How To Learn Everything About Any Polynomial

Calculus without calculus

1 Introduction

When you are given a function, probably as an expression, it is often desirable to be able to tell how its graph behaves, without actually graphing it. For example, a frequent question may be to find if and when the graph displays a "maximum" (it goes from increasing to decreasing) or a "minimum" (the converse). It may also be useful to know if, near a specific value of the input, the graph is increasing, decreasing or neither.

Calculus allows you to answer these and similar questions for very general functions. However, there is a class of functions that doesn't need sophisticated tools for this purpose: the polynomials (actually, this applies for the most part to rational functions too). As a matter of fact, if you go back to this paper once you have learned about the powerful tools that calculus provides, you might notice that the method we are going to discuss here leads to the very same equations that the general theory suggests.

Now, don't take this statements in the wrong way: what we are doing here is actually a technique deeply connected with calculus. However, it is an "elementary" technique, as it relies on a simple observation (which is presented in an intuitive way), and straightforward algebra.

2 Basic Fact

The following fact is the crucial starting point: for any number $x < 1$, $x > x^2 > x^3 > x^4 > \cdots$. This is easy to check: after all, any number multiplied by another number that is less than one, decreases. And the higher the power, the more you are decreasing your resulting product. To get an idea of what this entails, let's write a simple table, comparing the value of $x + 5x^2$ with the value of x, as x gets closer and closer to zero (the right hand column displays the *relative error* in approximating $x + 5x^2$ with x):

x	$x+5x^2$	$\left(x+5x^2\right)-x=5x^2$	$\frac{\left(x+5x^2\right)-x}{x}=\frac{5x^2}{x}=5x$
1	6	5	5
.1	.15	0.05	.5
.01	.0105	$5\cdot 10^{-4}$	.05
.001	.001005	$5\cdot 10^{-6}$	.005
.0001	.00010005	$5\cdot 10^{-8}$	.0005
.00001	.0000100005	$5\cdot 10^{-10}$	.00005
.000001	.000001000005	$5\cdot 10^{-12}$	.000005

The point to note here is that if we decide to approximate $x+5x^2$ with x, we will making a colossal error if $x=1$ (we will be off by 500%), but, as x decreases, not only does the difference between the two decrease, it's also the *relative* size of the error in the approximation that dwindles down to almost nothing. This allows us to ignore higher powers when calculating with small enough values of x, without much of a sacrifice in accuracy.

3 Studying A Polynomial For Values Of x Close To Zero

Consider a polynomial of any degree. To fix ideas, we'll take a couple of concrete examples, but the same argument applies to any other. We'll look at

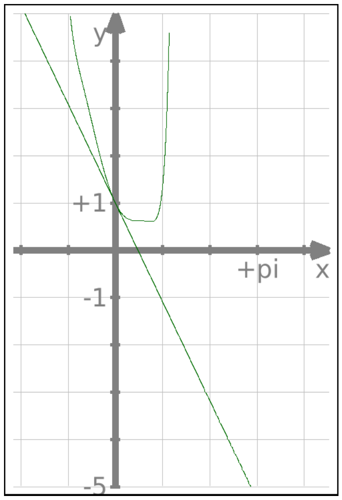
$$f(x)=x^{10}-2x^4+3x^2-2x+1 \tag{1}$$

and at

$$g(x)=x^{10}-2x^4+3x^2+1 \tag{2}$$

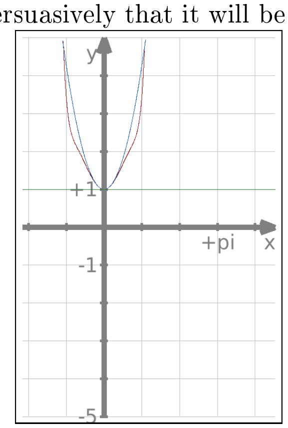
3.1 The Case of (1)

Suppose we are wondering what the graph looks like for values of x close to 0. Since x is assumed to be very small, we know that, for instance, $3x^2-2x\approx-2x$, following the discussion in section 2. Even more so, we will have that ignoring $x^{10}-2x^4$ will also cause an even more minimal error. Hence, we can say that, with a practically negligible error, if x is small enough, $f(x)\approx-2x+1$. Now, the right hand side represents a line going through $(0,1)$, with slope -2. The intuitive argument we are making (which can be made very rigorous, but we don't need this now) implies that the graph of $f(x)$ will be very close to this line as long as x does not deviate too much from 0. Hence, the graph will be decreasing, since so does this line (incidentally, this line is called the *tangent line* to the graph, at $x=0$).



3.2 The Case Of (2)

In this case, the lowest power of x is a square. Now, as before, we can still say that the function $g(x)$ will not be severely affected if we decide to ignore the higher terms. Note that ignoring the x^2 term would get us to a horizontal line $y = 1$. We will see in a minute that this is interesting: this line happens to be the tangent line to g at $x = 0$. Still, it makes sense to keep the lowest non constant term, ignoring $x^{10} - 2x^4$, and approximate $g(x) \approx 3x^2 + 1$. Now, $3x^2 + 1$ is a parabola, and you will notice that its vertex is precisely at $(0, 1)$. Hence the line $y = 1$ is indeed tangent to the parabola - and we can argue persuasively that it will be the tangent to g as well.



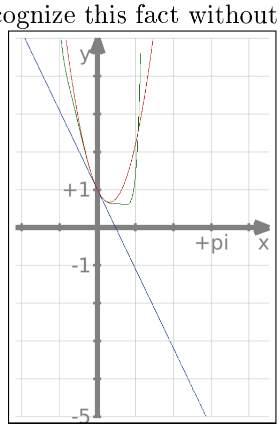
You can see how the red line (the graph of g) is hard to separate form the blue line (the graph of $3x^2 + 1$) when x is close to zero, and that the line $y = 1$ is tangent to both. Even without looking at the graph, since $3x^2 + 1$ has a minimum at $x = 0$, it is reasonable to conclude that the same applies to $g(x)$.

3.3 Going Further

You can push the study further. As an example, consider again $f(x)$ (example (1)). If we decided that ignoring $3x^2$ was too rough, we could try a better approximation for f, when x is small (but not necessarily minuscule): keep the square term, and drop the other two. This would lead us to consider

$$f(x) \approx 3x^2 - 2x + 1$$

Now, the right hand side is a parabola, and, just like f, it will go through $(0, 1)$ decreasing. Moreover, since it is a parabola, and it is "open up" (that is, it increases without bound when x gets farther from zero in both directions), it will display something like an outward "bulge" (see the picture). Well, the graph of f will display a similar "bulge" (this is called "concavity"), and we can recognize this fact without looking at the graph, by this argument.

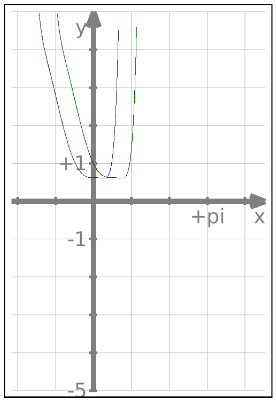


4 What If We Want To Work With $x \neq 0$?

The discussion above is simply a curiosity: sure, $x = 0$ might be a significant value in an application, but, hey, we work with the whole function, not just for small values of its input! Well, it turns out that it is easy to reduce the inspection of a function near *any* value of x to the previous case!

One trick we can use is to *shift the graph of the function horizontally.* That is, to study a different function, whose graph has exactly the same shape, and which has $x = 0$ correspond to the value of x we are interested in, in the original function. To make this less confusing, we will look how the graph of f (refer again to equation (1)) looks near $x = \frac{1}{2}$. If you check the figure, it is not completely obvious from the picture, so this exercise will tell us something more than looking at a graphing calculator.

If we consider a function with the same shape as f, but shifted to the left by 0.5, you will notice how the new function, let's call it $\widehat{f}$, looks near $x = 0$ exactly like f does near $x = \frac{1}{2}$:



Hence, what we find about this function near $x = 0$ applies to f near $x = \frac{1}{2}$! What is the equation of this function? You need to inspect the graph a bit, and also apply the logic discussed in the book about shifting graphs, to realize that $\widehat{f}(x)$ will have the same value as $f\left(x + \frac{1}{2}\right)$. In other words,

$$\widehat{f}(x) = (x + 0.5)^{10} - 2(x + 0.5)^4 + 3(x + 0.5)^2 - 2(x + 0.5) + 1$$

To apply our trick, we need to look at just the terms involving constants and x, ignoring higher powers. This makes the calculations simpler than they look at first sight, since we only need to look at the terms in x, without worrying about the higher powers. Thus, for instance, since

$$(x + 0.5)^4 = (x + 0.5)(x + 0.5)(x + 0.5)(x + 0.5)$$

we may notice that the constant term will be $0.5^4 = \frac{1}{16}$, and that the term in x will be given by

$$4 \cdot x \cdot \frac{1}{2^3} = \frac{x}{2}$$

The same consideration applies, with a bit more work, to the 10th power term, and we conclude that

$$\widehat{f}(x) \approx \frac{1}{2^{10}} + 10x\frac{1}{2^9} + \frac{x}{2} + \frac{1}{16} - 2x - 1 + 1 = \left(\frac{10 + 2^8 - 2^9}{2^9}\right)x + \frac{1}{16} + \frac{1}{2^{10}} = -\frac{246}{512}x + \frac{65}{1024} =$$

$$= -\frac{123}{256}x + \frac{65}{1024}$$

which is a line with slope $-\frac{123}{256} \approx -0.48$, going through the point $\left(0, \frac{65}{1024}\right)$. We conclude that $f(0.5) = \frac{65}{1024}$ (which we could find directly, of course), and, more interestingly, that the slope of its tangent line at $x = 0.5$ is approximately -0.48.

Note If you prefer, instead of thinking in terms of shifted graphs, we can look at the behavior of f near $x = 0.5$, by rewriting its expression as a function of $x - 0.5 = h$: $f(0.5 + h)$. Expanding the parentheses leads to the same expressions above, written with h in place of x.

4.1 Things may get more complicated - but not too much

What if the coefficient of x had turned out to be zero? Well, then we would have had to look one step further: if the coefficient of x^2 was not zero, then the point would have looked like a maximum or minimum, depending on its sign, since the function would be behaving like a parabola at its vertex. If even that had turned out to be zero, we would have had to move on to 3rd (and, if necessary, even higher) powers: the behavior of our function would have been similar to that of the corresponding polynomial. As an example, consider $h(x) = x^4 - 4x^3 + 4x^2$ near $x = 2$. If we shift the curve by 2 to the left, we get to study (ignoring all powers greater than 1)

$$\widehat{h}(x) = (x+2)^4 - 4(x+2)^3 + 4(x+2)^2 \approx 4x \cdot 2^3 + 2^4 - 4\left(3x \cdot 2^2 + 2^3\right) + 4(4x+4) =$$

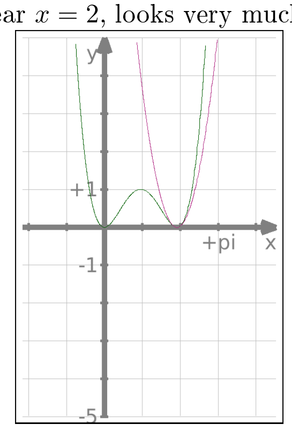
$$= 32x + 16 - 48x - 32 + 16x + 16 = 0 \cdot x + 0$$

This means that the function has the value 0 at $x = 2$ (as we can check immediately), and also, that to get more details, we better keep the x^2 terms in our approximation. If we do this, we get (note that we already know that the terms in x , and the constant are zero, so we just forget about them)

$$\widehat{h}(x) = (x+2)^4 - 4(x+2)^3 + 4(x+2)^2 \approx 6x^2 \cdot 2^2 - 4\left(3x^2 \cdot 2\right) + 4x^2 =$$

$$= 24x^2 - 24x^2 + 4x^2 = 4x^2$$

We conclude that the graph of $\widehat{h}(x)$ near $x = 0$, and hence the graph of $h(x)$ near $x = 2$, looks very much like that of a parabola with leading coefficient 4.



Remark: We needed an example where the minimum was attained at a point easy to express, hence the example was cooked up. In fact, an easy way to write the function is (just factor)

$$h(x) = x^2 (x-2)^2$$

Now it is easy to see that when $x \approx 0$ the function will look very much like $x^2 \cdot (0-2)^2 = 4x^2$, and when $x \approx 2$, it will look very much like $2^2 (x-2)^2 = 4(x-2)^2$, which is the red graph above.

Note: An equivalent way to perform the calculations in this section, to find out about the graph of f near a point $x = a$, is to rewrite the function as a function of $x = a + h$, and look at its behavior around $h = 0$. Try it out, ans see that you get the same calculations (change x to h as needed).

5 Looking For Maxima And Minima

The discussion above is interesting, but we might have more pressing questions. The one we can try to address is, given a function (a polynomial, more precisely: it's the only function type we can handle this way), how can we find its maxima and minima, if they exist? The answer is easier than we might expect (even if the actual calculations might turn out to be difficult to take to the end).

In fact, let us take, as an example, the function f in (1) - which is quite a complicated case indeed. We would like to find a value of x where the function reaches its minimum or maximum (though, from the picture, it's not too likely that the latter exists) value. We can work this way: let us shift the graph by a generic quantity - say a: we look at $\widetilde{f}(x) = f(x+a)$. We compute the expression for $\widetilde{f}(x+a)$, neglecting, as usual, all terms but the lowest powers of x, and try to pick an a such that the lowest surviving term is x^2:

$$\widetilde{f}(x) = f(x+a) = (x+a)^{10} - 2(x+a)^4 - 3(x+a)^2 - 2(x+a) + 1 \approx$$

$$\approx a^{10} + 10x \cdot a^9 - 2\left(a^4 + 4x \cdot a^3\right) - 3\left(a^2 + 2x \cdot a\right) - 2x - 2a + 1$$

The term in x is

$$10a^9 x - 8a^3 x - 6ax - 2x$$

and it will be zero if

$$10a^9 - 8a^3 - 6a - 2 = 0$$

Now this is quite a hard equation to solve, but, assuming we get hold of software able to do it, it will tell us where the polynomial attains its minimum (this requires also checking that the term in x^2has positive coefficient). Incidentally, a software package I have suggests, as approximate values, $x \approx 0.791$, and $y \approx 0.607978$. Also, incidentally, after a good chunk of your first Calculus course, applying the methods you will learn there, will lead to this very same equation!