Arithmetic of Polynomials

1 Integers and Polynomials

You will notice that the mechanics of addition, subtraction, and multiplication work on polynomials much like they do on positive integers. Actually, this is also true of division, even if we won't concentrate too much on that. This is strictly due to the way we write integers using positional notation.

Recall that when we write a number, say 47201, this means "4 ten-thousands, 7 thousands, 2 hundreds, No tens, and 1 unit". In formulas:

$$4 \cdot 10^4 + 7 \cdot 10^3 + 2 \cdot 10^2 + 0 \cdot 10^1 + 1$$

(1 can be thought as $1 \cdot 10^0$ if you want). Look again, and you'll notice that that's saying $p(10)$ where $p(x)$ is the polynomial $p(x) = 4x^4 + 7x^3 + 2x^2 + 1$ (we usually skip terms with 0 as a coefficient).

2 Adding Numbers and Adding Polynomials

If we add two polynomials, we apply the basic rules of addition and multiplication, and collecting common factors, we have, for example,

$$\begin{gathered}\left(4x^4 + 7x^3 + 2x^2 + 1\right) + \left(2x^4 + 2x^3 + 3x^2 + x + 3\right) = \\ = (4+2)\,x^4 + (7+2)\,x^3 + (2+3)\,x^2 + (0+1)\,x + (1+3) = \\ = 6x^4 + 9x^3 + 5x^2 + x + 4\end{gathered}$$

Of course, we also have that $47201 + 22313 = 69514$.

In a way (even if that's definitely not how the historical sequence happened), adding integers is an application of adding polynomials. There are some adjustments to make when transferring operations from polynomials to numbers, since positional notation requires each position to be occupied by a digit between 0 and 9, with no negative coefficients. That's where the usual "carry" rule kicks in. For example,

$$\left(4 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10 + 1\right) + \left(3 \cdot 10^3 + 9 \cdot 10^2 + 7 \cdot 10 + 2\right) =$$

$$= (4+3)\,10^3 + (2+9)\,10^2 + (3+7)\,10 + (1+2) = 7 \cdot 10^3 + 11 \cdot 10^2 + 10 \cdot 10 + 3$$

Now we note that $11 = 10+1$, so that $11 \cdot 10^2 = 10^3 + 10^2$, and that $10 \cdot 10 = 10^2$. The result becomes

$$7 \cdot 10^3 + 10^3 + 10^2 + 10^2 + 3 = 8 \cdot 10^3 + 2 \cdot 10^2 + 3$$

That is 4231+3972=8203.

3 Subtracting Numbers and Polynomials

Subtraction works just like addition does. Here the tweak to the polynomial version is due to the fact that we don't use negative coefficients when writing out numbers. That's where the usual "borrow" rule kicks in. For example,

$$\left(2x^3 + 5x^2 + 2x + 3\right) - \left(x^3 + 3x^2 + 4x + 1\right) =$$

$$= (2-1)\,x^3 + (5-3)\,x^2 + (2-4)\,x + (3-1) =$$

$$x^3 + 2x^2 - 2x + 2$$

Setting $x = 10$, results in

$$10^3 + 2 \cdot 10^2 - 2 \cdot 10 + 2$$

We cannot keep the negative coefficient of 10, so we borrow 10 from the 10^2 term: $2 \cdot 10^2 = 1 \cdot 10^2 + 1 \cdot 10^2 = 1 \cdot 10^2 + 10 \cdot 10$ and

$$10^3 + 1 \cdot 10^2 + 1 \cdot 10^2 - 2 \cdot 10 + 2 = 10^3 + 1 \cdot 10^2 + (10-2) \cdot 10 + 2 =$$

$$10^3 + 10^2 + 8 \cdot 10 + 2$$

so that 2523-1341=1182.

4 Multiplication of Numbers and Polynomials

Multiplication of polynomials (and, consequently, of integers) is based on the commutative and distributive rules. While you may be used to apply a FOIL method, it really does not matter, as long as each term in a factor is multiplied by each term in the other factor.

For example, with 3 terms in each polynomial, we'll have $3 \times 3 = 9$ term. Following, to make a point, a random order, we have, for example,

$$\left(3x^2 + x + 2\right) \cdot \left(5x^2 + 2x + 4\right) =$$

$$\left(3x^2 \cdot 5x^2\right) + (x \cdot 2x) + (2 \cdot 4) + \left(3x^2 \cdot 2x\right) + (x \cdot 4) +$$

$$+ \left(3x^2 \cdot 4\right) + \left(x \cdot 5x^2\right) + \left(2 \cdot 5x^2\right) + (2 \cdot 2x) =$$

$$= 15x^4 + 2x^2 + 8 + 6x^3 + 4x + 12x^2 + 5x^3 + 10x^2 + 4x =$$

$$= 15x^4 + (6+5)\,x^3 + (2+12+10)\,x^2 + (4+4)\,x + 8 =$$

$$= 15x^4 + 11x^3 + 24x^2 + 8x + 8$$

Setting $x = 10$ forces us to "carry" several terms in order to abide by the rule of using only digits from 0 to 9 in each position. Thus, $15 \cdot 10^4 = 10 \cdot 10^4 + 5 \cdot 10^4 = 10^5 + 5 \cdot 10^4$, $11 \cdot 10^3 = 10 \cdot 10^3 + 1 \cdot 10^3 = 10^4 + 10^3$, $24 \cdot 10^2 = 20 \cdot 10^2 + 4 \cdot 10^2 = 2 \cdot 10 \cdot 10^2 + 4 \cdot 10^2 = 2 \cdot 10^3 + 4 \cdot 10^2$. Bringing it all together, we end up with

$$10^5 + 5 \cdot 10^4 + 10^4 + 10^3 + 2 \cdot 10^3 + 4 \cdot 10^2 + 8 \cdot 10 + 8 =$$

$$= 10^5 + (5 + 1)\, 10^4 + (1 + 2)\, 10^3 + 4 \cdot 10^2 + 8 \cdot 10 + 8 =$$

$$= 10^5 + 6 \cdot 10^4 + 3 \cdot 10^3 + 4 \cdot 10^2 + 8 \cdot 10 + 8$$

That is, $312 \cdot 524 = 163488$. The usual mechanism that you learned in school to multiply two numbers simply tries to automate what we have done here in painstaking detail.

5 Why Set $x = 10$?

The only reason that people have come up with is that we have 10 fingers, and that is why we tend to group quantities in tens, ten times tens, and so on. A classic question in some college entrance tests ask to write a number using the notation employed by fictitious aliens who have 12 fingers (using the digits 0, 1, 2, ..., 9, A, B, for example, where A stands for 10 and B for 11). Non fictitious variations are everywhere in computer science, where numbers are usually represented in base 2 (so, in our method, $x = 2$ – binary numbers), 8 (octal numbers), and 16 (hexadecimal numbers). The rules are exactly the same as for our decimal numbers, with carry and borrow as above, making sure we have no negative coefficients, and the digits are the only one allowed (0 and 1 for binary, 0,1,... to 7, for octal, and 0,1,2,...,9,A,B,C,D,E,F, where F=15, E=14, and so on, for hexadecimal). In fact, once we expand to fractions, and their "decimal" representation, it turns out that 10 is a lousy choice, while 12 would have been nicer, since in the latter case, a very common fraction like $1/3$ would have terminating decimal representation. In fact, the first civilization to use positional notation[1] in Mesopotamia some 5000 years ago, used base 60, since most common fractions would have a simple representation this way (we use this system when considering 60 minutes to the hour and 60 seconds to the minute, while, as far as hours to the day go, we use 12, again a simpler choice than 10!

[1] Unfortunately, they didn't have the concept of 0, so that the notation was ambiguous. When an intermediate digit should have been 0 an empty space was left. However, when the 0 should appear at the end, as, in decimals, in 120, the result was ambiguous as 12 120, 1200, and so on all had the same notation and could only be discriminated by context. A "0" appears in Greek texts, but only as a starting level, for instance when describing a vertical structure. A proper 0 appears for the first time in Southeast Asia, in the Indian cultures in present-day Indonesia and Cambodia around the 7th Century.

6 Division

The same approach applies to the division of numbers and polynomials. Here the constraints when applying polynomial division to integers is in avoiding fractions as coefficients. You may recall long division from school, and the same algorithm works for polynomials (in fact, long division for integers is an adaptation of long division of polynomials, with these added constraints – even though it was developed for integers much earlier than for polynomials!). We could analyze it in great detail as we did with the other basic arithmetical operations, but since division of polynomials is not part of the Algebra curriculum, we'll defer this to the Precalculus section.

By the way, there is a simpler way to divide polynomials than long division, but it does not scale as well if we want to apply it to integers. You can read the details in the file "Division of Polynomials".