we will show in an example (but it is easy to consider the general case), the resulting equation will be of degree $n-2$, not $n-1$.

3 Example

As a simple example, consider the function

$$f(x) = \frac{2\,(x+3)\,(x+1)\,(x-2)\,(x-5)}{(x+2)\,x\,(x-3)}$$

(expanding the products, this is equal to

$$\frac{2x^4 - 6x^3 - 30x^2 + 38x + 60}{x^3 - x^2 - 6x}$$

We will have an oblique asymptote with slope 2 – to find it, we need to find the quotient of the two polynomials, sing one of the tools discussed in the file "Division of Polynomials":

$$\frac{2x^4 - 6x^3 - 30x^2 + 38x + 60}{x^3 - x^2 - 6x} = 2x - 4 + \frac{-22x^2 + 14x + 60}{x^3 - x^2 - 6x} \tag{1}$$

so that the oblique asymptote is $y = 2x - 4$)

Its zeros are

$$x = -3, x = -1, x = 2, x = 5$$

They are all simple, hence the function changes sign at each. Its vertical asymptotes are

$$x = -2, x = 0, x = 3$$

and these are all simple roots for the denominator - the function changes sign around each of these as well. Since the numerator is of degree 4, with positive coefficient for the highest term, for $|x|$ large it is positive. The denominator is of degree 3, with positive coefficient for the highest term, and hence for negative x, $|x|$ large, it will be negative, and for positive x, $|x|$ large, it will be positive. We have determined that the function changes sign at $x = -3, -2, -1, 0, 2, 3, 5$, so that we will have

$$f(x) \begin{cases} < 0 & x < -3 \\ = 0 & x = -3 \\ > 0 & -3 < x < -2 \\ < 0 & -2 < x < -1 \\ = 0 & x = -1 \\ > 0 & -1 < x < 0 \\ < 0 & 0 < x < 2 \\ = 0 & x = 2 \\ > 0 & 2 < x < 3 \\ < 0 & 3 < x < 5 \\ = 0 & x = 5 \\ > 0 & x > 5 \end{cases}$$

Now we study the oblique asymptote $y = 2x - 4$ in relation to f. To find intersections we solve

$$f(x) = 2x - 4$$

$$\frac{2(x+3)(x+1)(x-2)(x-5)}{(x+2)x(x-3)} = 2x - 4$$

$$2(x+3)(x+1)(x-2)(x-5) = \left(2x^2 - 4x\right)(x+2)(x-3)$$

We notice that the lead terms cancel (as they always do), and end up with the problem of solving

$$2x^4 - 6x^3 - 30x^2 + 38x + 60 = 2x^4 - 6x^3 - 8x^2 + 24x$$

$$22x^2 - 14x - 60 = 0$$

(of course, this is just the numerator in the remainder term in (1) – as it should, if you think about it).

We solve the equation as usual

$$11x^2 - 7x - 15 = 0$$

$$x = \frac{7 \pm \sqrt{49 + 44 \cdot 15}}{22} = \frac{7 \pm \sqrt{709}}{22}$$

The two solutions evaluate to, approximately, -0.89214 and 1.5285. Here is a graph, emphasizing all asymptotes: