Complex Roots of Quadratic Polynomials

The Quadratic Formula - More!

1 An Example

Let us consider a quadratic polynomial with no real roots, for example x^2+x+1. Using our usual tricks, we can easily check that the vertex of its graph (a minimum for the function) is at $\left(-\frac{1}{2}, \frac{3}{4}\right)$. The quadratic formula results in

$$\frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} \tag{1}$$

which (obviously) does not correspond to any real number.

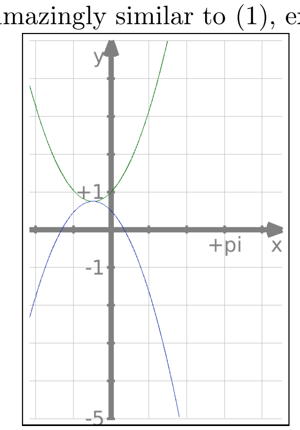
Now consider the function we get if we do a vertical "flip" around the vertex of this graph. As we know this can be obtained, for example, by combining the following steps:

1. Shift down by $\frac{3}{4}$, so the curve touches the $x-$axis: $x^2 + x + \frac{1}{4}$
2. Multiply by -1 to mirror the curve around the $x-$axis: $-x^2 - x - \frac{1}{4}$
3. Shift up by $\frac{3}{4}$ to take the vertex to its original position (but it is now a maximum): $-x^2 - x + \frac{1}{2}$

The roots of this new function are

$$\frac{1 \pm \sqrt{3}}{-2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \tag{2}$$

Amazingly similar to (1), except, of course, that they are real!



While this is not a standard observation, it gives a different (interesting?) twist on what the numbers we get when looking at "complex" roots may mean.

2 In General

Of course, the same holds in general (and in the opposite direction too, if you'd like to do that!). In fact, consider a generic quadratic function

$$f(x) = ax^2 + bx + c \tag{3}$$

Its graph has vertex at

$$\left(-\frac{b}{2a}, a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c\right) = \left(-\frac{b}{2a}, \frac{b^2}{4a} - \frac{b^2}{2a} + c\right) = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$$

(thus we also have a pre-cooked formula for the vertical coordinate of the vertex - except it's not really worth memorizing, since it's so fast and simple to calculate it every time).

Let's mimic the procedure above: if the vertex is above the x axis, $c - \frac{b^2}{4a} > 0$, and we want to shift the curve down, while, if it is below, in which case $c - \frac{b^2}{4a} < 0$, we want to shift the curve up. In both cases we want to shift by $-\left[c - \frac{b^2}{4a}\right] = \frac{b^2}{4a} - c$. We get to

$$ax^2 + bx + c + \frac{b^2}{4a} - c = ax^2 + bx + \frac{b^2}{4a}$$

Next, we mirror the curve around the $x-$axis, obtaining

$$-ax^2 - bx - \frac{b^2}{4a}$$

Finally, we shift back to the original vertex by adding $c - \frac{b^2}{4a}$. The result is

$$-ax^2 - bx + c - \frac{b^2}{2a} \tag{4}$$

The original roots of (3) were (ignoring the issue whether the discriminant is positive or negative)

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Our mirrored curve (4) has roots

$$\frac{b \pm \sqrt{b^2 + 4a\left(c - \frac{b^2}{2a}\right)}}{-2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 + 4ac - 2b^2}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}$$

Hence, if (3) had non real roots, (4) will have real roots, corresponding to the "complex" ones, in that the "imaginary part" has been "stripped" of the "i". On the other hand, if (3) had real roots, it is the mirrored function that has complex roots.