A Quadratic Demand Function

Applications of Quadratic Functions

This is an abbreviated illustration of a possible approach to the construction of a demand function $d = d(p)$, how much we expect to sell (d), if the price is set at p, which, for some reason, we decide to make quadratic.

Since a demand function is supposed to be decreasing, we'll be using half of the graph, so to speak. First, we'll need to decide on a few points:

- The shape around $p = 0$ (as well as its positioning)
- The shape around $d = 0$ (as well as its positioning)

The technical name for "open up" is, actually, *concave up, in th*e sense of how the slope changes. Which way it goes at the extremes, defines our choice.

The choice of shape is deeper than the choice of positioning: indeed, the value of $d\,(0)$ is very much a choice of units: you might even choose an arbitrary value, like 100, and describe all the remaining values as relative to this conventional maximum (recall that $d\,(0)$ is the "demand" corresponding to a price of 0, hence the maximum possible absorption of your product by the market, when you just give it away for free).

Similarly, the value of p such that $d\,(p) = 0$, is also very much a choice of units: prices need not be expressed in dollars, and you might also express them as fractions of the maximum possible price, the one that will result in zero sales. For the purpose of this example, we'll set this value to 100 as well.

These choices already limit our freedom: indeed, thinking of our model as of the form

$$d\,(p) = ap^2 + bp + c \tag{1}$$

with a, b, c to be chosen to satisfy our specifications, we have already fixed $c = 100$ (the vertical intercept). Also, having declared that $p = 100$ is a *root* (a zero) of our quadratic function, we are specifying that (1) can be written, equivalently, as

$$a\,(p - 100)\,(p - r) \tag{2}$$

where r is the other root of our function. Matching (1) and (2), we also have

$$ap^2 + bp + 100 = ap - (100 - p)\,ap + 100ar = ap^2 + 99a + 100ar$$

which implies $ar = 1$, or $a = \frac{1}{r}$. We now have only one more choice to make. This is not very surprising: we started with three parameters to determine, and we imposed two conditions. With these two conditions in force, we are down to one free parameter only.

Note that with our conventions, the domain of our function (as restricted by the application) is given by $[0, 100]$. We need our function to be decreasing over this interval. This poses some constraints on r.

Suppose first that $a > 0$, and hence $r = \frac{1}{a} > 0$. Now the second root r can't be positive and less than 100 (that would imply negative demand for $r < p < 100$)! Since we want d to be positive throughout $(0, 100)$, r should be larger than 100, when $a = \frac{1}{r} > 0$, and the function will be open (or concave) up.

We may choose $r = \frac{1}{a} < 0$ as well. In that case, the function will be open down. Note that, since we want the function to be always decreasing, we now need, in this last case, the vertex to be situated no farther to the right than 0. By the symmetry of the position of the vertex with respect to the roots, it follows that we need $r \leq -100$ (if $r > -100$, the vertex would be situated at $\frac{r+100}{2} > 0$, and the function would be increasing between $x = 0$ and the vertex).

The choice between the two options depends very much on what we assume the behavior of the market will be when faced with a price increase from 0, and when the price approaches its upper limit.

Two interesting "limit" cases are the following:

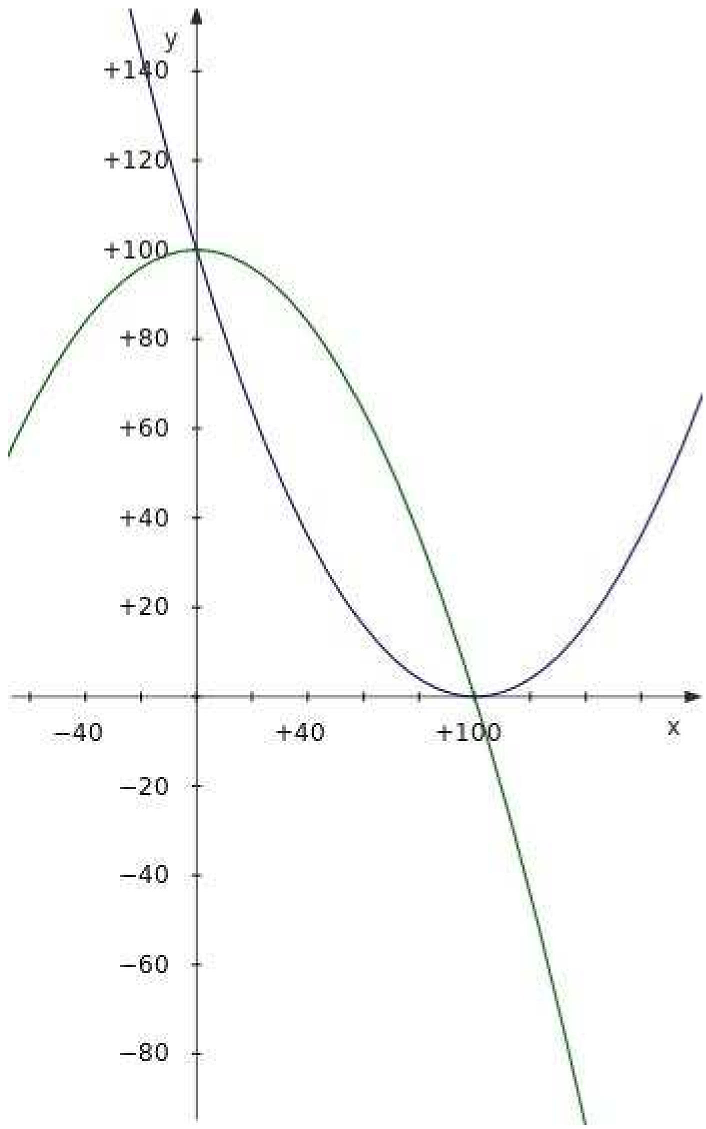
- We may choose, in the case $a > 0$, to have the vertex (a minimum) at $p = 100$. In this case, $r = 100$, and $a = \frac{1}{100}$. The curve "slides" very slowly towards 0:

$$d(p) = \frac{1}{100}(p - 100)^2 = \frac{p^2}{100} - 2p + 100$$

- We may choose, in the case $a < 0$, to have the vertex (a maximum) at $p = 0$. This amounts to set $r = -100$, since the vertex will be the midpoint between the roots. Now it is $a = -\frac{1}{100}$, and the slow behavior occurs at the top of the curve:

$$d(p) = -\frac{1}{100}(p - 100)(p + 100) = -\frac{p^2}{100} + 100$$

In the first case, the curve drops down with a slope of -2 near $p = 0$, in other words fairly sharply, but it "slides" towards 0 when p approaches 100. In the second case, the curve starts a very slow decrease form its maximum at $p = 0$, while it goes though zero, at $p = 100$ with a slope of -2 (to see this, we may use our "shifting" trick). You can't help but notice a certain symmetry between these two cases, can you?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