Simple Curves in Polar Coordinates

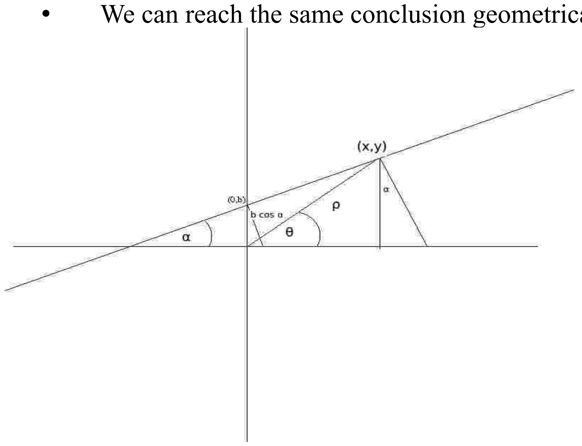
- Circle of radius R, centered at the origin: $\rho = R$
- Half-line through the origin: $\theta = u$
- General line: We can argue geometrically or algebraically.
  - Algebraically: take a line $y = mx + b$. Substituting,

$$\rho\sin\theta = m\rho\cos\theta + b \text{ or } \rho(\sin\theta - m\cos\theta) = b$$

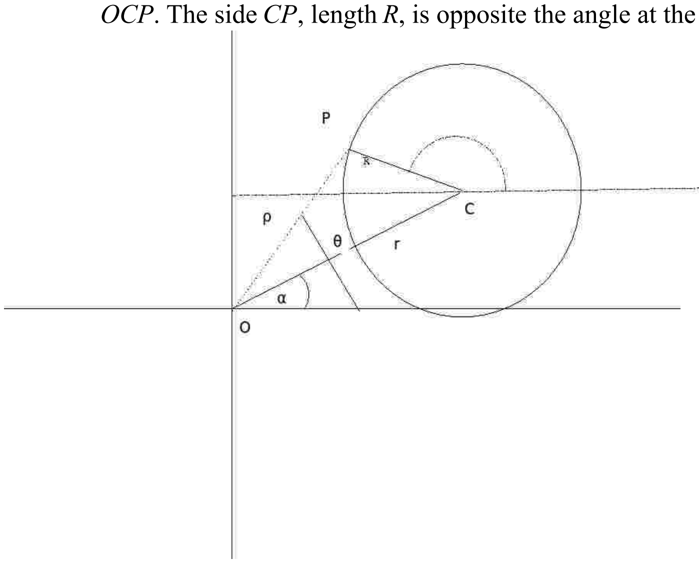
Since $m = \tan\alpha$ is the slope, this becomes

$$\rho\left(\sin\theta - \frac{\sin\alpha\cos\theta}{\cos\alpha}\right) = b$$
$$\rho(\cos\alpha\sin\theta - \sin\alpha\cos\theta) = b\cos\alpha$$
$$\rho\sin(\theta - \alpha) = b\cos\alpha$$

  - We can reach the same conclusion geometrically:



- General circle: we can argue geometrically or algebraically.
  - Geometrical construction: if the center is at (r, α), we can argue as follows. Consider the triangle OCP. The side CP, length R, is opposite the angle at the vertex O, whose size is $\theta - \alpha$.



By the Law of Cosines,

$$R^2 = \rho^2 + r^2 - 2r\rho\cos(\theta - \alpha)$$
$$\rho^2 - 2r\rho\cos(\theta - \alpha) = R^2 - r^2$$

This is an implicit equation for our circle. An example in the book has $r = R = 2$ and $\alpha = \frac{\pi}{2}$. Our equation now becomes ($\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$)

$$\rho^2 = 2R\rho\sin\theta \text{ or } \rho = 2R\sin\theta = 4\sin\theta$$

- Algebraic construction: the circle is the set of points (*x,y*) at distance *R* from the center (*a,b*). Hence it can be represented by the equation
$$(x-a)^2+(y-b)^2=R^2$$
Substituting polar coordinates, we get
$$(\rho\cos\theta-r\cos\alpha)^2+(\rho\sin\theta-r\sin\alpha)^2=$$
$$=\rho^2\cos^2\theta+\rho^2\sin^2\theta+r^2\cos^2\alpha+r^2\sin^2\alpha-2r\rho\cos\theta\cos\alpha-2r\rho\sin\theta\sin\alpha=$$
$$=\rho^2+r^2-2r\rho\cos(\theta-\alpha)=R^2$$
which is the equation we found before.