Slope of Tangents to Rational Functions

More Calculus without Calculus

1 Polynomials

Let's quickly review how we can find the slope of the tangent to the graph of a polynomial at, say, $x = a$. We start by noting that, for very small $|x|$, $x^2 \ll |x|$ (and, of course, $\left|x^3\right| \ll x^2$, and so on). This allows us to think that in a polynomial of any degree (say, $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, when we are looking at the $y-$intercept, and the graph right there, it will have to look very much like that of the straight line $a_1 x + a_0$, which, reasonably enough, should b the tangent line to the graph.

To find the tangent at any other point, we just shift the point so that it becomes the $y-$intercept, and apply the preceding argument. Since we only performed a horizontal shift, the shape of the curve did not change, and the slope of the tangent to the new $y-$intercept will be the same as the slope of the tangent to our original point.

As an example, the slope of the tangent to

$$3x^3 - 4x + 1$$

at $x = 3$, requires us to shift the graph by 3 *to the left*. This is done by substituting $x + 3$ to x everywhere:

$$3(x+3)^3 - 4(x+3) + 1 = 3x^3 + 9x^2 + 27x + 81 - 4x - 12 + 1 = 3x^3 + 9x^2 + 23x + 69$$

and the slope of the tangent will be 23.

2 Reciprocals of Polynomials

To get the same information on Rational Functions, let's start with the reciprocal of a polynomial, say

$$\frac{1}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}$$

If $|x|$ is very small, it is reasonable to think that we may neglect everything but the lowest terms:

$$\approx \frac{1}{a_1 x + a_0}$$

Now, if this has to have a tangent, it should very much like a *linear* function - but what linear function?

2.1 The Mathematical Argument

If $\frac{1}{a_1x+a_0}$ was a (linear) Polynomial, which it surely isn't, it would be a function $p(x) = ux + v$, such that $p(x)(a_1x + a_0) = 1$. Well, there is no such function, but then we are systematically treating higher powers as if they didn't exist, so that, for example, $1 + cx^2$ on the right hand side would be just as good. Now, that's easy to arrange (thanks, special products!). In fact we can observe that

$$a_1x + a_0 = a_0\left(1 + \frac{a_1}{a_0}x\right)$$

and

$$a_0\left(1 + \frac{a_1}{a_0}x\right) \cdot \frac{1}{a_0}\left(1 - \frac{a_1}{a_0}x\right) = 1 - \frac{a_1^2}{a_0^2}x^2$$

which fits our bill! Hence, we should be able to say that

$$\frac{1}{a_1x + a_0} \approx \frac{1}{a_0}\left(1 - \frac{a_1}{a_0}x\right) = \frac{1}{a_0} - \frac{a_1}{a_0^2}x$$

2.2 The Sale Discount Argument

This is a delightful argument, due to an insightful instructor at Shoreline CC. Rather than a formal proof, it shows that you may have known about the approximation above all along...

In a nutshell, the argument goes like this: suppose you buy an item on sale, offered with a discount of r. If you paid p, what was the original price?

The correct argument is: let's call the original price x. Then $p = x \cdot (1 - r)$, or

$$x = \frac{p}{1 - r}$$

The wrong argument that is very commonly argued, is that the original price was $p(1 + r)$. This is incorrect, since $1 + r \neq \frac{1}{1-r}$. However, it is not horrendously wrong: it is just a little off (at least, if the discount was not too high). With a little patience, you will realize that it is wrong, because if $p(1 + r)$ was the original price, the discounted price would be

$$p(1 + r)(1 - r) = p\left(1 - r^2\right) \neq p$$

While we could add a correction to our mistaken estimate to improve the result, here we can be satisfied observing that the mistake consisted in neglecting r^2p. If this is a sufficiently small number, we may get away with our carelessness...

3 Rational Functions

Now, we can proceed full blast, starting with the slope at the y axis. We look at the lowest terms in both numerator and denominator, and apply the arguments above. Here is a simple example:

$$\frac{4x^3-3x^2+2x+1}{3x^2-2x-2} \approx \frac{2x+1}{-2x-2} = -\left(2x+1\right)\frac{1}{2x+2} \approx -\left(2x+1\right)\left(\frac{1}{2}-\frac{2}{4}x\right) =$$

$$= -\frac{1}{2}\left(2x+1\right)\left(1-x\right) = -\frac{1}{2}\left(2x-2x^2+1-x\right) =$$

$$= x^2-\frac{x}{2}-\frac{1}{2} \approx -\frac{1}{2}-\frac{x}{2}$$

which tells us that this rational function has a y−intercept of $-\frac{1}{2}$ (which we could see directly, of course: just set $x=0$ in the original function), and that the slope of its tangent at $x=0$ is $-\frac{1}{2}$.

Of course, to get the slope of the tangent at any other point, we will have to shift the function again, just as we did before. The calculations are a bit long, but very simple.