Measuring Angles

1 Different Methods

We all know what an angle is, and we are most likely familiar with the most common practical unit of measure: the "degree" (divided in "minutes" and "seconds"). This unit (a full circle is conventionally assigned a measure of $360°$) dates back thousands of years, and is adapted to the Mesopotamian numbering system. It is also convenient in that 360 (and the subsequent subdivisions of 60) is a number with many, many divisors - hence most common simple fractions of a full circle correspond to an even number of degrees.

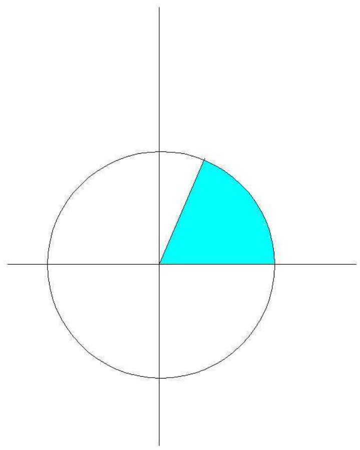
This is a good reason why the usage of other similar units, but based on a decimal scale, like assigning 100 "grades" to a right angle (and hence a full circle is 400 grades) have very little application.

There is however another unit that has actually wider application in the sciences. Its value can be appreciated once we consider trigonometric *functions,* rather than trigonometric *ratios*[1] as when dealing with triangles and simple geometric objects.

2 The Unit Circle

Consider a circle - for convenience the circle of radius 1, centered at the origin of the coordinate plane, whose equation is $x^2 + y^2 = 1$. Given an angle size, we can draw an angle of that size, with the first side on the horizontal (x) axis, anchored at the origin, and the other, also anchored at the origin, rotated by the angle size in counterclockwise direction (the choice of the direction is purely conventional, but universal):

[1] Recall that, historically, trigonometry was created to deal with triangles – the name means "measuring triangles" – starting with right triangles, so that "sine" was defined as the ratio of the leg opposite to the angel and the hypotenuse, and cosine as the ratio of the adjacent leg to the hypotenuse.



From “elementary” geometry, we know that the angle is proportional to the length of the arc it cuts on the circle (whatever unit we may use for either). Hence, we are allowed to use the length of the arc as the measure of the angle! For example, since we assumed the radius to be 1, the length cut by a right angle will be $\frac{1}{4}$ of a full circle, i.e. $\frac{1}{4} \cdot 2\pi \cdot 1 = \frac{\pi}{2}$. When using this measure, we say we are measuring an angle in *radians*. Hence, a right angle measures $\frac{\pi}{2}$ radians.

Since the length is proportional to the angle size, we have a direct rule to convert angle measures from degrees to radians and back: $360°$ is 2π, $180°$ is π, and so on. An angle of $\alpha°$ measures then $x = \frac{\alpha}{180} \cdot \pi$ radians, and an angle of x radians, measures $\alpha = \frac{x}{\pi} \cdot 180$ degrees.

Once we have this definition at hand, we can start using radians as our standard unit for angles, and start considering things like $\sin x, \cos x, \tan x$, and so on, where, say $\sin x$ is the sine of an angle measuring x radians. **In fact, this is the definition of the sine *function* (as well as cosine, tangent, and all other trigonometric functions)**.

Since, by their definition, $\sin 0 = \sin 2\pi, \cos 0 = \cos 2\pi, \tan 0 = \tan 2\pi$, etc. We can extend the notion of these functions from the interval $[0, 2\pi]$ to all real number: given an x, just add or subtract enough multiples of 2π, say n, so that $x + 2n\pi = y \in [0, 2\pi]$, and define, for example, $\sin x = \sin y$. With this convention, all trigonometric functions become *periodic functions, with period* 2π: for example, for any $n = 0, \pm1, \pm2, \ldots$

$$\sin(x + 2n\pi) = \sin x$$

This gives us a powerful modeling tool for most *periodic phenomena*. For example, the oscillation of a friction less spring can be described, with appropriate choice of the constants A, ω, ϕ, by a function of time t like

$$l(t) = A\cos(\omega t + \phi)$$

Why, though, should we first switch to radians to extend our use of these tools? The reason is that these tools become really significant after we have the methods of calculus available to us, and it turns out that it is way easier, and more natural, to use radians as units, when applying the methods of calculus to trigonometric functions.